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An impressive method for adjoint of linear and nonlinear operators

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Abstract In this paper, we have obtained the adjoint of an arbitrary operator (linear and nonlinear) in Hilbert space by introducing an n-dimensional Riemannian manifold. This general formalism covers every linear operator (non-differential) in Hilbert space. In fact, our approach shows that instead of directly using an operator's adjoint definition, it can be obtained directly by relying on a suitable generalized space according to the action of the operator in question. In the case of nonlinear operators, we must change the definition of the linear operator adjoint. However, here, we have obtained an adjoint of these operators concerning the definition of the derivative of the operator. We have shown one of the straight applications of "Frechet derivative" in the algebra of the operators.

1 Introduction

This paper consists of two main parts. In the first part, we look for a general relationship for the adjoint of the linear operators. We intend to achieve a universal formula for the adjoint of the linear operators with a generalized space, such as the Riemannian manifold. Although the self-adjoint extensions of differential operators on the Riemannian manifold have been studied [1], we focus on non - differential operators. Recently, it has been reported that linear operators are unitary in curved space [2].

In the second part, we seek a special approach for the nonlinear operators.

Consider a linear operator O such that [3]

$$Of(x) = f(3x^2 + 1). \quad (1)$$

Even though we can calculate the adjoint of the operator O, we show in the next section that it is possible to do it directly by introducing the generalized space. Now, we define the linear operator $\Im_\varepsilon$ by

$$\Im_\varepsilon f(x) = f\big(x + \varepsilon h(x)\big), \quad (2)$$

where ε is an infinitesimal quantity. Obviously, considering $h(x) = 1$, the operator $\Im_\varepsilon$ is the translation operator in standard quantum mechanics [4]. But, in general, the arbitrary function $h(x) = 1$ could be the space metric, such that the operator $\Im_\varepsilon$ will be the translation operator in generalized space. In section II, we introduce a general approach to define the adjoint of these operators.

Most of the operators we deal with in quantum mechanics are linear. By definition, every linear operator must have the following two conditions [5]:

$$A[f(x) + g(x)] = A[f(x)] + A[g(x)], \quad (3)$$

$$A\big(af(x)\big) = aA\big(f(x)\big), \quad (4)$$

where a is a complex constant.

There are two different fundamental classes of non-linear operators: One is homogeneous non-linear operators, which do not satisfy the condition of equation (4), and the second one is nonhomogeneous nonlinear operators, which do not satisfy the condition of equation (3), [6]. As an example, for a homogeneous nonlinear operator, we can write

$$Af = |f| \int_0^{2\pi} e^{ia} B e^{-ia} \frac{f}{|f|} da, \quad (5)$$

where the domain of A is the same as the domain of B.

The operator A is not differentiable, so as we will see further, the adjoint of this operator cannot be defined. In fact, the point is that any differential operator that has the property of (4) is a linear operator. The argument is very simple.

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In the mathematical literature, the anti-linear or conjugate linear operator is a renowned example that does not satisfy the equation (4). This operator is reminiscent of the time reversal operator in quantum mechanics. It is well known that Dirac's bra-ket notation is not suitable for these operators [4]. A report examines this problem with a special approach [7].

Now consider a non-linear operator B which can be defined by

$$B[f(x)] = [f(x)]^2. \tag{6}$$

In section III, we discuss the general approach for the adjoint of the operator B.

2 Adjoint of linear operator

Consider the Riemannian manifold M with an atlas consisting of only one chart. This manifold is such that a global one-to-one correspondence exists between the points of M and $\mathbb{R}^n$.

So, the inner product of two arbitrary functions ψ and ϕ are defined through.

$$\langle\varphi|\psi\rangle = \int \zeta(x)\overline{\phi(x)}\psi(x), \tag{7}$$

where ζ is the invariant volume element $\zeta(x) = g(x)d^n x$, and g is the square root of the absolute value of the determinant of the metric matrix. We define an arbitrary operator O by

$$\langle x|O|\psi\rangle = \psi[h(x)], \tag{8}$$

so that

$$\langle\phi|O|\psi\rangle = \int \zeta(x)\bar{\phi}(x)\,\psi[h(x)], \tag{9}$$

where $h(x)$ is smooth, invertible, and differentiable function. Now we define z through $z = h(x)$ and using

$$\zeta[h^{-1}(z)] = \left[\frac{(h^{-1})^* g}{g}\right](z)\zeta(z), \tag{10}$$

where h^* is the pullback of h[1], one arrives at

$$\langle\phi|O|\psi\rangle = \int \zeta(z)\left[\frac{(h^{-1})^* g}{g}\right](z)\bar{\phi}[h^{-1}(z)]\,\psi[z], \tag{11}$$

The equation (11) shows that

$$\langle x|O^\dagger|\phi\rangle = \left[\frac{(h^{-1})^* g}{g}\right](x)\bar{\phi}[h^{-1}(x)]. \tag{12}$$

This is correct if h is one-to-one, otherwise, there would be several inverses for h.

Denoting these by ${h_i}^{-1}$, hence we can write:

$$\langle x|O^\dagger|\phi\rangle = \sum_i \left[\frac{({h_i}^{-1})^* g}{g}\right](x)\,\phi[{h_i}^{-1}(x)]. \tag{13}$$

Equation (13) can calculate the adjoint of any arbitrary linear operator (non-differential) with any representation or definition. For two simple examples, see Appendix A.

Now, we can compute operators (1) and (2) adjoints. For the first case, we have

$$h(x) = 3x^2 + 1, h^{-1}(z) = \pm\sqrt{\frac{z-1}{3}}, \left|\frac{d[h^{-1}(z)]}{dz}\right| = \frac{1}{6\sqrt{\frac{z-1}{3}}}, \tag{14}$$

So, we can write

$$\langle z|O^\dagger|\phi\rangle = \frac{1}{6\sqrt{\frac{z-1}{3}}}\left[\phi\left(\sqrt{\frac{z-1}{3}}\right) + \phi\left(-\sqrt{\frac{z-1}{3}}\right)\right]. \tag{15}$$

For the operator (2), at first, we consider an infinitesimal diffeomorphism f_ε, with

$$f_\varepsilon = x + \varepsilon h(x). \tag{16}$$

Note that, in this equation x and f are members of the manifold M. Their coordinates are, of course, scalars, as they are the components of the $h(x)$, which is itself a vector.

[1] According to the definition, the pullback of the function h is equal to:
$(h^* f)_{\mu\nu}(x) = \{f_{\alpha\beta}[h(x)]\}\frac{\partial h^\alpha}{\partial x^\mu}\frac{\partial h^\beta}{\partial x^\nu}$

The definition of pullback f gives the following equation:

$$(f^*g)_{\mu\nu} = g_{\mu\nu} + \varepsilon\left[\left(\nabla_\mu h\right)_\nu + (\nabla_\nu h)_\mu\right], \tag{17}$$

where ∇ is the covariant derivative corresponding to the Levi-Civita connection in relation to the metric g. So, by using (17) we can obtain immediately

$$f^*g = [1 + 2\varepsilon\nabla . h]g. \tag{18}$$

According to equation (2) and after some calculations we get the adjoint of the operator $\Im_\varepsilon$ in the form of

$$\Im_\varepsilon{}^\dagger f(x) = f\left[\left(x - \varepsilon h(x)\right)\left(1 - \varepsilon(\nabla . h)(x)\right)\right]. \tag{19}$$

3 Adjoint of nonlinear operator

For some reasons that will be revealed later we provide another definition in equations (3) and (4), for arbitrary linear operator A.

Definition: An arbitrary operator is linear if and only if its "Frechet derivative" is constant number or constant matrix. In another statement, operator A which acting on a function f is linear if its "Frechet derivative" does not depend on function f.

To clarify this definition, we consider two examples. In our opinion, probably, the equivalency of our definition with the common definition (equations (3) and (4)) is also true for other examples.

At first, consider linear operator A such that $Af(x) = f(g(x))$, where $f(x)$ and $g(x)$ are two arbitrary linear functions. So, we can write:

$$\begin{aligned} A[f(x) + h(x)] &= f(g(x)) + h(g(x)) \\ &= Af(x) + \int h(y)\delta(g(x) - y)d \end{aligned} \tag{20}$$

The derivative of A or $(DA)f(x)$ is given by

$$(DA)f(x) = \delta(g(x) - y). \tag{21}$$

Obviously, the equation of (21) shows that the derivative does not depend on the function f. It is simple to show that conditions (3) and (4) both, are consistent for this operator.

Now, consider another example:

$$Bf(x) = ln(f(x)). \tag{22}$$

We can write:

$$\begin{aligned} B[f(x) + h(x)] &= ln[f(x) + h(x)] \\ &= Bf(x) + \frac{h(x)}{f(x)}, \end{aligned} \tag{23}$$

where, we used the first order of h. Therefore, the derivative of B is given by

$$(DB)f(x) = \frac{1}{f(x)}. \tag{24}$$

The derivative of nonlinear operator B depends on function f. Again, it is simple to show that the conditions (3) and (4) aren't consistent for this operator.

As the complex conjugate operator is not differentiable and linear, in an orthodox manner, its adjoint is not defined. However, perhaps one could extend the definition of the adjoint operator to include this case as well. This definition is attributed to Wigner [8]. If we consider the usual definition for adjoint of an operator A as following

$$\langle u|A^\dagger v\rangle = \langle Au|v\rangle, \tag{25}$$

So for the anti-linear operators we should change the definition (25) to following form:

$$\langle u|A^\dagger v\rangle = \langle v|Au\rangle. \tag{26}$$

There are some references concerning nonlinear operator algebra [9, 10], with no specific suggestion to define the adjoint of nonlinear operators. In one reference, [11], for the special class of nonlinear operators in Banach space, which most of its operators are similar to linear operators, the adjoint of these operators is introduced on the basis of their derivatives.

At first, we notice that for the nonlinear operator B, whatever $B^\dagger$ would be, the statement $\langle u, B^\dagger v\rangle$ should be anti-linear in terms of u. So we need to construct some function like $\Re$ of u, v and B in such a way that by using the inner product, we get, namely

$$\langle u, B^\dagger v\rangle = \Re(u, v, B). \tag{27}$$

But if the definition somehow resembles the definition of the adjoint operator, we expect that some operation like the action of B on u occurs in $\Re$. The problem is that if the action of B on u is nonlinear, then it does n't seem to be a natural way to construct something anti-linear in terms of u, say from (Bu) and the inner product. The reason that one could construct such a thing for linear or anti-linear B's, is that $\langle Bu, . \rangle$ is anti-linear in terms of u if B is linear, and if B is anti-linear, $\langle . , Bu \rangle$ would be anti-linear in u. So, the problem is to construct something quasilinear (linear or anti-linear) in u, from the action of something related to B on u.

One way to do so, is to use the derivative of B instead of B itself. According to our definition in section I, if B is linear, then its derivative is a constant matrix, which its action on a vector u is the same as $B(u)$, namely,

$$(DB)u = B(u). \tag{28}$$

For the general case where B is not linear, of course above relation does not hold. Then in the general case of nonlinear operator B, let's define $B^\dagger$ as $(DB)^\dagger$, that is

$$\langle u, B^\dagger v \rangle = \langle (DB)u, v \rangle. \tag{29}$$

But then, the problem is that (DB) is no longer a constant, if B is nonlinear operator. So, the correct form of the above relation should be

$$\langle u, [B^\dagger(f)]v \rangle = \langle [(DB)(f)]u, v \rangle, \tag{30}$$

where f is some point.

Now we come back to the equation (5), then

$$\begin{aligned}[B(f+\delta f)](x) &= [(f+\delta f)(x)]^2 \\ &= [f(x)]^2 + 2[f(x)][(\delta f)(x)] + \ldots \\ &= [f(x)]^2 + \int dy \{2[f(x)]\delta(x-y)\} \\ &\qquad \times [(\delta f)(y)] + \ldots,\end{aligned} \tag{31}$$

which means that

$$[(DB)(f)](x,y) = 2[f(x)]\delta(x-y). \tag{32}$$

The left-hand side is the matrix element of $[(DB)(f)]$. So,

$$\begin{aligned}\{[(DB)(f)]u\}(x) &= \int dy\, \{[(DB)(f)](x,y)\} u(y) \\ &= 2f(x)u(x).\end{aligned} \tag{33}$$

Then ,

$$\begin{aligned}\langle [(DB)(f)]u, v \rangle &= \int dx \overline{[2f(x)u(x)]}\, v(x) \\ &= \int dx \overline{[u(x)]}\, \overline{[2f(x)]} v(x).\end{aligned} \tag{34}$$

Therefore,

$$\{[B^\dagger(f)v]\}(x) = \overline{[2f(x)]} v(x), \tag{35}$$

or

$$[B^\dagger(f)](x,y) = \overline{[2f(x)]}\delta(x-y). \tag{36}$$

4 Conclusion

In this paper we have searched two original questions in two parts. 1) Is there a universal formula for the adjoint of the linear operator? 2) Is there a definition for the adjoint of the nonlinear operators? Regarding the first question, one can generalize the metric space and perform the adjoint computation on a Riemannian manifold to reach a universal formula. The equation (13) could be a universal formula for the adjoint of the linear operators. Regarding the second question, we have argued and indicated by using the definition of the derivative of the operator, we can obtain an adjoint of the nonlinear operator. Our method in this paper is suitable for nonlinear operators that are differentiable. It seems to nonlinear operators that are not differentiable do not exist in a "natural way" to define adjoint[2]. Our meaning of the "natural way" is preserving some of the properties of the standard definition of the adjoint. For instance, in the new definition, keeping the inner product's absolute value is necessary. A central concept in the linear operator's theory is the concept of the inner product. That is why we have used the operator's derivative to define the adjoint of nonlinear operators.

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[2] For example, consider an operator such as A which $A\psi(x) = |\psi(x)|^2$ which is not differentiable. One can write:
$|\psi(x) + h(x)|^2 = |\psi(x)|^2 + \psi(x)h^*(x) + \psi^*(x)h(x) + O(h)$
but the sum of the second and third terms (the pseudo-linear part) is neither linear nor anti-linear. Therefore, we don't know yet a "natural way" to define the adjoint of this operator.

Appendix A:

Here, we consider two examples with respect to the equation (13).

Example 1:

Consider $h(x)=\frac{1}{x}$ and $h^{-1}(z)=\frac{1}{z}$ so we can write $\left|\frac{d[h^{-1}(z)]}{dz}\right|=\frac{1}{z^2}$, therefore we get $\langle z|O^{\dagger}|\phi\rangle=\frac{1}{z^2}\phi(\frac{1}{z})$.

Example 2:

Suppose $h(x)=-x^2$ and $h_{\pm}{}^{-1}(z)=\pm\sqrt{-z}$ then we can write $\left|\frac{d[h_{\mp}{}^{-1}(z)]}{dz}\right|=\frac{1}{2\sqrt{-z}}$ therefore we obtain

$$\langle z|O^{\dagger}|\phi\rangle=\frac{1}{2\sqrt{-z}}\left[\phi(\sqrt{-z})+\phi(-\sqrt{-z})\right]$$

References

1. S. O. Milatovic, F. Truc, Ann.Glob.Anal.Geom **49** (2016)
2. M. Jafari Matehkolaee, Chinese Phys. B **30** (2021).
3. S. Gasiorowicz, *Quantum Physics* (John Wiley & Sons, 2003)
4. J.J. Sakurai, *Modern Quantum Mechanics*, (edited by San Fu Tuan Benjamin, Cumminges, Menlo Park, Ca. 1985)
5. Weidmann,J, *Linear operators in Hilbert Spaces* (Springer-Verlag, New York, 1980)
6. S. Bugajski, Int.J.Theor.Phys. **30**, 7 (1991)
7. A.Royer, Am. J. Phys. **62**, 8 (1994)
8. A. Uhlmann, Sci. China: Phys. Mech. Astron. **59**, 3 (2016)
9. C. Schwartz I, J. Math. Phys. **38**, 7 (1997)
10. C. Schwartz II, J. Math. Phys. **38**, 7 (1997)
11. V. Buryskova, The Rocky M. J. of Math. **28**, 1 (1998)