



A new approach to momentum operators in curved space

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Abstract In this paper, we obtained momentum representation in curved space using a distinct procedure from previous reports. Then, as a straight application for these operators, we have investigated them on the compact manifold as a sphere. Our investigation indicates that the spectrum of generalized momentum operators on the compact manifold is discrete.

1 Introduction

The "relativity revolution" and the "quantum revolution" are among the greatest successes of twentieth-century physics, yet the theories they produced appear to be fundamentally incompatible. General relativity remains a purely classical theory and describes the geometry of space and time as smooth and continuous, whereas quantum mechanics divides everything into discrete pieces. Many attempts have been made to unify two theories, and one of the ideas for unifying these theories is quantized gravity [1-4]. But in some references instead of trying to quantize the gravity, the effects of the metric have been proposed [5].

For a comprehensive understanding of the quantum mechanics in curved space, it is essential to establish a representation of the generalized momentum operators [6]. These operators must satisfy the Heisenberg commutation:

$$[X_i, P_j] = i\hbar\delta_j^i, \quad (1)$$

$$[P_i, P_j] = 0. \quad (2)$$

In a renowned paper [7], considering the aforementioned equations and concept of the delta function in curved space, the Hermitian form of the generalized momentum operator is obtained

$$P_i = -i\hbar\left(\frac{\partial}{\partial x^i} + \frac{1}{2}\Gamma_{ji}^j\right) = -i\hbar\frac{1}{\sqrt[4]{g}}\frac{\partial}{\partial x^i}\sqrt[4]{g}, \quad (3)$$

where $\Gamma_{ji}^j(x) = \frac{\partial}{\partial x^i} \ln(\sqrt{g(x)})$ and g is the metric of the curved space under consideration.

In ref. [8] and [9] the non-Hermitian form of momentum operator in curved space is presented. In an approach near the report [7], equation (3) is derived without employing the delta function in curved space. However, in reference [10], a distinct approach is taken. By utilizing the matrix representation of the momentum generalized coordinates transformation, the non-Hermitian form of equation (3) is obtained. The covariant and contravariant components of the familiar form of the momentum operator $-i\hbar\vec{\nabla}$ have been formulated for curvilinear coordinates in three-dimensional space directly [11]. Recently, the representation of the inverse momentum operator in curved space has been reported [12].

This paper is organized as follows: first, in section II, we obtain equation (3) with a different procedure, which is mentioned in the various references. In Section III, as an application of this relation, we examine a quantum particle on a sphere.

2 Momentum operator in curved space

To any vector $|u\rangle$ in the Hilbert space, corresponds a function u such that

$$u(x) = \langle x|u\rangle. \quad (4)$$

The operators K_j , A_j and B_j operate on the vectors of the Hilbert space, while D_j represents the standard partial differentiation that operates on functions. Obviously, it is desired that K_j 's are anti-Hermitian and satisfy

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$$[X^j, K_l] = -\delta_l^j. \quad (5)$$

Defining A_j 's through

$$\langle x|A_j|u\rangle = (D_j u)(x), \quad (6)$$

we have

$$\langle x|X^j A_l|u\rangle = x^j \langle x|A_l|u\rangle = x^j (D_l u)(x). \quad (7)$$

Also, one can write

$$\langle x|A_l X^j|u\rangle = (D_l u^j)(x), \quad (8)$$

where $|u^j\rangle = X^j|u\rangle$ and $u^j(x) = x^j u(x)$. It follows:

$$(D_l u^j)(x) = \delta_l^j u(x) + x^j (D_l u)(x). \quad (9)$$

We can also write

$$\langle x|[X^j, A_l]|u\rangle = -\delta_l^j u(x) = -\delta_l^j \langle x|u\rangle, \quad (10)$$

which shows that

$$[X^j, A_l] = -\delta_l^j. \quad (11)$$

The lastest equation resembles to equation (5) where A is replaced by K , although A is not necessarily Hermitian. By defining B through $B_j = K_j - A_j$ it becomes evident that

$$[X^j, B_l] = 0. \quad (12)$$

Therefore, B_j 's are functions of X , namely $B_j = F_j(X)$. The remaining task is to determine B_j 's so that K_j 's becomes anti-Hermitian.

If the inner product is

$$\langle u|v\rangle = \int d^n x g^{\frac{1}{2}}(x) \overline{u(x)} v(x), \quad (13)$$

where g is the metric of the curved space under consideration. Assuming $g^{\frac{1}{2}}$ being real and positive, we can write

$$\langle u|K_j|v\rangle = \int d^n x g^{\frac{1}{2}}(x) \overline{u(x)} [(D_j v)(x) + F_j(x)v(x)]$$

$$= \int d^n x g^{\frac{1}{2}}(x) \left\{ -\overline{D_j u(x)} - (g^{\frac{-1}{2}} D_j g^{\frac{1}{2}})(x) \overline{u(x)} + F_j(x) \overline{u(x)} \right\} v(x). \quad (14)$$

Anti-Hermiticity of K_j means

$$\langle u|K_j|v\rangle = -\overline{\langle v|K_j|u\rangle}. \quad (15)$$

So, using

$$\overline{\langle v|K_j|u\rangle} = \int d^n x g^{\frac{1}{2}}(x) \overline{u(x)} [(D_j v)(x) + F_j(x)v(x)] v(x), \quad (16)$$

one can see that

$$F_j + \overline{F_j} = g^{\frac{-1}{2}} D_j g^{\frac{1}{2}}. \quad (17)$$

That is

$$Re(F_j) = g^{\frac{-1}{4}} D_j g^{\frac{1}{4}}. \quad (18)$$

The determination of the imaginary part of F_j isn't dictated by this method. A straightforward option is to set it to zero. In this case K_j 's do commute with each other. The final result is

$$\langle x|K_j|u\rangle = g^{\frac{-1}{4}}(x) \left[D_j \left(g^{\frac{1}{4}} u \right) \right] (x). \quad (19)$$

The equations (3) and (19) are completely consistent.

3 Application of momentum operator in curved space

Applying equation (3) we consider a particle constrained to the surface of a sphere an established example for quantum mechanics in curved space. The motion of constrained particle on two-dimensional sphere have been investigated in [13]. The coherent states for a particle on a sphere can be seen that in [14]. In general, there has also been a discussion about the behavior of the operator (3) on boundary conditions [15].

The adequate generalized coordinates to describe the dynamics on a sphere are $\varphi \in [0, 2\pi]$, the polar angle $\theta \in [0, \pi]$ in spherical coordinates and also the radial coordinate r remains restricted to fixed radius $r = R$. The classical Hamiltonian function for a particle of mass M moving freely on this surface is then given by

$$H = \frac{1}{2MR^2} \left(p_\theta^2 + \frac{p_\varphi^2}{\sin^2 \theta} \right), \quad (20)$$

where $p_\varphi = \hat{k} \cdot \vec{L}$ and $p_\theta = -\hat{\varphi} \cdot \vec{L}$ describe the components of the angular momentum \vec{L} with respect to the unit vectors \hat{k} and $\hat{\varphi}$.

The consistent quantum mechanical Hamiltonian to the equation (20) is as following [16,17]

$$H = \frac{1}{2MR^2} (g^{ij}(\theta, \varphi) p_i p_j). \quad (21)$$

Comparing the equation (21) with the equation (20), we can identify the metric components

$$g^{\theta\theta}(\theta, \varphi) = 1, \quad (22)$$

$$g^{\varphi\varphi}(\theta, \varphi) = \frac{1}{\sin^2 \theta}, \quad (23)$$

$$g^{\theta\varphi}(\theta, \varphi) = g^{\varphi\theta}(\theta, \varphi) = 0. \quad (24)$$

From above equations we can infer the covariant components $g_{\theta\theta}(\theta, \varphi) = 1$, $g_{\varphi\varphi}(\theta, \varphi) = \sin^2 \theta$ and $g_{\theta\varphi}(\theta, \varphi) = g_{\varphi\theta}(\theta, \varphi) = 0$ which yield the metric determinant

$$g(\theta, \varphi) = \begin{vmatrix} 1 & 0 \\ 0 & \sin^2 \theta \end{vmatrix} = \sin^2 \theta. \quad (25)$$

The identity operator expressed in terms of the coordinate basis reads as [7,8]

$$1 = \int dx \sqrt{g(x)} |x\rangle \langle x|. \quad (26)$$

Therefore to the equation (26), the identity operator in the coordinate basis $|\theta, \varphi\rangle$ is given by

$$1 = \int_0^\pi d\theta \int_0^{2\pi} d\varphi \sin \theta |\theta, \varphi\rangle \langle \theta, \varphi|. \quad (27)$$

In addition to this, from the equation (3), conjugate momentum operators are

$$p_\varphi = \frac{\hbar}{i} \frac{\partial}{\partial \varphi}, \quad p_\theta = \frac{\hbar}{i} \left(\frac{\partial}{\partial \theta} + \frac{1}{2} \cot \theta \right). \quad (28)$$

As we can obtain the eigenfunctions of the equation (3) [7], the eigenfunctions of (28) are given by

$$\langle \theta, \varphi | m_\theta, m_\varphi \rangle = \frac{e^{2im_\theta\theta}}{\sqrt{\pi \sin \theta}} \frac{e^{im_\varphi\varphi}}{\sqrt{2\pi}}, \quad (29)$$

where m_θ, m_φ are the corresponding to related eigenvalues and $m_\theta, m_\varphi \in \mathbb{Z}$, so we have

$$p_\theta |m_\theta, m_\varphi\rangle = 2\hbar m_\theta |m_\theta, m_\varphi\rangle, \quad (30)$$

$$p_\varphi |m_\theta, m_\varphi\rangle = \hbar m_\varphi |m_\theta, m_\varphi\rangle. \quad (31)$$

The eigenvectors form a discrete orthonormal basis of the Hilbert space

$$1 = \sum_{m_\theta \in \mathbb{Z}} \sum_{m_\varphi \in \mathbb{Z}} |m_\theta, m_\varphi\rangle \langle m_\theta, m_\varphi|, \quad (32)$$

$$\langle m_\theta, m_\varphi | m'_\theta, m'_\varphi \rangle = \delta_{m_\theta, m'_\theta} \delta_{m_\varphi, m'_\varphi}. \quad (33)$$

Unlike in the unbounded configuration space, the discrete momentum spectrum implies a discrete momentum subspace coordinate. This characteristic is a generic feature of compact coordinates and also arises, e.g., in the case of a single angle variable (motion on a circle) [18] or the orientation state of a rigid body. We emphasize that this discreteness does not arise due to the subspace formalism; rather, it emerges as a necessary physical consequence, as also reflected in the discrete measurement outcomes of the corresponding momentum observables.

4 Conclusion

We were able to obtain equation (3) using a completely different approach from what has been done so far. Subsequently, as a straight application of operator (3), we examined their eigenstates on the sphere. It is interesting that these eigenstates are discrete while this does not happen on unbounded spaces.

References

1. L. Viola, R. Onofrio, Phys. Rev. D **55**, (1996)
2. J. F. Doonoghue and B. R. Holstein, Am. J. Phys. **54**, 9 (1986)
3. Y. Q. Cai, G. Papini, Gen. rel. Grav. **22**, 3 (1990)
4. A. Karamatskou, H. Klinert, Int. J. Geom. Methods. Phys. **11**, 8 (2014)
5. C. C. Barros Jr, Eur. Phys. J. C **42** (2005)

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6. W. Pauli, *General Principles of Quantum Mechanics*, (Springer – Verlag, New York, 1980)
 7. B. S. DeWitt, T. Stanev, *Phys. Rev.* **85**, 653 (1952)
 8. M. Carreau, *Phys. Rev. D* **40**, 6 (1989)
 9. C. S. Wang, *Am. J. Phys.* **57**, 1 (1989)
 10. P. V. Gonzalez, J. C. Parra, *Am. J. Phys.* **49**, 8 (1981)
 11. B. Leaf, *Am. J. Phys. A* **47**, 9 (1979)
 12. M. Jafari Matehkolae, *Pramana J. Phys.* **93**, 84 (2019)
 13. L. Jahangiri, H. Panahi, *Ann. Phys.* **375** (2016)
 14. K. Kowalski, J. Rembielinski, *J. Phys. A.* **33** (2000)
 15. M. Jafari Matehkolae, *Pramana J. Phys.* **95**, 131(2021)
 16. G. R. Gruber, *Found. Phys.* **6**, 1 (1976).
 17. J. M. Domingos, M. H. Caldeira, *Found. Phys.* **14**, 7 (1984)
 18. I. Rigas, L. L. Sanchez-Soto, A. B. Klimov, J. Rzehacek, Z. Hradil, *Ann. Phys.* **326**, 426 (2011)