TTMP vol 1(1), 2024
doi: 10.30511/TTMP.2024.709795

Transactions in Theoretical and Mathematical Physics

An effective technique for solving generalized Cahn-Hilliard (C-H) problems

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Received: 12 August 2023/ Accepted: 04 November 2023/ Published: 05 February 2024

Abstract Throughout this paper, we apply the Optimal Homotopy Asymptotic Method (OHAM) to find out the numerical solutions of the fractional Cahn-Hilliard (C-H) equation. We examine fractional order time-dependent partial differential equations to assess the method's competency. In the Caputo sense, fractional-order derivatives have been applied with numerical values in the closed interval [0, 1]. The biggest advantage of this method is that it contains parameters that strongly control the solution series convergence. Additionally, this method greatly simplifies calculations because it does not require any linearization, discretization, or little perturbations. Approximate solutions of the C-H equation were compared with the exact solutions; moreover, the results of the suggested method have been compared with those of other widely used numerical techniques, such as the Adomian decomposition analysis method. A comparison of these solutions with the exact solution shows that our method is more effective and accurate for solving nonlinear differential equations. MATLAB R2021b is utilized to generate the numerical results.

Keywords: Cahn–Hilliard equation, Fractional calculus, Optimal Homotopy Asymptotic Method, Numerical solutions

1 Introduction

According to the historical standpoint, fractional calculus has always been a classical calculus. Even so, in the present era, fractional calculus has wide-ranging solicitations in many technological fields, and as a result, it has more attentiveness [1,2]. Since analytical frameworks are often hard to find, very few researchers have examined their mathematical approximation approaches using the effective application of fractional systems in these disciplines. Fractional-order differential equations are useful for modelling a wide range of real-world issues. Numerous fields, including biological sciences, electromagnetic theory, electric grids, diffuse transport, groundwater problems and fluid mechanics, can benefit from the usage of these equations [3-11]. Finding the approximation of a nonlinear problem's solution is an alternative to trying to discover the exact solution, which is extremely challenging to achieve. To solve linear and nonlinear problems of FDEs, several numerical methods are used [12-22], like the Adomian Decomposition Method (ADM), Variational Iteration Method (VIM), homotopy perturbation method (HPM), and others. The previously mentioned techniques work for simple nonlinear while others work even for complex nonlinear problems. In 2008, to solve a nonlinear problem, Marinca and Herisanu [23] presented a numerical method called OHAM. The OHAM is valid not only for solving small parameters but also for nonlinear problems in science and engineering [23-27]. This strategy was then used to develop a number of solutions to large nonlinear problems across several researchers. It was shown in several of these studies that this approach is a reliable, simple, and effective tool for providing accurate analytical approximations to numerous severely nonlinear problems

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[28, 29]. Additionally, it was discovered that its essential advantage is that it's able to manage the convergence of approximate series solutions in a perfect manner. The OHAM needs only two or three terms to get an accurate solution thus, it guarantees a prompt convergence. In fact, it is the real power of this technique. The auxiliary constants (Convergence control parameters provide us with a reasonable way to ensure the convergence of OHAM series solution. By 1958, the classical Cahn–Hilliard equation (C-H equation) was introduced by the American scientists JW Cahn and J Hilliard [30]. In the field of mathematical physics, this model is one of the most studied models. Many physical phenomena, such as spinodal decomposition, phase separation, and phase ordering dynamics, are associated with this equation. Such mathematical physics equation characterizes the phase separation process through which the two components of the binary fluid are automatically separated [31-32]. Hence, this paper is organized according to the following: In section 2, an illustration of the fractional calculus definitions. Section 3 is devoted to the analysis of OHAM, and in section 4, we tackle the numerical of two problems of (CH) in the sense of the Caputo order operator. The conclusion is presented in Section 5.

2 Description of Fractional Calculus

Below I will discuss some important definitions and formulas of the theory of fractional derivatives and integrals that will be employed in this article [1,2]; we adopt the two commonly used definitions: the Caputo and its reverse operator Riemann-Liouville. That is due to Caputo fractional derivative permits traditional initial condition assumption and boundary conditions. In the following, we are going to provide the necessary remarks and basic definitions:

2.1 Riemann Liouville fractional integral

The Riemann–Liouville fractional integral of order $\alpha > 0$ of a function is $f: R^+ \to R$ given by [13]

$$J^\alpha f(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1} f(t)dt, \alpha > 0, x > 0,$$

$$J^0 f(x) = f(x).$$

Hence, we have:

$$J^\alpha t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} t^{\alpha+\gamma},\ \ \alpha > 0,\ \gamma > -1,\ t > 0.$$

2.2 Riemann–Liouville fractional derivative

Riemann–Liouville fractional derivatives of order α of a continuous function $f: R^+ \to R$ is obtained consecutively by

$$D^\alpha f(x) = D^m\left(J^{m-\alpha} f(x)\right),$$

$$D_*^\alpha f(x) = J^{m-\alpha}\left(D^m f(x)\right),$$

where $m-1 < \alpha \le m\,, m \in N.$

The Riemann–Liouville derivative has certain drawbacks when attempting to model real-world phenomena with fractional differential equations. Therefore, we will propose a modified fractional differential operator proposed by Caputo. Fractional-order differential equations, at least, are as stable as their integer-order counterpart [19].

2.3 The Caputo fractional derivative

The Caputo fractional integral of order α of a function $f: R^+ \to R$ is given by

$$D^\alpha f(x) = J^{m-\alpha}\left(D^m f(x)\right) = \frac{1}{\Gamma(m+\alpha)}\int_0^x (x-t)^{m-\alpha-1} f^m(t)dt,$$

where $m-1 < \alpha \le m, m \in N, x > 0.$

The following properties are some of the essential fractional derivatives and integrals for $\alpha, \beta \in R^+$[35]:

$$J^\alpha J^\beta f(x) = J^{\alpha+\beta} f(x),$$

$$J^\alpha J^\beta f(x) = J^\beta J^\alpha f(x),$$

$$J^\alpha t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} t^{\alpha+\gamma},\ \ \alpha > 0, \gamma > -1, t > 0.$$

Lemma. If $-1 < \alpha \le m\,, m \in N$, f(x) $\in C_\mu^m, \mu \ge -1$ it holds

$$D^\alpha J^\alpha f(x) = f(x),$$

$$J^\alpha D^\alpha f(x) = f(x) + \sum_{k=0}^{m-1} f^k(0^+)\frac{x^k}{k!},\qquad x > 0.$$

3 The basic idea of the optimal homotopy analysis method (OHAM)

OHAM for fractional partial differential equations is presented in steps as follows:
Consider the following partial differential equation:

$$D\left(u(x,t) + g(x,t)\right) = 0\,, x\epsilon\Gamma\ \ , t \ge 0\,, \tag{1}$$

$$B\left(u, \frac{\delta u}{\delta t}\right) = 0, \tag{2}$$

where D denotes the operator which may be an integer or fractional order differential operator. x, t indicate an independent variable, $u(x,t)$ is an unknown function, B is a boundary operator and Γ is the boundary of the domain Ω, and $g(x,t)$ denotes known expression in Eq. (1).

Now, we can split the differential operator D into the terms of L, N differential operators so that:

$$L(u(x,t) + N(u(x,t) + g(x,t)) = 0, \ x\epsilon\Gamma \ . \quad (3)$$

Here, L denotes the simpler linear differential operator, which may be the linear and non-complicated portion of the Eq. (1) so that it would be solvable via any auxiliary analytical method, whereas N the operator denotes the differential operator which would be a non-linear and complicated portion of the Eq. (1).

We first construct the homotopy as:

$$(1-q)L(\phi(x,t;q)) = H(q)(L(\phi(x,t;q) + N(\phi(x,t;q)), \quad (4)$$

$$B\left(\phi(x,t;q), \frac{\delta\phi(x,t;q)}{\delta t}\right) = 0, \quad (5)$$

where $q \in [0,1]$ is an embedding parameter, $H(q)$ is a nonzero function for $q \neq 0$ and $H(0) = 0$, $\phi(x,t;q)$ is an unidentified function.

Clearly, when $q = 0$ and $q = 1$ it holds:

$$\phi(x,t;0) = u_0(x,t), \quad \phi(x,t;1) = u(x,t),$$

respectively. Therefore, when q increases from 0 to 1, the solution $\phi(x,t)$ varies from $u_0(x,t)$ to the solution $u(x,t)$ so it guarantees a rapid convergence to the exact solution.

The auxiliary function $H(q)$ gives us a simple way to manage and control the convergence while increasing the precision of the method's results and effectiveness.

The auxiliary function $H(q)$ is chosen in the form

$$H(q) = q\,c_1 + q^2 c_2 + q^3 c_3 + \cdots, \quad (6)$$

where c_i, $i = 1,2,3,\ldots$ are auxiliary convergence control parameters.

Expand $\phi(x,t;q,c)$ in Taylor's series about q, to find out approximate solutions as:

$$\phi(x,t;q,c_i) = u_0(x,t) + \sum_{k=1}^{\infty} u_k(x,t;c_i)\,q^k. \quad (7)$$

It was noted that the auxiliary convergence-control parameters were among the main causes of series (7) convergence.

When we substitute $\phi(x,t;q,c_i)$, and $H(q)$ in Eqs.(1-2) and equal coefficient of the same powers of q, then we get Zeroth-order and series problems, respectively:

$$L(u_0(x,t)) = 0 \ , \qquad B\left(u_0(x,t), \frac{\partial u_0(x,t)}{\partial t}\right) = 0, \quad (8)$$

$$L(u_1(x,t)) = c_1\,N_0(u_0(x,t)), B\left(u_1(x,t), \frac{\partial u_1(x,t)}{\partial t}\right) = 0, \quad (9)$$

$$L(u_2(x,t)) = c_2\,N_0(u_0(x,t)) + c_1\,N_1(u_0(x,t), u_1(x,t)) + (1 + c_1)\,L(u_1(x,t)),$$
$$B\left(u_2(x,t), \frac{\partial u_2(x,t)}{\partial t}\right) = 0\,. \quad (10)$$

The general governing k^{th} - order problem of the analytical solution is $u_k(x,t)$ in the form:

$$L(u_k(x,t)) - L(u_{k-1}(x,t)) = c_k\,N_0(u_0(x,t)) + \sum_{i}^{k-1} c_i\big[L(u_{k-i}(x,t)) + N_{k-i}(u_0(x,t), u_1(x,t), \ldots, u_{k-1}(x,t))\big], \quad k = 2,3,\ldots, \quad (11)$$

$$B\left(u_k(x,t), \frac{\partial u_k(x,t)}{\partial t}\right) = 0, \quad (12)$$

where N_i, $i > 0$ is the coefficient of q^i in the nonlinear operator.

$$N\,(u(x,t)) = N_0(u_0(x,t)) + qN_1(u_0(x,t), u_1(x,\,t)) + q^{\,2} N_2(u_0(x,t), u_1(x,t), u_2(x,t)) + \cdots. \quad (13)$$

If series (4) converges at q = 1, one has:

$$\tilde{u}(x,t,c_i) = u_0(x,t) + \sum_{k=1}^{\infty} u_k\,(x,t;c_i), \ i = 1,\ldots. \quad (14)$$

Substituting Eq. (14) into Eq. (1), we get the residual as follows:

$$R(x,t;c_i) = L(\tilde{u}(x,t,c_i)) + N(\tilde{u}(x,t,c_i)), \quad i = 1,\ldots. \quad (15)$$

As $R(x,t;c_i) = 0$ then $\tilde{u}(x,t,c_i)$ happens to be the same solution. Identifying the auxiliary convergence-control parameters c_1, c_2, c_3,… using a method of the following as Ritz, Least square, Collocation, and Galerkin's Method [33].

In the presentation below, to obtain the optimal values of auxiliary convergence-control parameters, the least square method is used as follows:

$$J(c_i) = \int_0^t \int_\Omega R^2(x,t,c_i)\, dx\, dt, \qquad (16)$$

where R is the residual. The unidentified constants $c_i (i = 1,2,3,\ldots,m)$ is optimally identifiable from the conditions:

$$\frac{\delta J}{\delta c_1} = \frac{\delta J}{\delta c_2} = \cdots = \frac{\delta J}{\delta c_m} = 0. \qquad (17)$$

Obviously, in the case that the low order of m, the nonlinear algebraic system can be solved easily but the larger m is, the more difficult to solve.

4 Numerical simulations

Through this part, we are going to apply OHAM for two problems of the Time-Fractional Cahn-Hilliard Equation.

Example 1 Consider the following form of the time-fractional C-H equation [34]

$$D_t^\alpha u = u_x + 6\,u\,(u_x)^2 + (3u^2 - 1)\ ;\ 0 < \alpha \le 1, \qquad (18)$$

with the initial condition

$$u(x,0) = \tanh\left(\frac{\sqrt{2}}{2}\,x\right). \qquad (19)$$

The exact solution of Eq. (18)

$$u(x,t) = \tanh\left(\frac{\sqrt{2}}{2}\,(x+t)\right). \qquad (20)$$

Following the OHAM formulation presented in Section 3, we have

$$L(\phi(x,t;q) = D_t^\alpha u,$$

$$N(\phi(x,t;q) = -(u_x + 6\,u\,(u_x)^2 + (3u^2-1)u_{xx} - u_{xxxx}),$$

with the initial condition:

$$\phi(x,0;q) = \tanh\left(\frac{\sqrt{2}}{2}\,x\right). \qquad (21)$$

Collecting the same powers of q, and equating each coefficient of q to zero, we get the zeroth order problem:

$$\frac{\delta^\alpha u_0}{\delta t^\alpha} = 0. \qquad (22)$$

Table 1 Auxiliary convergence-control parameters (c_1, c_2) of Example 1 for different values of α

α	c_1	c_2
0.8	-0.0211939932470	0.0900871779811
0.9	-2.2025E-20	0.0667761610830
0.9	-1.3814E-20	0.0686531670168
1	4.4134E-20	0.0706631567497

We begin with an initial approximation $u_0(x,t)$= $u(x,0)$ = Tanh$\left(\frac{\sqrt{2}}{2}\,x\right)$, and using the initial approximation (19), we get the first order problem as:

$$\frac{\delta^\alpha u_1}{\delta t^\alpha} = -c_1\left(\frac{1}{\sqrt{2}}\left(\operatorname{sech}\left(\frac{x}{\sqrt{2}}\right)\right)^2 - (\operatorname{sech}\left(\frac{x}{\sqrt{2}}\right))^4\ \tanh\left(\frac{x}{\sqrt{2}}\right) - \left(\operatorname{sech}\left(\frac{x}{\sqrt{2}}\right)\right)^2 (\tanh\left(\frac{x}{\sqrt{2}}\right))^3 + \left(\operatorname{sech}\left(\frac{x}{\sqrt{2}}\right)\right)^2 \tanh\left(\frac{x}{\sqrt{2}}\right)\right). \qquad (23)$$

$$u_1 = \frac{-c_1\,t^\alpha}{\Gamma(\alpha+1)}\left(\frac{1}{\sqrt{2}}\left(\operatorname{sech}\left(\frac{x}{\sqrt{2}}\right)\right)^2 - (\operatorname{sech}\left(\frac{x}{\sqrt{2}}\right))^4\ \tanh\left(\frac{x}{\sqrt{2}}\right) - (\tanh\left(\frac{x}{\sqrt{2}}\right))^3 + \tanh\left(\frac{x}{\sqrt{2}}\right)\right). \qquad (24)$$

Using the J^α operator which is the inverse operator of D^α in (23) we obtain:

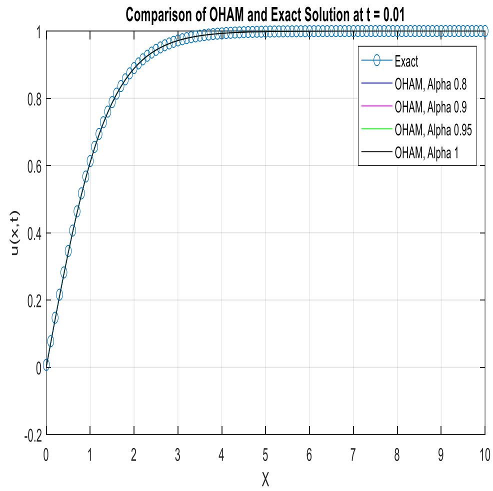

Fig.1. The numerical solutions are provided by expression (26) for Eq. (18) with different values of alpha at $t = 0.01$

By Similarity we compare the coefficient of q^2 and take J^α operator for the two sides of $\frac{\delta^\alpha u_2}{\delta t^\alpha}$ to get the value of the second-order problem:

$$u_2 = -c_1(1+c_1)\frac{t^\alpha}{\Gamma(\alpha+1)}\left(\frac{1}{\sqrt{2}}\, sech^2\left(\frac{x}{\sqrt{2}}\right) - sech^4\left(\frac{x}{\sqrt{2}}\right)\tanh\left(\frac{x}{\sqrt{2}}\right) - tanh^3\left(\frac{x}{\sqrt{2}}\right) sech^2\left(\frac{x}{\sqrt{2}}\right) + sech^2\left(\frac{x}{\sqrt{2}}\right)\tanh\left(\frac{x}{\sqrt{2}}\right)\right)$$
$$-c_2\,\frac{t^\alpha}{\Gamma(\alpha+1)}\left(\frac{1}{\sqrt{2}}\, sech^2\left(\frac{x}{\sqrt{2}}\right)\right) - 3c_2\frac{t^\alpha}{\Gamma(\alpha+1)}\; sech^4\left(\frac{x}{\sqrt{2}}\right)\tanh\left(\frac{x}{\sqrt{2}}\right) - (c_1)^2\,\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\left(sech^2\left(\frac{x}{\sqrt{2}}\right)\tanh\left(\frac{x}{\sqrt{2}}\right)\right)$$
$$-\frac{12}{\sqrt{2}}\,(c_1)^2\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\, sech^4\left(\frac{x}{\sqrt{2}}\right) tanh^2\left(\frac{x}{\sqrt{2}}\right) + 3\,(c_1)^2\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\; sech^4\left(\frac{x}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\, sech^2\left(\frac{x}{\sqrt{2}}\right)\right.$$
$$\left.-sech^4\left(\frac{x}{\sqrt{2}}\right)\tanh\left(\frac{x}{\sqrt{2}}\right) - tanh^3\left(\frac{x}{\sqrt{2}}\right) sech^2\left(\frac{x}{\sqrt{2}}\right) + sech^2\left(\frac{x}{\sqrt{2}}\right)\tanh\left(\frac{x}{\sqrt{2}}\right)\right)$$
$$-(c_1)^2\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\left(52\, sech^4\left(\frac{x}{\sqrt{2}}\right) tanh^3\left(\frac{x}{\sqrt{2}}\right) - 34\, sech^6\left(\frac{x}{\sqrt{2}}\right)\tanh\left(\frac{x}{\sqrt{2}}\right) - 4\, sech^2\left(\frac{x}{\sqrt{2}}\right) tanh^5\left(\frac{x}{\sqrt{2}}\right)\right)$$
$$-c_2\frac{t^\alpha}{\Gamma(\alpha+1)}\left(sech^2\left(\frac{x}{\sqrt{2}}\right)\tanh\left(\frac{x}{\sqrt{2}}\right)\right) - (c_1)^2\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\left(-2\, sech^2\left(\frac{x}{\sqrt{2}}\right) tanh^3\left(\frac{x}{\sqrt{2}}\right) + 4\, sech^4\left(\frac{x}{\sqrt{2}}\right)\tanh\left(\frac{x}{\sqrt{2}}\right)\right)$$
$$-6\,(c_1)^2\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\, sech^2\left(\frac{x}{\sqrt{2}}\right) tanh^2\left(\frac{x}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\, sech^2\left(\frac{x}{\sqrt{2}}\right) - sech^4\left(\frac{x}{\sqrt{2}}\right)\tanh\left(\frac{x}{\sqrt{2}}\right) - tanh^3\left(\frac{x}{\sqrt{2}}\right) sech^2\left(\frac{x}{\sqrt{2}}\right)\right.$$
$$\left.+sech^2\left(\frac{x}{\sqrt{2}}\right)\tanh\left(\frac{x}{\sqrt{2}}\right)\right) + 3\,(c_1)^2\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} tanh^2\left(\frac{x}{\sqrt{2}}\right)\left(-2\, tanh^3\left(\frac{x}{\sqrt{2}}\right) sech^2\left(\frac{x}{\sqrt{2}}\right) + 4\, sech^4\left(\frac{x}{\sqrt{2}}\right)\tanh\left(\frac{x}{\sqrt{2}}\right)\right)$$
$$+3\,c_2\,\frac{t^\alpha}{\Gamma(\alpha+1)}\, tanh^3\left(\frac{x}{\sqrt{2}}\right) sech^2\left(\frac{x}{\sqrt{2}}\right) \qquad (25)$$

By using the initial condition (19), Eq. (24), and Eq. (25), we find out second-order approximate solution of Eq. (18)

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + \cdots \qquad (26)$$

Through the least squares method mentioned in Section 3, we obtain the values of constants c_1, c_2 for different values of α. See Table 1.

Example 2 We will tackle the nonlocal fractional order (C-H equation) with terms of advection and reaction [35], which is defined as

$$D_t{}^\alpha u - 6\,u\,(u_x)^2 - (3u^2-1)u_{xx} + u_{xxxx} - \beta\, u_x - k(u-u^2) = 0, \qquad (27)$$

where k reaction term and β advection term. With the initial condition

$$u(x,0) = x. \qquad (28)$$

Without exact solutions we will apply the OHAM steps.

Table 2 The results of the second order solution (26) of OHAM method at different values α

t	x	Exact	OHAM α = 0.8	OHAM α = 0.9	OHAM α= 0.95	OHAM α = 1	Error OHAM α = 1	Error ADM α = 1
0.002	0.1	0.072	0.07034	0.07041	0.07045	0.07049	0.00150	0.00281
0.002	0.2	0.1418	0.14024	0.14031	0.14035	0.14039	0.00148	0.00278
0.002	0.3	0.2103	0.20876	0.20883	0.20888	0.20891	0.00144	0.002709
0.004	0.1	0.0734	0.07015	0.07025	0.07033	0.07039	0.00301	0.005645
0.004	0.2	0.14326	0.14005	0.14015	0.14023	0.14029	0.002967	0.0055630
0.004	0.3	0.21171	0.20858	0.20868	0.20876	0.20882	0.002894	0.0054236
0.006	0.1	0.07481	0.069986	0.070104	0.070211	0.070295	0.004518	0.0084799
0.006	0.2	0.14464	0.13989	0.14000	0.14011	0.14019	0.004450	0.0083498
0.006	0.3	0.21306	0.20842	0.20854	0.20864	0.20872	0.004340	0.0081448
0.01	0.1	0.07762	0.06968	0.069819	0.069972	0.070096	0.007529	0.0141747
0.01	0.2	0.14741	0.13959	0.13972	0.13987	0.14	0.007414	0.0139581
0.01	0.3	0.21576	0.20813	0.20826	0.20841	0.20853	0.00722	0.013609

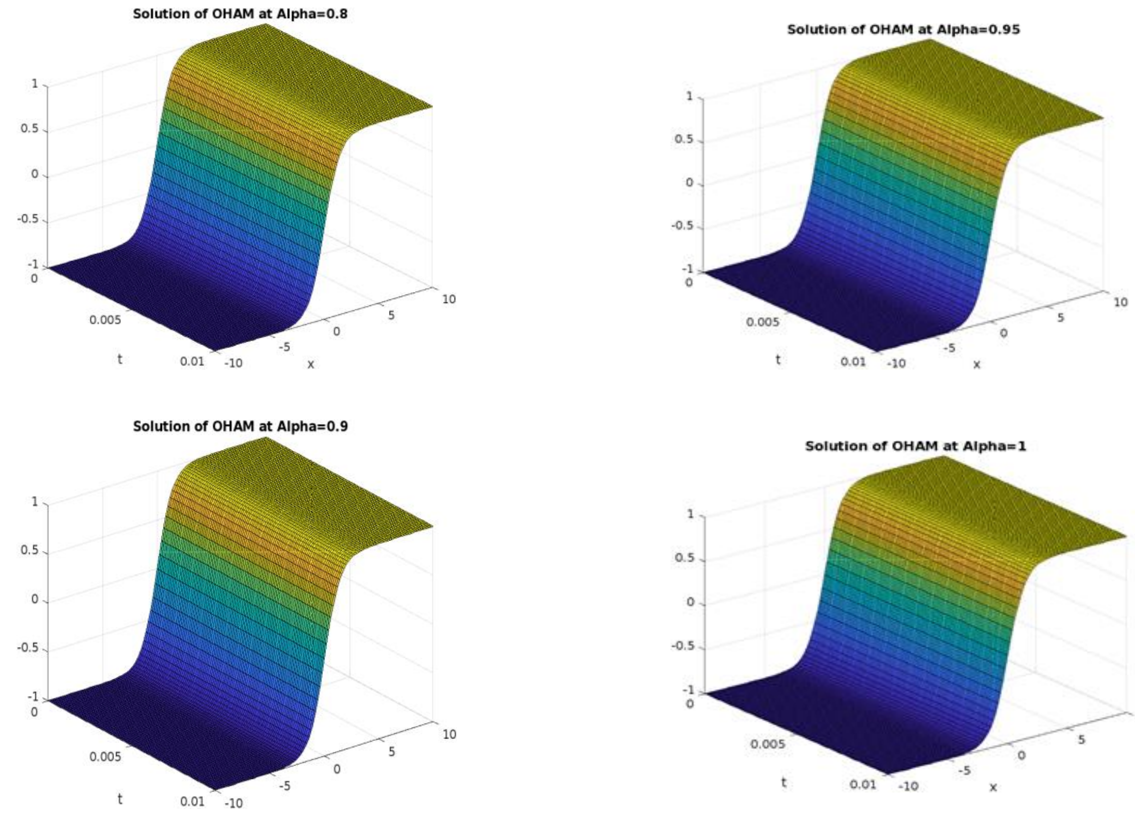

Fig.2. The numerical solutions are presented by expression (26) for Eq.18 with different values of Alpha at $t = 0.01$.

We are considering differential operators for Eq. (27) as:

$$L(\phi(x,t;q) = D_t^{\alpha} u,$$

$$N(\phi(x,t;q) = -6\,u\,(u_x)^2 - (3u^2-1)u_{xx} + u_{xxxx} - \beta\,u_x - k(u - u^2).$$

With the initial condition:

$$\phi(x,0;q) = x. \tag{29}$$

The zeroth order problem is the first one of these and it is written as follows:

$$\frac{\delta^{\alpha} u_0}{\delta t^{\alpha}} = 0. \tag{30}$$

We begin with an initial approximation $u_0(x,t)$= $u(x,0)$ = x, we get the first order problem is:

$$\frac{\delta^{\alpha} u_1}{\delta t^{\alpha}} = -c_1\,(\,6x + \beta + \mathrm{k}\,x - \mathrm{k}x^2\,)\,. \tag{31}$$

By using the J^{α} operator for the two sides of (31) we get:

$$u_1 = \frac{-\,c_1\,t^{\alpha}}{\Gamma(\alpha+1)}\,(\,6x + \beta + \mathrm{k}\,x - \mathrm{k}x^2\,). \tag{32}$$

By Similarity we compare the coefficient of q^2 and take J^{α} operator for the two sides of $\frac{\delta^{\alpha} u_2}{\delta t^{\alpha}}$ to get the value of the second-order problem:

$$u_2 = \frac{t^{\alpha}}{\Gamma(\alpha+1)}\,(\,-c_1(1+c_1) + 6x + \beta + \mathrm{k}\,x - \mathrm{k}x^2 - \beta c_2 - 6c_2\,x - kc_2\,x + kc_2\,x^2\,) + {c_1}^2\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}(\,(2k) + 12\beta + 2\mathrm{k}\,\beta + (2k)x^3 + x^2(-48k - -3k^2) + x(108 + 24k + k^2 - 4\mathrm{k}\,\beta). \tag{33}$$

By using the initial condition (27), Eq. (32), and Eq. (33), we find out second-order approximate solution of Eq. (27)

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + \cdots, \tag{34}$$

while through using the least squares method mentioned in Section 3, we obtain the values of constants c_1, c_2 for different values of k and α. Check Tables 3-5.

Table 3 Auxiliary convergence-control parameters (c_1, c_2) of Example 2 for different value of α at $\beta = -1, x = 1$

α	k	c_1	c_2
0.8		-8.0452e-15	0.16381
0.9	1	2.9374e-15	0.17061
1		1.7577e-15	0.17854

Table 4 Auxiliary convergence-control parameters (c_1, c_2) of Example 2 for different values of α at $\beta = -1, x = 1$

α	k	c_1	c_2
0.8		-1.8637e-14	0.20398
0.9	-1	-5.2333e-15	0.21185
1		3.4992e-16	0.22094

Table 5 Auxiliary convergence-control parameters (c_1, c_2) of Example 2 for different values of α at $\beta = -1, x = 1$

α	k	c_1	c_2
0.8		2.9374e-15	0.21274
0.9	0	-5.7462e-14	0.22459
1		6.4766e-15	0.2363

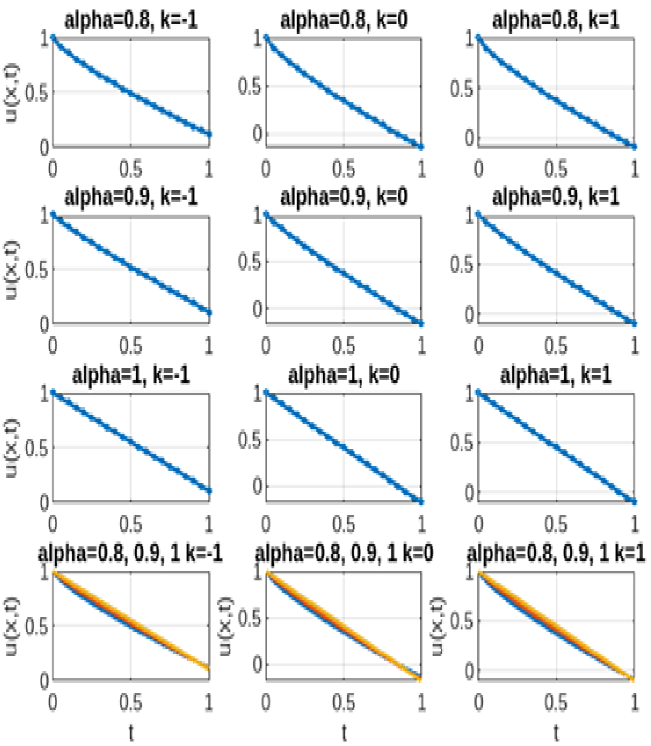

Fig.3. The numerical solutions are shown in expression (34) for Eq. (27) in different values of alpha and various values of k, $\beta = -1, x = 1$.

5 Conclusion

In this study, we implement OHAM to accurately solve the C-H equations. The accuracy of these results has been shown in Tables 1-5 and they have been shown graphically in Figures 1-3 in order to highlight the efficiency and distinction of this method. The technique convergence is regulated by a flexible function known as the auxiliary function. The Caputo derivative fractional-order and the well-known least squares technique are used to determine the values of the unknown arbitrary constants in the auxiliary function. In the Caputo meaning, fractional-order derivatives are taken with results in the closed interval [0, 1]. The proposed technique is immediately applicable to Cahn-Hilliard equations, and no small or large parameter assumptions are required. Also, studies on this topic may lead to more interesting conclusions and results. Thus, it offers more realistic solutions to real physical problems.

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