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Numerical study of the spot size changes of a guided laser pulse through a plasma channel with a density profile with radial and longitudinal variation

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Abstract: In this paper, guiding of Gaussian laser pulse in plasma channel is numerically investigated. We assumed that the plasma channel has radial and longitudinal changes. We obtained the matched condition for guiding of high-intensity laser pulses through the plasma channel. Using the source-dependent expansion (SDE) method, we extracted four paired equations for pulse amplitude, phase, spot size and inverse of the radius of wave front curvature. Numerical results of equations were obtained using the Runge-Kutta numerical method. Numerical results showed that the normalized laser spot size during propagation in the plasma channel is constant in the matched mode and oscillates for the mismatched mode. The Runge-Kutta method of order 4 has good results for well posed problems. We have considered the functional form of the model and in this work we have shown that our problem is well posed, so the proposed method has satisfactory numerical results. The numerical results confirm this concept.

Keywords: Guiding, Runge-Kutta, Matched condition, Plasma channel, Spot size

1 Introduction

Propagation of high intensity lasers in plasma is closely related to wide range of applications such as x-ray lasers, high harmonics generation and wake field acceleration [1-3]. These applications have motivated researchers to study the underlying physics of the interaction between high-intensity laser and matter. In a medium, the propagation coefficients of a high-intensity laser can be significantly different from that of a vacuum. The characteristic length associated with the laser beam diffraction is in which λ corresponds to wavelength and R_{s0} is laser spot size in vacuum. Optical guiding is a method for increasing the propagation's distance in plasma. Optical guiding based on refraction change, however, is only possible when the radial index of refractive index n has a maximum on axis; that is $\partial n/\partial r<0$. When this condition is met, phase velocity $v_p=c/n$ on axis (r=0) becomes smaller than that of the off-axis. This, in turn, converges the laser's phase front in a way that the laser beam focuses toward the axis. The refractive index of the center of an inhomogeneous plasma channel will be higher than its periphery, as long as the density of electrons in center remains smaller than the periphery density. A plasma channel with such a refractive index causes optical guiding of the laser pulse. Several methods of guiding, such as relativistic guiding [4, 5], Z-pinch waveguide [6], grazing incidence waveguide [7, 8], gas-filled capillary discharge [9, 10] have been successfully examined. In this paper, the optical guiding of a laser pulse inside a plasma channel, along with the variation of radius and longitude of the laser, will be discussed.

The organization of this paper is as follows. The equations for the laser wave envelope and matched conditions are derived in Sec. 2. In Sec. 3, numerical results of the guided laser pulse propagating through the plasma channels are presented. In Sec. 4, we show that the equations (12-15) define a well-posed problem, and the Runge-Kutta method of order 4 is applicable to them. Finally, in Section 5, the results and discussions are given.

2 Guiding a high-intensity Gaussian laser pulse through the plasma channel

Laser electric field formula obtained from Maxwell equations is defined as follows:

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$$(\nabla_\perp^2 + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2})\vec{E}(\vec{r},t) = \vec{S}(\vec{r},t), \quad (1)$$

in which $s(\vec{r},t)$ represents the source term. Electric field $E(\vec{r},t)$ and $s(\vec{r},t)$ can be expressed via complex amplitude $\vec{A}(\vec{r},t)$ and $S(\vec{r},t)$, respectively, according to the following equations:

$$\vec{E}(\vec{r},t) = \frac{1}{2}A(\vec{r},t)e^{i(k_0 z-\omega_0 t)}\hat{e}_x + c.c., \quad (2)$$

$$\vec{S}(\vec{r},t) = \frac{1}{2}S(\vec{r},t)e^{i(k_0 z-\omega_0 t)}\hat{e}_x + c.c.. \quad (3)$$

The amplitude of electric field is $A(r,z,\tau) = B(z,\tau)e^{i\psi(z,\tau)}e^{-(1-i\alpha)r^2/r_s^2}$, where $\tau = t - z/v_g$, v_g and $B(z,\tau)$ are group velocity and field's amplitude, respectively. Moreover $\Psi(z,\tau)$ corresponds to phase and $\alpha(z,\tau)$ represents the radius of curvature. Finally, $r_s(z,\tau)$ is the laser spot size. The source terms in the group velocity co-moving frame are as follows [4]:

$$S = (\frac{\omega_p^2}{c^2} - \frac{4}{r_{s0}^2})A(r,z,\tau). \quad (4)$$

The first term of source, corresponding to pre-formed parabolic plasma channel, demonstrates inhomogeneous density of plasma while second term indicates the finite transverse effect of the pulse. In equation (4), $\omega_p^2 = 4\pi n_e e^2/m$ and $n_e(r) = n_{e0} + n_{e0}\tan\left(\frac{z}{d}\right) + \frac{\Delta n_e r^2}{R_{ch}^2}$ in which density in the center of channel n_{e0}, Δn_e depth of channel, R_{ch} or channel's radius and d as a fixed parameter can be seen. Plasma profile density as a function of radius coordinate, and longitude coordinate of z in figure 1 is illustrated.

We utilize source-dependent expansion (SDE) method [4] to solve equation (1). As can be seen in below, integrals of source are used in this method:

$$F_{0,0}(z,\tau) = \frac{1}{2k_0}\int_0^\infty d\left(2\frac{r^2}{r_s^2}\right)S(r,z,\tau)e^{-(1+i\alpha(z,\tau))\frac{r^2}{r_s^2}}e^{-i\psi(z,\tau)} \quad (5-a)$$

$$F_{1,0}(z,\tau) = \frac{1}{2k_0}\int_0^\infty d\left(2\frac{r^2}{r_s^2}\right)S(r,z,\tau)\left(1 - 2\frac{r^2}{r_s^2}\right)e^{-(1+i\alpha(z,\tau))\frac{r^2}{r_s^2}}e^{-i\psi(z,\tau)}. \quad (5-b)$$

To obtain 4 coupled equations for amplitude of laser $B(z,\tau)$, phase $\Psi(z,\tau)$, the inverse of curvature's radius and spot size $r_s(z,\tau)$.

$$\frac{\partial\psi}{\partial z} = \frac{-2}{k_0 r_s^2} + \frac{2}{k_0 r_{s0}^2} - \frac{2\pi e^2 n_{e0}}{k_0 mc^2}\left(1 + \tan\left(\frac{z}{d}\right)\right), \quad (6)$$

$$\frac{\partial r_s}{\partial z} = \frac{2\alpha}{k_0 r_s}, \quad (7)$$

$$\frac{\partial\alpha}{\partial z} = \frac{2(1+\alpha^2)}{k_0 r_s^2} - \frac{2\pi e^2\Delta n_e}{k_0 mc^2}\cdot\frac{r_s^2}{R_{ch}^2}, \quad (8)$$

$$\frac{\partial B}{\partial z} = -\frac{2\alpha}{k_0 r_s^2}B. \quad (9)$$

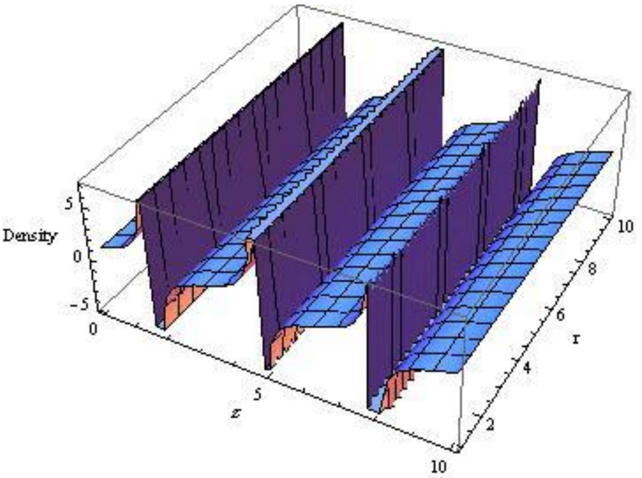

Fig. 1 Plasma density profile with radial and longitudinal variations in terms of radial coordinate r and longitudinal coordinate z.

To guide laser's pulse inside plasma channel, equation (6) to (9) should be zero. Equating equations (7) and (9) to zero, results in α=0 and B=cte, which means that amplitude of laser and curvature of wave front inside channel remain unaffected. Finally, compatibility condition for guiding Gaussian beam with TEM_{00} mode, expressed below, can be obtained by equating equation (8) to zero.

$$r_s = \left(\frac{R_{ch}^2}{\pi r_e \Delta n_e}\right)^{1/4}, \quad (10)$$

$r_e = e^2/mc^2$ is classical radius of electron.

Coupled equations (6) to (9) can become dimensionless by using change of variables as follows:

$$\tilde{r}_s = \frac{r_s}{r_{s0}}, \qquad \frac{\partial}{\partial\tilde{z}} = Z_R\frac{\partial}{\partial z}, \qquad \tilde{n}_{e0} = \frac{n_{e0}}{N_n},$$

$$\Delta\tilde{n}_e = \frac{\Delta n_e}{N_n}, \quad \tilde{B} = \sqrt{\frac{n_0 c}{8\pi}}\sqrt{\frac{\pi r_{s0}^2}{2P_{in}}}B, \tag{11}$$

r_{s0}, Z_R, N_n, ω_0 and P_{in} are laser's spot size, Rayleigh length, density of neutral gas, and input power of laser, respectively. Therefore equations (5-8) are transformed into the following equations:

$$\frac{\partial\psi}{\partial\tilde{z}} = 1 - \frac{1}{\tilde{r}_s^2} - \frac{2\pi e^2 N_n Z_0 \tilde{n}_{e0}}{mc\omega_0}\left(1 + \tan\left(\frac{z_0\tilde{z}}{d}\right)\right), \tag{12}$$

$$\frac{\partial\tilde{r}_s}{\partial\tilde{z}} = \frac{\alpha}{\tilde{r}_s}, \tag{13}$$

$$\frac{\partial\alpha}{\partial\tilde{z}} = \frac{(1+\alpha^2)}{\tilde{r}_s^2} - \frac{2\pi e^2 N_n}{mc\omega_0}\cdot\frac{Z_0 r_{s0}^2}{R_{ch}^2}\Delta\tilde{n}_e\tilde{r}_s^2\,, \tag{14}$$

$$\frac{\partial B}{\partial z} = -\frac{\alpha}{\tilde{r}_s^2}\tilde{B}. \tag{15}$$

3 Numerical Results

In this section, we provide numerical solutions of unitless equations (12-15) which are obtained from Runge-Kutta method of order 4 (which we will explain in the next section) for Gaussian laser beam with incident wavelength λ = 800 nm. Figure 2 illustrates normalized laser pulse spot size r_s/r_{s0} as a function of normalized propagation distance z/z_0 in plasma channel for matched and mismatched modes. These line charts show that although the normalized laser spot size does not vary when the wave propagates inside plasma channel in the case of matched conditions, it oscillates periodically when matched conditions are not met.

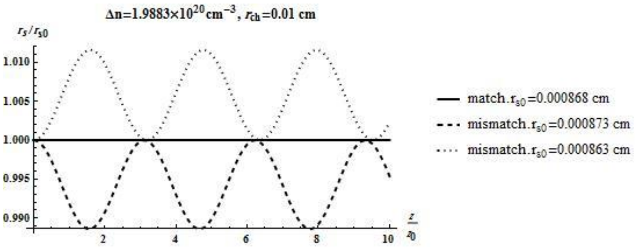

Fig. 2 Normalized spot size r_s/r_{s0} laser pulse in terms of normalized propagation distance z / z_0 during propagation in the plasma channel for matched and mismatched modes.

4 Runge-Kutta method

To show that the Rung-Kutta method of order 4 [11] can be applied to the equations (12) - (15), we write the dimensionless format of this equations as follows.

$$\dot{X}(z) = f(z, X(z)), \tag{16}$$

where

$$X(z) = \begin{bmatrix} x_1(z) \\ x_2(z) \\ x_3(z) \\ x_4(z) \end{bmatrix} = \begin{bmatrix} r_s(z) \\ \alpha(z) \\ B(z) \\ \psi(z) \end{bmatrix}, \tag{17}$$

$$f(z, X) = \begin{bmatrix} \frac{x_2}{x_1} \\ \frac{1+x_2^2}{x_1^2} - c_1 x_1^2 - c_2 x_3^2 \\ -\frac{x_2 x_3}{x_1^2} \\ 1 - \frac{1}{x_1^2} + c_3\left(1 + \tan\left(\frac{z_0 z}{d}\right)\right) \end{bmatrix}, \tag{18}$$

c_1, c_2 and c_3 are constants. According to Chapter 5 of [11], the 4th order Runge-Kutta method can be applied to well-posed systems. This means f must be continuous and satisfies a Lipschitz condition on its domain with Lipschitz constant $L < \infty$. Value of f satisfies the following inequality [12, 13]:

$$\left|\frac{\partial f}{\partial x_i}\right| \le L, \quad i = 1,2,3,4. \tag{19}$$

Now suppose

$$0 < m_1 = \min_{0\le z} x_1(z), M_i := \max_{0\le z} x_i(z), i = 1,2,3.$$

Then for the function f in (17) we have

$$\left|\frac{\partial f}{\partial x_1}\right| = \sqrt{\left(\frac{x_2}{x_1^2}\right)^2 + 4\left|c_1 x_1 + \frac{1+x_2^2}{x_1^2}\right|^2 + 4\left(\frac{x_2 x_3}{x_1^2}\right)^2 + \frac{4}{x_1^6}}$$
$$\le \sqrt{\left(\frac{M_2}{m_1^2}\right)^2 + 4\left|c_1 M_1 + \frac{1+M_2^2}{m_1^2}\right|^2 + 4\left(\frac{M_2 M_3}{m_1^2}\right)^2 + \frac{4}{m_1^6}}$$
$$=: L_1,$$

$$\left|\frac{\partial f}{\partial x_2}\right| = \sqrt{\left(\frac{1}{x_1}\right)^2 + 4\left|\frac{x_2}{x_1^2}\right|^2 + 4\left(\frac{-x_3}{x_1^2}\right)^2}$$

$$\leq \sqrt{\frac{1}{m_1^2} + 4\frac{M_2^2}{m_1^4} + \frac{M_3^2}{m_1^4}} =: L_2,$$

$$\left|\frac{\partial f}{\partial x_3}\right| = \sqrt{\left(\frac{-x_2}{x_1^2}\right)^2 + 4c_2^2 x_3^2} \leq \sqrt{\frac{M_2^2}{m_1^4} + 4c_2^2 M_3^2} =: L_3,$$

$$\left|\frac{\partial f}{\partial x_4}\right| = 0.$$

This means f has a Lipschitz condition with Lipschitz constant $L = \underset{i=1,2,3}{\mathrm{Max}}\, L_i < \infty$ and since f continuous on its domain, then the system (15) is well posed.

5 Conclusions

In this paper, we investigated the Gaussian laser pulse guidance in the plasma channel numerically. It was assumed that the plasma channel's length and radius change over time. Matching conditions for guiding high intensity laser was also acquired. Using SDE method, the equations governing laser pulse parameters were solved by Runge-Kutta numerical method. Numerical results showed that in the case of matching condition, normalized laser spot size does not change. However, when these conditions are not met, the size of laser spot oscillate periodically.

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