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Modified geometrical optics in a curved spacetime

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Abstract In this article, the propagation of high-frequency monochromatic beam of circularly polarized electromagnetic waves in a curved spacetime has been studied. At first, the standard geometrical optics is investigated; That is, we consider the waves with infinitely high frequency. In this step, it is found that the trajectories of light are null geodesics. Secondly, the geometrical optics is modified. For this end, by considering the polarization of light, helicity-dependent correction on the geometrical optics is included. As a result, we realize that the modified wave vector is null. Furthermore, the trajectories of light are null non-geodesic paths.
Keywords: *Circular Polarization, Helicity, Curved Spacetime, Geometrical Optics, Modified Geometrical Optics.*

1 Introduction

The propagation of circularly polarized beam of light in a gravitational field has been a matter of study in the past several years [1–6]. As we know, the propagation of electromagnetic waves in general relativity is obtained by investigating Maxwell equations in a curved spacetime. But, finding an exact solution to Maxwell equations in such spaces is a formidable problem. When the electromagnetic wave is highly monochromatic over a region of spacetime, we use an asymptotic short-wave approximation. In quantum mechanics, this method is known as WKB approximation and in wave optics is called geometrical optics approximation. This approximation is valid when the reduced wavelength (wavelength/2π) is much smaller than any characteristic scales (such as the curvature of the wave front, the size and duration of the radiation beam and the radius of the spacetime curvature) in the problem. we begin with the Maxwell equations in a curved spacetime. we write the Lorenz condition and wave equation for the potential 1-form. we select an ansatz for the potential and put it in the Lorenz condition and wave equation. Investigating these equations, we conclude that in the leading order of the geometrical optics approximation, light ray paths are null geodesics [7]. But, if the light frequency is very high but it is finite, we modify the geometrical optics by including helicity-dependent corrections on phase function of the potential ansatz. What we have done here is somehow different from what has been presented in [5] in that we directly modify the wave vector, and we find that this modified vector, corrected up to the first order of expansion parameter, is null; Also, we find that the ray trajectories of circularly polarized light in this approach are null but not geodesic.

In this article, the metric has signature $(-,+,+,+)$, vectors and differential forms are denoted by boldface letters, the inner product of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is defined as $(\boldsymbol{a},\boldsymbol{b}) = g_{\mu\nu}a^{\mu}b^{\nu}$, with $\boldsymbol{a}^2 = (\boldsymbol{a},\boldsymbol{a})$ and we shall use geometrized units $c = G = 1$.

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2 Null Tetrads and Maxwell Equations

2.1 Null tetrads and Polarization forms

Here, we use the Newman-Penrose formalism, a tetrad formalism with a special choice of the basis vectors $\{\boldsymbol{l},\boldsymbol{n},\boldsymbol{m},\bar{\boldsymbol{m}}\}$, of which $\boldsymbol{l}$,$\boldsymbol{n}$ are real and $\boldsymbol{m}$,$\bar{\boldsymbol{m}}$ are complex conjugates of one another. They are required to satisfy the orthogonality conditions,

$$(\boldsymbol{l},\boldsymbol{m}) = (\boldsymbol{l},\bar{\boldsymbol{m}}) = (\boldsymbol{n},\boldsymbol{m}) = (\boldsymbol{n},\bar{\boldsymbol{m}}) = 0, \tag{1}$$

besides the conditions,

$$(\boldsymbol{l},\boldsymbol{l}) = (\boldsymbol{n},\boldsymbol{n}) = (\boldsymbol{m},\boldsymbol{m}) = (\bar{\boldsymbol{m}},\bar{\boldsymbol{m}}) = 0, \tag{2}$$

that the vectors be null. We impose further normalization conditions as

$$(\boldsymbol{l},\boldsymbol{n}) = -1, \quad (\boldsymbol{m},\bar{\boldsymbol{m}}) = 1. \tag{3}$$

It is necessary to mention that there are some freedoms in selecting such null tetrads. For example, to construct the vector $\boldsymbol{m}$, we have the freedom $\boldsymbol{m} \to e^{i\psi(\boldsymbol{x})}\boldsymbol{m}$, where ψ is an arbitrary function. But, the validity of our framework to measure the changes of quantities is guaranteed when the size and angle between the basis vectors do not change along the trajectory of light; Also, they are required to be rotation free, which means if the basis vector is initially tangent to the ray, it will remain tangent during the motion. But, the operator that can satisfy our needs from a basis vector is the Fermi-Walker derivative. Fermi-Walker transportation of the bases imposes some conditions which fix gauge ambiguity in the choice of these bases. Volume 4-form and three polarization 2-forms $\pi^{(a)}$, $a = 0,1,2$, are defined as [5]:

$$\boldsymbol{e} = i\boldsymbol{l}\wedge\boldsymbol{m}\wedge\bar{\boldsymbol{m}}\wedge\boldsymbol{n}, \tag{4}$$

$$\pi^{(0)} = \bar{\boldsymbol{m}}\wedge\boldsymbol{n}, \;\; \pi^{(1)} = -(\boldsymbol{l}\wedge\boldsymbol{n} - \boldsymbol{m}\wedge\bar{\boldsymbol{m}}), \;\; \pi^{(2)} = \boldsymbol{l}\wedge\boldsymbol{m}. \tag{5}$$

For the electromagnetic field 2-form $\boldsymbol{F}$, Maxwell equations in the absence of electric currents can be written as $d\boldsymbol{F} = \delta\boldsymbol{F} = 0$, in which d is differential operator, $\delta := \star d\star$ is codifferential operator and $\star$ is the Hodge star operator. We define [5]:

$$\mathscr{F}^\sigma = \frac{1}{2}\left[\boldsymbol{F} - i\sigma(\star\boldsymbol{F})\right], \tag{6}$$

in which $\sigma = +1$ is the helicity parameter of the field related to right-handed and $\sigma = -1$ is related to left-handed circularly polarized waves. Since in our 4-dimensional spacetime with the defined signature we have $\star\star\boldsymbol{F} = -\boldsymbol{F}$, we can write $\star\mathscr{F}^\sigma = i\sigma\mathscr{F}^\sigma$. Then, $\mathscr{F}^{+1}$ and $\mathscr{F}^{-1}$ are self-dual and anti-self-dual complex electromagnetic fields, respectively. For these fields we have $d\mathscr{F}^\sigma = \delta\mathscr{F}^\sigma = 0$. Using the coefficients Φ_a^σ, we can express $\mathscr{F}^\sigma$ in terms of the basis 2-forms $\pi^{(a)}$ as [5]:

$$\mathscr{F}^\sigma = \sum_{a=0}^{2} \Phi_a^\sigma \pi^{\sigma(a)}. \tag{7}$$

Since $\mathscr{F}^\sigma$ is a closed form, assuming the region we are working on is simply connected, by the use of Poincare lemma we can define the complex potential 1-form $\mathscr{A}^\sigma$ as $\mathscr{F}^\sigma = d\mathscr{A}^\sigma$. The Lorenz gauge condition is $\delta\mathscr{A}^\sigma = 0$.

2.2 Field equations

We start with the following ansatz for the potential 1-form of the electromagnetic field:

$$\mathscr{A}^\sigma = \boldsymbol{a}^\sigma e^{\frac{iS^\sigma}{\varepsilon}}, \tag{8}$$

in which $\boldsymbol{a}^\sigma$ is the complex amplitude 1-form, S is the real phase function (eikonal function) and $\varepsilon \ll 1$ is a dummy expansion parameter that helps to track order of terms: a term with ε^n, for some integer n, varies as $(\lambda\!\!\!^{-}/l_{min})^n$, where $\lambda\!\!\!^{-}/l_{min} \ll 1$. Here $\lambda\!\!\!^{-}$ is the reduced wavelength (wavelength/2π) and l_{min} is the minimal of the characteristic scales of the problem. It is important to mention that we skip the helicity index σ and our calculation will be for the wave with right-handed polarization. For left-handed one, it is necessary to change $\varepsilon \to -\varepsilon$ and $\boldsymbol{a} \to \bar{\boldsymbol{a}}$. Putting the ansatz (8) into Lorenz gauge condition gives:

$$\star(\boldsymbol{P}\wedge\star\boldsymbol{a}) - i\varepsilon\star d\star\boldsymbol{a} = 0, \tag{9}$$

in which $\boldsymbol{P} := dS$ is the wave 1-form. Also, the field strength $\mathscr{F}$ can be written as:

$$\mathscr{F} = \frac{i}{\varepsilon}\mathscr{Z} e^{\frac{iS}{\varepsilon}}, \tag{10}$$

in which

$$\mathscr{Z} := \mathscr{B} - i\varepsilon\mathscr{C}, \quad \mathscr{B} := \boldsymbol{P}\wedge\boldsymbol{a}, \quad \mathscr{C} := d\boldsymbol{a}. \tag{11}$$

It is easy to show that:

$$\delta\mathscr{F} = -\frac{1}{\varepsilon^2} e^{\frac{iS}{\varepsilon}} \boldsymbol{j}, \tag{12}$$

in which

$$\boldsymbol{j} \overset{1}{=} -\star[\boldsymbol{P}\wedge\star(\boldsymbol{a}\wedge\boldsymbol{P})] - i\varepsilon\left[\star(\boldsymbol{P}\wedge\star d\boldsymbol{a}) - \star d\star(\boldsymbol{a}\wedge\boldsymbol{P})\right], \tag{13}$$

is the truncated current 1-form up to the first order ε. Note that the symbols $\overset{0}{=}$ and $\overset{1}{=}$ indicate that we have kept the equations up to zero order and first order of the parameter ε, respectively. Maxwell equations in current free spaces are satisfied if $\boldsymbol{j} = 0$. This point reaches us to the following field equation:

$$\boldsymbol{P}^2\boldsymbol{a} - i\varepsilon\Big((\nabla^\nu P_\nu)\,\boldsymbol{a} + 2P^\nu\left(\nabla_\nu a_\mu\right) \boldsymbol{e}^\mu \Big) \overset{1}{=} 0, \tag{14}$$

in which ∇_ν is the covariant derivative associated with the spacetime metric $g_{\mu\nu}$ and $\boldsymbol{e}^\mu$ are co-frame 1-forms. It is necessary to mention that some conditions are needed to be imposed on the fields depending on the point that either they are self-dual or anti-self-dual. For the self-dual field we should have:

$$\mathscr{F}^{+1}\circ\bar{\pi}^{(a)} = 0, \tag{15}$$

in which "$\circ$" indicates the inner product of two differential 2-forms [8]. It is necessary to note that the gauge condition $\mathscr{A} \to \mathscr{A} + d\Psi$ which $\Psi := \varepsilon \psi e^{\frac{iS}{\varepsilon}}$ and ψ is a scalar function, preserves the physics of the problem. This gauge condition will help us to find vector polarization in geometrical optics approximation.

3 Geometrical Optics

At first, we keep the equation (14) up to zero order of the parameter of expansion. so, we have:

$$\boldsymbol{P}^2 \stackrel{0}{=} 0. \tag{16}$$

This means that $\boldsymbol{P}$ is null. If we interpret $P_\alpha = \nabla_\alpha S$ as the momenta canonically conjugated to x^α, we can identified the equation (16) with the Hamilton-Jacobi equation with an effective Hamiltonian defined as follows [9]:

$$H(x^\mu, P_\mu) := \frac{1}{2} g^{\mu\nu} P_\mu P_\nu. \tag{17}$$

Let $x^\alpha(\lambda)$, which λ is an affine parameter, be an integral curve of P^α, $P^\mu = dx^\mu / d\lambda$, then the Hamiltonian equations give the trajectories of light in the geometrical optics limit:

$$\frac{D^2 x^\lambda}{d\lambda^2} := \ddot{x}^\lambda + \Gamma^\lambda_{\kappa\nu} \dot{x}^\kappa \dot{x}^\nu = 0, \tag{18}$$

in which $\Gamma^\lambda_{\kappa\nu}$ are Christoffel symbols. So, in the geometrical optics approximation, the trajectories of light are null geodesics.

4 Modified Geometrical Optics

In the next step, we modify the geometrical optics by including first order correction in the wave vector. Here, different from what has been done in [5], we directly correct the wave vector as $\boldsymbol{P} = \boldsymbol{P}_0 + \varepsilon \boldsymbol{P}_1$. If we put this in the equation (14), we have:

$$\begin{aligned} &(\boldsymbol{P}_0 + \epsilon \boldsymbol{P}_1)^2 a_{0\mu} + \varepsilon (\boldsymbol{P}_0 + \epsilon \boldsymbol{P}_1)^2 a_{1\mu} \\ &- 2i\varepsilon \left[(\boldsymbol{P}_0 + \epsilon \boldsymbol{P}_1)^\nu \nabla_\nu a_{0\mu} + \frac{1}{2} a_{0\mu} \nabla^\nu (\boldsymbol{P}_0 + \epsilon \boldsymbol{P}_1)_\nu \right] \stackrel{1}{=} 0. \end{aligned} \tag{19}$$

Now, we split this equation order by order in ε. At first, we have $\boldsymbol{P}_0^2 = 0$. Here, we put $\boldsymbol{P}_0 = \boldsymbol{l}$. If we put this result into (19) and use the relation $a_{0\mu} = f_0 z_{0\mu}$, by some simplifications we obtain:

$$(\boldsymbol{l}, \boldsymbol{P}_1) - i l^\nu \bar{z}_0{}^\mu \nabla_\nu z_{0\mu} = 0, \tag{20}$$

Using the condition (15) and the mentioned gauge condition, we can imply that $\boldsymbol{z}_0 = \boldsymbol{m}$. Therefore, we get:

$$(\boldsymbol{l}, \boldsymbol{P}_1) - i l^\nu \bar{m}^\mu \nabla_\nu m_\mu = 0. \tag{21}$$

Using the property of Fermi-Walker transportation, the second term on the left hand side is zero, then we have $(\boldsymbol{l}, \boldsymbol{P}_1) = 0$. Therefore, we can write:

$$\boldsymbol{P}^2 \stackrel{1}{=} \boldsymbol{l}^2 + 2\varepsilon (\boldsymbol{l}, \boldsymbol{P}_1) = 0. \tag{22}$$

This indicate that in the modified geometrical optics, the wave vector correction up to the first order of ε is null. Up to now, we find that $\boldsymbol{P} = \boldsymbol{l} + \varepsilon \boldsymbol{P}_1$, so we have $(\boldsymbol{P} - \varepsilon \boldsymbol{P}_1)^2 = 0$. Like the geometrical optics section, we introduce the effective Hamilton-Jacobi equations, we obtain:

$$H(x^\mu, P_\mu, \varepsilon) := \frac{1}{2} (\boldsymbol{P} - \varepsilon \boldsymbol{P}_1)^2. \tag{23}$$

Inspecting the Hamilton equations, we obtain:

$$\frac{D^2 x^\mu}{d\lambda^2} = \varepsilon \sigma \left(\nabla^\mu P_{1\nu} - \nabla_\nu P_1^\mu \right) \dot{x}^\nu, \tag{24}$$

Which means in modified geometrical optics the trajectories of light rays are null non-geodesics.

5 Conclusion

In this paper, the propagation of circularly polarized high-frequency electromagnetic waves in a curved spacetime was studied. It is found that in the geometrical optics limit, the trajectories of light rays are null geodesics. But, modifying the wave vector up to first order correction, a process different from what has been presented in [5], we can conclude that the modified wave vector is null and the ray trajectories of light are null non-geodesics.

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