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Study of all possible interactions in the quantum three-body problem with Calogero-Sutherland Hamiltonian

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Abstract In the quantum three-body problem, There are three possible models: 1. Each particle interacts independently with the two others, forming three interacting pairs. Pairwise interactions refer to this model. 2. Each particle interacts with the centre of mass of the two others, called the pure three-body interaction. 3. Each particle interacts with the centre of mass of the two others. This kind of interaction is called the pure three-body interaction. 3. Each particle interacts simultaneously with the two others and with the centre of mass of the two others. This latest is called the full-three-body interaction, which combines pairwise and pure three-body interactions. Therefore, knowing which of these models prevails in a given physical context is essential. In this paper, we choose the integrable Calogero Sutherland Hamiltonian, and we explicitly write the wave functions of the ground states in each of these modellings for three fermions and three bosons on a ring. These wave functions reveal regions of the null probability of the presence of particles. This theoretical distribution of probability is to be compared with observations to discover which of these models prevails in the observed physical context. As a physical context, we suggest cold atoms' ring-shaped optical lattices. In addition, we clarify several points about the relation between trigonometric and inverse square interactions on a line and a ring.

Keywords: *Three-body quantum Problem; two-body interactions; Calogero-Sutherland Hamiltonian; three-body interactions; ring-shaped optical lattices*

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1 Introduction

In recent decades, significant efforts have been made to consider convenient Hamiltonians to describe material properties properly. In this context, one of the efficient procedures is, for a given potential, to classify the way particles interact as a many-body system. Namely, pure two-body interactions, two and three-body interactions simultaneously, and two or three-body interactions separately. [1]. Among recent works using this approach, one can mention [2, 3] based on two-body interactions and [4, 5] taking into account three-body interaction, for various potentials.

In parallel, solvable models often provide insights to initiate further approximations or numerical investigations. In the case of many-body quantum systems, the knowledge of the ground state is of great importance, not only because it determines the complete structure of the spectrum [6, 7], but also, as we are going to show in the case of three-body quantum systems, it reveals essential correlations among particles. For this reason, in the particular case of three-body quantum systems, many efforts have been made, in the particular case of the solvable model of Calogero-Sutherland (CS), to generalize the pairwise interactions, to pure three-body and full three-body interactions [8–12]. Let us remember that the pure three-body interactions are the interaction between each particle of the system with the centre-of-mass of the two others. The full three-body interactions are the simultaneous two and three-body interactions.

In this paper, we consider the one-dimensional CS model, which, beyond its solvability, plays an important role in various domains of physics [13, 14] as different as the physics of black holes[15] or the quantum Hall effects [16]. In the case of three identical particles on a ring described by the CS model, we give each model's explicit wave functions for fermions and bosons: full three-body interactions, pairwise interactions and pure three-body interactions.

Although experimental setups to highlight inverse square or trigonometric interactions are a real challenge for experimentalists, various experiments have been realized in the context of cold atoms ring-shaped optical lattices [17], some others are suggested to be realized [18], and a global review can be found in [19]. Based on our present study, the experimental observation of particles' probability of presence can inform which model best describes the system correctly and accurately: the full three-body interactions or the pure pairwise or triplet interactions.

The paper is organized as follows: In section (2), we make some rigorous considerations about the relation between spatial and angular coordinates of particles on a line and on a ring, as well as some essential clarification on the relation between inverse square and trigonometric interactions in the case of the three-body problem. In section (3), we briefly recall the method of obtaining the wave function of the full trigonometric C-S Hamiltonian. In section (4), we write the system's wave function for full three-body interactions for fermions and bosons. We obtain all the system's forbidden configurations by solving the null density probability equation. In Sections (5) and (6), the same work is done for pure three-body and pairwise interactions, respectively. In section (7), we conclude and discuss the outlooks.

2 Rigorous considerations

The pioneer study [20] reports on the N-body quantum system online in pairwise interaction with an inverse square potential $V(r) = g/r^2$, where r represents the distance between two particles and g is a real constant. This work has been generalized to the periodic case for N particles [21], by considering N particles on a ring with a circumference equal to L, where the following identity is introduced:

$$V(r) = \frac{g}{r^2} = g\sum_{-\infty}^{+\infty}(r+nL)^{-2} = \frac{g\pi^2}{L^2}\left[\sin\left(\frac{\pi r}{L}\right)\right]^{-2}. \quad (1)$$

Eq. (1) can be justified as follows: at the first stage, the periodicity is obtained by placing after the first pair of particles r apart, N particles at regular distances L apart from each other. Hence, we have N pairwise interactions between the first and the second particle r apart, the first and the third particle $r+L$ apart, the first and the fourth particle $r+2L$ apart, etc. In the second stage, the line is wrapped into a circle. Consequently, the distances in the trigonometric form of the potential are angular. The relation between angular and linear distances is clarified in Eq. (3). However, in the thermodynamics limit, the equalities of Eq. (1) are valid rigorously.

The particular case of three particles *on line*, has been considered in [8] and [22] where polar coordinates are defined to study the problem with the potential $V(r) = g/r^2$.

Later, the interaction of three particles via three body trigonometric potential [10] was considered, and in a more abstract approach, interactions of type r^{-2} and $\left[\sin\left(\frac{\pi r}{L}\right)\right]^{-2}$ were studied as two distinguished cases, and were classified in two distinguishable mathematical categories [9][11].

In this section, we intend to dissipate two ambiguities. First, as mentioned above, the trigonometric potential is only valid in the thermodynamics limit for many particles. Therefore, a legitimate question is how rigorously one can consider a three-body trigonometric potential, far from the thermodynamics limit, and still use Eq. (1). The second point, directly related to the first, concerns the relation between trigonometric and inverse square interactions.

The standard way to justify using the trigonometric interaction for three particles is to consider the limit as the ring's radius grows and tell that the trigonometric and inverse square interactions coincide in this limit.

We intend to go beyond this standard justification and clarify these points without considering the ring's radius limit. To this end, let $\overline{x_ix_j}$, be the distance of the straight line between particles labelled i and j, with $i,j = 1,2,3$, and $\widehat{x_ix_j}$, the length of the arc between particles i and j, on the ring. It is easy to see that $\overline{x_ix_j} = 2Ox_i \sin(\theta/2)$, where θ is the angle between the two radius Ox_i and Ox_j of the ring, (see Figure 1). Using fundamental trigonometric relations, we have

$$\overline{x_ix_j} = \frac{L}{\pi}\sin\left(\frac{\pi}{L}\,\widehat{x_ix_j}\right). \quad (2)$$

It follows,

$$\frac{g\pi^2}{L^2}\frac{1}{\left[\sin\left(\frac{\pi}{L}(\widehat{x_ix_j})\right)\right]^2} = \frac{g}{[\overline{x_ix_j}]^2}. \quad (3)$$

Hence, taking into account that $r = a^{-1}\sin[a(x_i - x_j)]$, where $r = \overline{x_ix_j}$, and $a = \pi/L$, Eq. (3) shows that the trigonometric potential and the inverse square potential represent both, the same type of interactions, in the sense that, the strength of the potential, is proportional to the inverse of the square of the distance

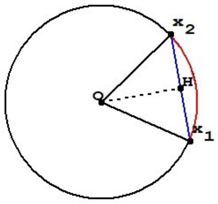

Fig. 1 The relation between the distance $\overline{x_i x_j}$ and the length of the arc $\widehat{x_i x_j}$ leading to Eq. (3) can be seen in this panel.

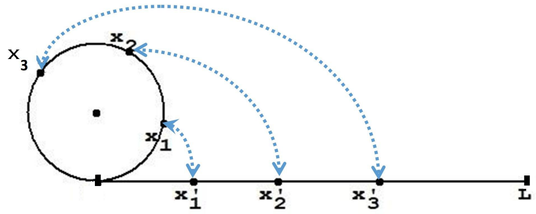

Fig. 2 The case of particles on a ring is *not* a simple winding of the segment of length L on a circle with the same circumference, but a mapping which *does not conserve the length*.

between particles. Consequently, trigonometric and inverse square potentials can be mapped from one to the other. However, we must remember that this mapping is not isometric and does not conserve the length. Indeed, as illustrated in Figure 2, the distance of the arc $\widehat{x_i x_j} = \overline{x'_i x'_j} \neq \overline{x_i x_j}$, $i,j=1,2,3$. This is the fundamental care that must be taken when using the trigonometric interaction far from the thermodynamic limit or without considering a ring with an infinitely large radius.

3 Transformation of CS Hamiltonian into Laplace-Beltrami Hamiltonian

We consider the full trigonometric Hamiltonian with full three-body interactions of CS type for three identical distinguishable fermions with mass m on a ring. This Hamiltonian reads [12]:

$$\begin{aligned} H^{full} = &\frac{-\hbar^2}{2m}\sum_{i=1}^{3}\left(\frac{\partial}{\partial x_i}\right)^2 \\ &+\frac{\hbar^2}{m}\nu(\nu-1)a^2 \sum_{\substack{i,j=1\\ i\neq j}} \frac{1}{\sin^2[a(x_i-x_j)]} \\ &+\frac{\hbar^2}{m}\mu(\mu-1)a^2 \sum_{\substack{i,j,k=1\\ i\neq j\neq k}} \frac{1}{\sin^2[a(x_i+x_j-2x_k)]}, \end{aligned} \quad (4)$$

where $a(x_i-x_j)$ is the angular distance between particles, while $a(x_i+x_j-2x_k)$ is the angular distance between one of the particles and the centre-of-mass of the two others. $0<\mu<1$ and $0<\nu<1$ are real coefficients, $\hbar$ is the Planck constant, and a is a constant homogeneous to the inverse of length, namely $a=\pi/L$, where L is the circumference of the ring.

Notice that the pairwise interactions vanish for $\nu=0$ or $\nu=1$. The Hamiltonian (4) coincides with the Sutherland Hamiltonian [21], whilst the three-body interactions vanish for $\mu=0$ or $\mu=1$, in which case, the Hamiltonian in Eq. (4) covers the three-body inverse-square potential on a line studied in [8]. Also, notice that the constraints $0<\mu<1$ and $0<\nu<1$ ensure the system's stability. Indeed, within these ranges of values, the interactions are attractive. Otherwise, they are repulsive and cause the instability of the system.

The method to find the wave function of the Hamiltonian (4) is exposed in [12] and consist of separating the Hamiltonian (4) as the sum of H^{full}_{CM} and H^{full}_{rel}, the Hamiltonian of the centre-of-mass and the relative Hamiltonian, respectively. Then, the following change of variables:

$$z_j = \exp[2\mathrm{i}a(R-x_{kl})], \quad (5)$$

where $x_{kl}=x_k-x_l$, $k,l=1,2,3$ and $R=\frac{1}{3}(x_1+x_2+x_3)$ transform H^{full}_{CM} and H^{full}_{rel} into $\widetilde{H}^{full}_{CM}$ and $\widetilde{H}^{full}_{rel}$, as follows:

$$\widetilde{H}^{full}_{cm} = \frac{4}{3}\frac{\hbar^2 a^2}{m}\left(\sum_{j=1}^{3} z_j\frac{\partial}{\partial_{z_j}}\right)^2, \quad (6)$$

$$\begin{aligned} \widetilde{H}^{full}_{rel} = &6\frac{\hbar^2 a^2}{m}\left[\sum_{j=1}^{3}\left(z_j\frac{\partial}{\partial_{z_j}}\right)^2 \right. \\ &\left. +(\nu+\mu)\sum_{\substack{j,k=1\\ j\neq k}}^{3}\frac{z_j+z_k}{z_j-z_k}\left(z_j\frac{\partial}{\partial_{z_j}}\right)\right]. \end{aligned} \quad (7)$$

Notice that $\widetilde{H}^{full}_{rel}$ expressed in Eq. (7) is the so-called Laplace-Beltrami Hamiltonian.

4 Forbidden configurations in the case of full three-body interactions

4.1 three identical distinguishable fermions with full three-body interactions

The eigenfunction of $\widetilde{H}^{full}_{rel}$ for three fermions reads [12]:

$$\psi_{rel}^{full,fermions}(z_j) = J_{\{2,1,0\}}^{(\mu+\nu)^{-1}}(z_1, z_2, z_3) \times \prod_{\substack{j,k=1 \\ j\neq k}}^{3} (z_j - z_k)^{(\nu+\mu)}, \qquad (8)$$

where

$$J_{\{2,1,0\}}^{(\mu+\nu)^{-1}}(z_1, z_2, z_3) = z_1^2 z_2 + z_2^2 z_1 + z_1^2 z_3 + z_3^2 z_1 + z_2^2 z_3 + z_3^2 z_2. \qquad (9)$$

$J_{\{2,1,0\}}^{(\mu+\nu)^{-1}}$ is the Jack polynomial for the partition $\{2,1,0\}$ which is the only possible partition for three fermions in the Fock space. For more details, see [12] and related references therein. Hence, in terms of the variable z_j, Eq. (8) reads

$$\psi_{rel}^{full,fermions}(z_1, z_2, z_3) = \left(z_1^2 z_2 + z_1^2 z_3 + z_2^2 z_1 + z_2^2 z_3 + z_3^2 z_1 + z_3^2 z_2\right) \times [(z_1 - z_2)(z_2 - z_3)(z_3 - z_1)]^{(\nu+\mu)}. \qquad (10)$$

Back to the initial coordinates x_1, x_2 and x_3 by using Eq. (5) and, relative distances $x_{ij} = x_i - x_j$, $i, j = 1, 2, 3$, Eq. (10) becomes:

$$\begin{aligned}\psi_{rel}^{full,fermions} &= \left[e^{-4iax_{23}-2iax_{31}} + e^{-4iax_{23}-2iax_{12}} + e^{-4iax_{31}-2iax_{23}}\right. \\ &\left.+ e^{-4iax_{31}-2iax_{12}} + e^{-4iax_{12}-2iax_{23}} + e^{-4iax_{12}-2iax_{31}}\right] \\ &\times \left[(e^{-2iax_{23}} - e^{-2iax_{31}})(e^{-2iax_{31}} - e^{-2iax_{12}})\right. \\ &\left.\times (e^{-2iax_{12}} - e^{-2iax_{23}})\right]^{(\nu+\mu)} \qquad (11)\end{aligned}$$

It is worth noticing that μ and ν does not appear separately in Eq. (11), but appears only as the sum $\mu+\nu$. Therefore, the probability density of particles depends on $\mu+\nu$. As seen in Eq. (4), ν and μ determine the strength of pure two and pure three-body interactions, respectively. Consequently, it is impossible to distinguish the strength of each interaction separately.

Straight forward calculations yield the density of probability $|\psi_{fermion}|^2$ as follows:

$$\begin{aligned}|\psi_{rel}^{full,fermions}|^2 = &2^{(1+3\nu+3\mu)}[3 + 2\cos 2a(x_{21} + x_{31}) \\ &+ 2\cos 2a(x_{13} + x_{23}) 2\cos 2a(x_{12} + x_{32}) \\ &+ \cos 4a(x_{12} + x_{32}) + \cos 4a(x_{23} + x_{13}) \\ &+ \cos 4a(x_{31} + x_{21}) + 2\cos 6ax_{12} \\ &+ 2\cos 6ax_{23} + 2\cos 6ax_{31}] \\ &\times [(1 - \cos 2ax_{12})(1 - \cos 2ax_{23}) \\ &\times (1 - \cos 2ax_{31})]^{(\nu+\mu)} \qquad (12)\end{aligned}$$

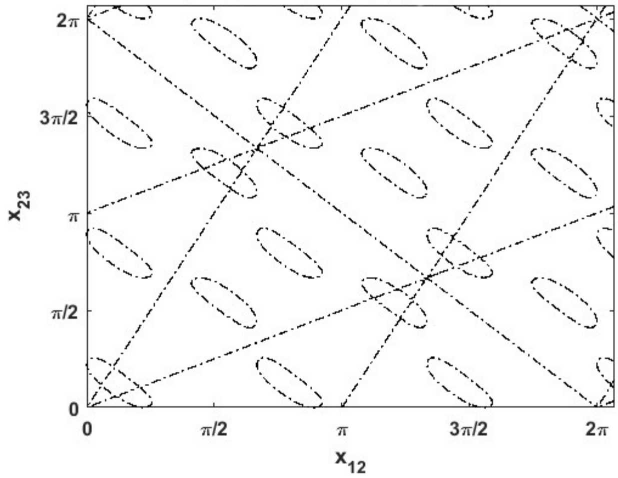

Fig. 3 The configurations with null probability for $a = 0.5$, in the case of three identical distinguishable fermions with the full 3-body interactions. x_{12} is the relative distance between particles labelled 1 and 2 and, x_{23} is the relative distance between particles labelled 2 and 3.

The configurations with null probability corresponding to $|\psi|^2 = 0$ for $a = 0.5$, are shown in Figure 3. Notice that $a = 0.5$ corresponds to the circumference $L = 2\pi$.

Figure 3 is a superposition of lines and ellipses. Any point of these lines or ellipses constitutes a forbidden configuration for fermions on the ring. For instance, the equation of one of these lines reads $x_{23} = \pi x_{12}$. Therefore $x_{12} = \frac{\pi}{2}$ and $x_{23} = \frac{\pi^2}{2}$ constitute a forbidden configuration. i.e. the angular distance $\widehat{x_1 x_2} = 45$ and $\widehat{x_2 x_3} \approx 103$ degrees is one of the infinite number of forbidden configurations in this case.

4.2 three identical distinguishable bosons with full three-body interactions

The suitable partition for three bosons is $\{3,0,0\}$. The Jack polynomial for this partition reads

$$J_{\{3,0,0\}}^{(\mu+\nu)^{-1}}(z_1, z_2, z_3) = 2(z_1^3 + z_2^3 + z_3^3). \qquad (13)$$

Therefore, the eigenfunctions of $\widetilde{H}_{rel}^{full}$ for three bosons reads

$$\psi_{rel}^{full,boson}(z_j) = (z_1^3 + z_2^3 + z_3^3 + z_1^3 + z_2^3 + z_3^3) \times [(z_1 - z_2)(z_2 - z_3)(z_3 - z_1)]^{(\nu+\mu)}. \qquad (14)$$

Back to the initial coordinates x_1, x_2 and x_3 by using Eq. (5) and, relative distances $x_{ij} = x_i - x_j$, $i, j = 1, 2, 3$, Eq. (14) yields

After straight forward calculations, we obtain the density of probability $|\psi_{rel}^{full,boson}|^2$ as follows:

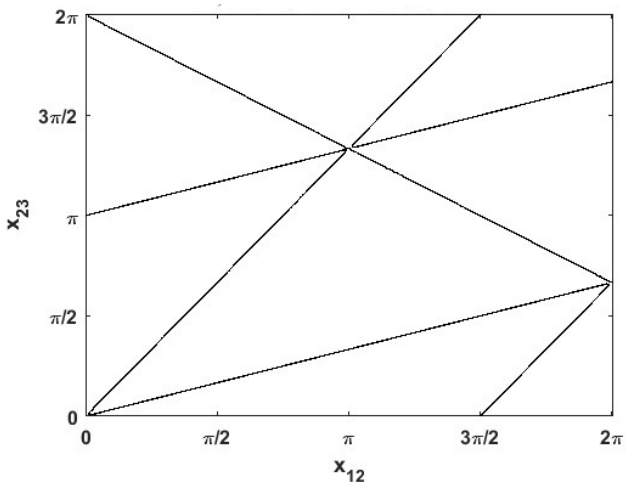

Fig. 4 The configurations with null probability for $a = 0.5$, in the case of three identical distinguishable bosons with the full 3-body interactions. x_{12} is the relative distance between particles labelled 1 and 2 and, x_{23} is the relative distance between particles labelled 2 and 3.

$$|\psi_{rel}^{full,boson}|^2 = 2^{2+3\nu+3\mu}[3+2\cos(6a(x_{12}+x_{32})) + 2\cos(6a(x_{12}+x_{13})) + 2\cos(6a(x_{23}+x_{13}))] \times [(1-\cos 2ax_{12})(1-\cos 2ax_{23}) \times (1-\cos 2ax_{31})]^{(\nu+\mu)}. \quad (15)$$

$$\psi_{rel}^{full,boson} = 2\left[e^{-6iax_{12}} + e^{-6iax_{23}} + e^{-6iax_{31}}\right] \times \left[(e^{-2iax_{23}} - e^{-2iax_{31}})(e^{-2iax_{31}} - e^{-2iax_{12}}) \times (e^{-2iax_{12}} - e^{-2iax_{23}})\right]^{(\nu+\mu)}. \quad (16)$$

The null probability configurations are obtained by letting $|\psi_{rel}^{full,boson}|^2 = 0$. These configurations for $a = 0.5$, are plotted in Figure 4. The lines of Figure 4 are similar to those of Figure 3. These two figures differ only due to the ellipses of Figure 3 that are not present in Figure 4.

5 Forbidden configurations in the case of pure three body interactions

In the case of pure three body interactions, by setting $\nu = 0$ in (4), the Hamiltonian of the system reads:

$$H^{(pure-3body)} = \frac{-\hbar^2}{2m}\sum_{i=1}^{3}\left(\frac{\partial}{\partial x_i}\right)^2 + \frac{\hbar^2}{m}\mu(\mu-1)a^2 \sum_{\substack{i,j,k=1\\ i\neq j\neq k}} \frac{1}{\sin^2[a(x_i+x_j-2x_k)]}, \quad (17)$$

the Hamiltonian (17) is the sum of system's Center of Mass Hamiltonian $H_{CM}^{(pure-3body)}$, and the Hamiltonian of the system in the framework of its Center of Mass $H_{rel}^{(pure-3body)}$. Following the method exposed in [12] by applying the change of variable (5), $H_{rel}^{(pure-3body)}$ is transformed into $\widetilde{H}_{rel}^{(pure-3body)}$:

$$\widetilde{H}_{rel}^{(pure-3body)} = \frac{2\hbar^2 a^2}{m}\left[\sum_{j=1}^{3}\left(z_j\frac{\partial}{\partial_{z_j}}\right)^2 + \mu\sum_{\substack{j,k=1\\ j\neq k}}^{3}\frac{z_j+z_k}{z_j-z_k}\left(z_j\frac{\partial}{\partial_{z_j}}\right)\right]. \quad (18)$$

5.1 three identical distinguishable fermions with pure three-body interactions

The eigenfunction of $\widetilde{H}_{rel}^{(pure-3body)}$ for three distinguishable fermions read [12]:

$$\psi_{rel}^{(pure-3body),fermions}(z_j) = J_{\{2,1,0\}}^{\mu^{-1}}(z_1,z_2,z_3) \times \prod_{\substack{j,k=1\\ j\neq k}}^{3}(z_j-z_k)^{\mu}, \quad (19)$$

where

$$J_{\{2,1,0\}}^{\mu^{-1}}(z_1,z_2,z_3) = z_1^2 z_2 + z_2^2 z_1 + z_1^2 z_3 + z_3^2 z_1 + z_2^2 z_3 + z_3^2 z_2. \quad (20)$$

In terms of the variable z_j, Eq. (19) reads

$$\psi_{rel}^{(pure-3body),fermions}(z_1,z_2,z_3) = \left(z_1^2 z_2 + z_1^2 z_3 + z_2^2 z_1 + z_2^2 z_3 + z_3^2 z_1 + z_3^2 z_2\right) \times [(z_1-z_2)(z_2-z_3)(z_3-z_1)]^{\mu}. \quad (21)$$

Back to the initial coordinates x_1, x_2 and x_3 by using Eq. (5) and, relative distances $x_{ij} = x_i - x_j$, $i,j = 1,2,3$, Eq. (21) reads:

$$\psi_{rel}^{(pure-3body),fermions} = \left(e^{2ia(2x_1-x_2+2x_3)} + e^{6iax_3} + e^{2ia(2x_2-x_3+2x_1)} + e^{6iax_1} + e^{2ia(2x_3-x_1+2x_2)} + e^{6iax_2}\right) \times \left[e^{\frac{2}{3}ia(x_1-2x_2+4x_3)} - e^{\frac{2}{3}ia(x_2-2x_3+4x_1)}\right]^{\mu} \times \left[e^{\frac{2}{3}ia(x_2-2x_3+4x_1)} - e^{\frac{2}{3}ia(x_3-2x_1+4x_2)}\right]^{\mu} \times \left[e^{\frac{2}{3}ia(x_3-2x_1+4x_2)} - e^{\frac{2}{3}ia(x_1-2x_2+4x_3)}\right]^{\mu}. \quad (22)$$

After straight forward calculations, the density of probability reads:

$$
\begin{aligned}
&|\psi_{rel}^{(pure-3body),fermions}|^2 = 2^{(1+3\mu)}\big(3+2\cos 6ax_{23}\\
&+2\cos 6ax_{12}+2\cos 6ax_{31}+2\cos 2a(x_{13}+x_{23})\\
&+2\cos 2a(x_{31}+x_{21})+2\cos 2a(x_{12}+x_{32})\\
&+\cos 4a(x_{13}+x_{23})+\cos 4a(x_{31}+x_{21})\\
&+\cos 4a(x_{12}+x_{32})\times((1-\cos 2a(x_{13}+x_{23}))\\
&\times(1-\cos 2a(x_{31}+x_{21}))(1-\cos 2a(x_{12}+x_{32}))^{\mu} \qquad (23)
\end{aligned}
$$

The configurations with null probability corresponding to $|\psi_{rel}^{(pure-3body),fermions}|^2 = 0$ for $a = 0.5$, are shown in Figure 5.

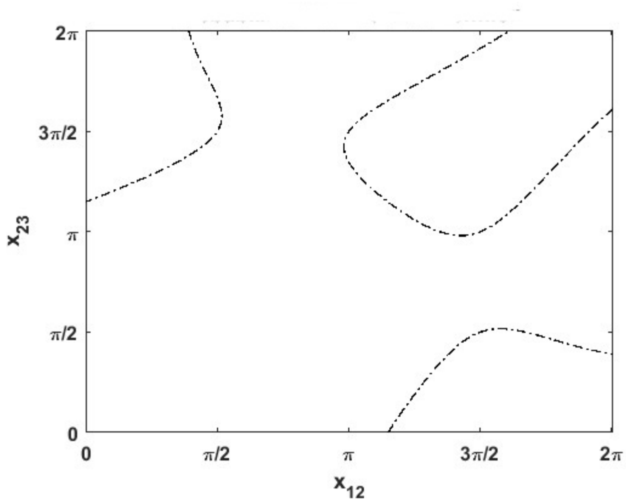

Fig. 5 The configurations with null probability for $a = 0.5$, in the case of three identical distinguishable fermions with pure 3-body interactions. x_{12} is the relative distance between particles labelled 1 and 2 and, x_{23} is the relative distance between particles labelled 2 and 3.

Each of the points of Figure one is a forbidden configuration of the system. For instance the point $x_{12} \approx \frac{3\pi}{2}$, $x_{23} \approx \frac{\pi}{2}$ is a forbidden configuration where angular distances $\widehat{x_1x_2} \approx 270$ and $\widehat{x_2x_3} \approx 90$ degrees.

5.2 three identical distinguishable bosons with pure three-body interactions

In the case of three identical bosons, the eigenfunction of Eq. (18) is

$$
\psi_{rel}^{(pure-3body),bosons}(z_j) = J_{\{3,0,0\}}^{\mu^{-1}}(z_1,z_2,z_3) \times \prod_{\substack{j,k=1\\ j\neq k}}^{3}(z_j - z_k)^{\mu}. \qquad (24)
$$

Back to the initial coordinates x_1, x_2 and x_3 by using Eq. (5) and, relative distances $x_{ij} = x_i - x_j$, $i, j = 1, 2, 3$, Eq. (24) becomes

$$
\begin{aligned}
&\psi_{rel}^{(pure-3body),bosons}(z_j) = \big[2e^{2\mathrm{i}a(x_1-2x_2+4x_3)}\\
&+2e^{2\mathrm{i}a(x_2-2x_3+4x_1)}+2e^{2\mathrm{i}a(x_3-2x_1+4x_2)}\big]\times\\
&\big[e^{\frac{2}{3}\mathrm{i}a(x_1-2x_2+4x_3)}-e^{\frac{2}{3}\mathrm{i}a(x_2-2x_3+4x_1)}\big]^{\mu}\times\\
&\big[e^{\frac{2}{3}\mathrm{i}a(x_2-2x_3+4x_1)}-e^{\frac{2}{3}\mathrm{i}a(x_3-2x_1+4x_2)}\big]^{\mu}\times\\
&\big[e^{\frac{2}{3}\mathrm{i}a(x_3-2x_1+4x_2)}-e^{\frac{2}{3}\mathrm{i}a(x_1-2x_2+4x_3)}\big]^{\mu}. \qquad (25)
\end{aligned}
$$

Straight forward calculations yields:

$$
\begin{aligned}
&|\psi_{rel}^{(pure-3body),bosons}|^2 = 2^{(2+3\mu)}\big(3\\
&+2\cos 6a(x_{13}+x_{23})+2\cos 6a(x_{31}+x_{21})\\
&+2\cos 6a(x_{12}+x_{32})\big)\big((1-\cos 2a(x_{13}+x_{23}))\\
&\times(1-\cos 2a(x_{31}+x_{21}))(1-\cos 2a(x_{12}+x_{32}))\big)^{\mu}. \quad (26)
\end{aligned}
$$

The configurations with null probability corresponding to $|\psi_{rel}^{(pure-3body),bosons}|^2 = 0$ for $a = 0.5$, are shown in Figure 6.

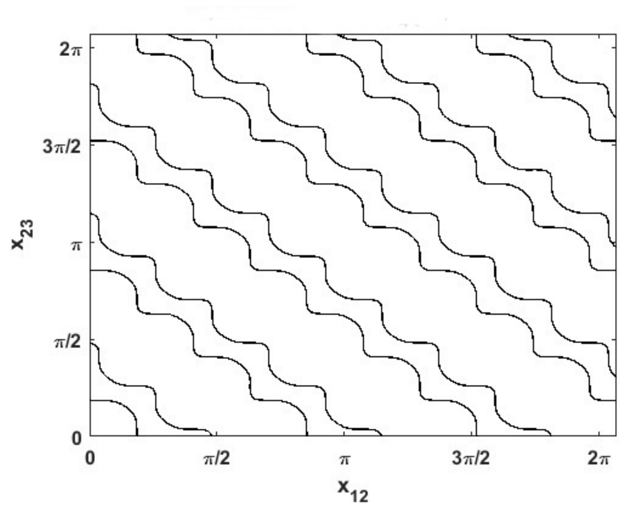

Fig. 6 The configurations with null probability for $a = 0.5$, in the case of three identical distinguishable bosons with pure 3-body interactions. x_{12} is the relative distance between particles labelled 1 and 2 and, x_{23} is the relative distance between particles labelled 2 and 3.

As in previous figures, In Figure 6, any point corresponds to a forbidden configuration. For instance, the point $x_{12} \approx \pi$, $x_{23} \approx \frac{\pi}{4}$ is a forbidden configuration where angular distances $\widehat{x_1x_2} \approx 180$ and $\widehat{x_2x_3} \approx 45$ degrees.

6 Forbidden configurations in the case of pairwise interactions

In the case of pure pairwise interaction, the Hamiltonian of the system (4) is reduced to [23]:

$$H^{pairwise} = \frac{-\hbar^2}{2m}\sum_{i=1}^{3}\left(\frac{\partial}{\partial x_i}\right)^2 + \frac{\hbar^2}{m}\nu(\nu-1)a^2 \sum_{\substack{i,j=1\\ i\neq j}} \frac{1}{\sin^2[a(x_i-x_j)]}. \quad (27)$$

Eq. (27) can be written as the sum of $H^{pairwise}_{CM}$ and $H^{pairwise}_{rel}$, Hamiltonian of the centre-of-mass and Relative Hamiltonian, respectively. The relative Hamiltonian can be written as follows [24]:

$$H^{pairwise}_{rel} = -\frac{\hbar^2}{2m}\sum_{j=1}^{3}\left(\frac{\partial^2}{\partial x_j^2}\right) - \frac{\hbar^2}{m}a\nu\sum_{j\neq k}^{3}\cot(ax_{jk})\left(\frac{\partial}{\partial x_j}-\frac{\partial}{\partial x_k}\right) \quad (28)$$

Letting $z_j = \exp(2iax_j)$, Eq. (28) becomes

$$\widetilde{H}^{pairwise}_{rel} = \frac{2\hbar^2 a^2}{m}\left[\sum_{j=1}^{3}\left(z_j\frac{\partial_j}{\partial z_j}\right)^2 + \nu\sum_{j\neq k}^{3}\frac{z_j+z_k}{z_j-z_k}\left(z_j\frac{\partial_j}{\partial z_j} - z_j\frac{\partial_k}{\partial z_k}\right)\right]. \quad (29)$$

6.1 three identical distinguishable fermions

For three identical fermions, the eigenfunction of Eq. (28) is

$$\psi^{pairwise,fermions}_{rel}(z_j) = (z_1^2z_2 + z_1^2z_3 + z_2^2z_1 + z_2^2z_3 + z_3^2z_1 + z_3^2z_2) \times [(z_1-z_2)(z_2-z_3)(z_3-z_1)]^{\nu}, \quad (30)$$

Back to the initial coordinates x_1, x_2 and x_3 by using Eq. (5) and, relative distances $x_{ij} = x_i - x_j$, $i,j = 1,2,3$, the wave function in Eq. (30) reads:

$$\psi^{pairwise,fermions}_{rel} = e^{-2ia(2x_1+x_2)} + e^{-2ia(2x_1+x_3)} + e^{-2ia(2x_2+x_1)} + e^{-2ia(2x_2+x_3)} + e^{-2ia(2x_3+x_1)} + e^{-2ia(2x_3+x_2)} \times \left[(e^{-2iax_1} - e^{-2iax_2}) \times (e^{-2iax_2} - e^{-2iax_3})(e^{-2iax_3} - e^{-2iax_1})\right]. \quad (31)$$

Straight forwards calculations yields

$$|\psi^{pairwise,fermions}_{rel}|^2 = 2^{(1+3\nu)}\big(3 + 2\cos 2ax_{12} + 2\cos 2ax_{23} + 2\cos 2ax_{31} + \cos 4ax_{12} + \cos 4ax_{23} + \cos 4ax_{31} + 2\cos 2a(x_{31}+x_{21}) + 2\cos 2a(x_{12}+x_{32}) + 2\cos 2a(x_{23}+x_{13})\big) \times \big((1-\cos 2ax_{12})(1-\cos 2ax_{23})(1-\cos 2ax_{31})\big)^{\nu} \quad (32)$$

The configurations with null probability corresponding to $|\psi^{pairwise,fermions}_{rel}|^2 = 0$ for $a = 0.5$, are shown in Figure 7.

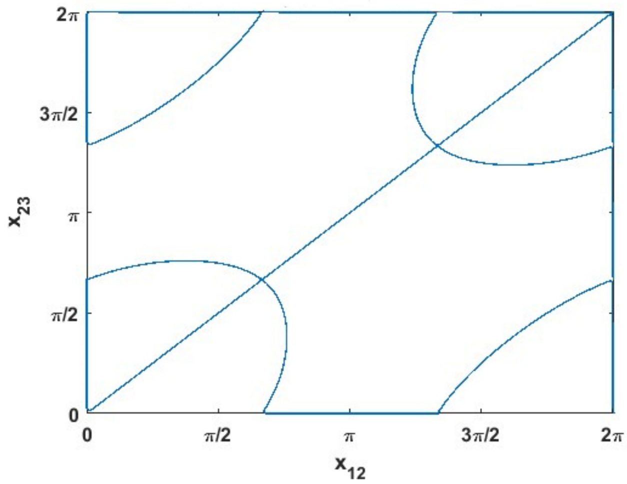

Fig. 7 The configurations with null probability for $a = 0.5$, in the case of three identical distinguishable fermions with pairwise interactions. x_{12} is the relative distance between particles labelled 1 and 2 and, x_{23} is the relative distance between particles labelled 2 and 3.

Similar to previous figures, In Figure 7, any point corresponds to a forbidden configuration. For instance, the point $x_{12} = \frac{\pi}{2}$, $x_{23} = \frac{\pi}{2}$ is a forbidden configuration where angular distances $\widehat{x_1x_2} = \widehat{x_2x_3} = 90$ degrees.

6.2 three identical distinguishable bosons

For three identical bosons, the eigenfunction of Eq. (28) is

$$\psi^{pairwise,bosons}_{rel} = 2\big[e^{-6iax_1} + e^{-6iax_2} + e^{-6iax_3}\big] \times \big[(e^{-2iax_1} - e^{-2iax_2}) \times (e^{-2iax_2} - e^{-2iax_3})(e^{-2iax_3} - e^{-2iax_1})\big] \quad (33)$$

Back to the initial coordinates x_1, x_2 and x_3 by using Eq. (5) and, relative distances $x_{ij} = x_i - x_j$, $i,j =$

1, 2, 3, Eq. (33) reads:

$$|\psi_{rel}^{pairwise,bosons}|^2 = 2^{(2+3\nu)}\left[3+2\cos 6ax_{12} + 2\cos 6ax_{23} + 2\cos 6ax_{31}\right] \times (1-\cos 2ax_{12})(1-\cos 2ax_{23})(1-\cos 2ax_{31}) \quad (34)$$

The configurations with null probability corresponding to $|\psi_{rel}^{pairwise,bosons}|^2 = 0$ for $a = 0.5$, are shown in Figure 8.

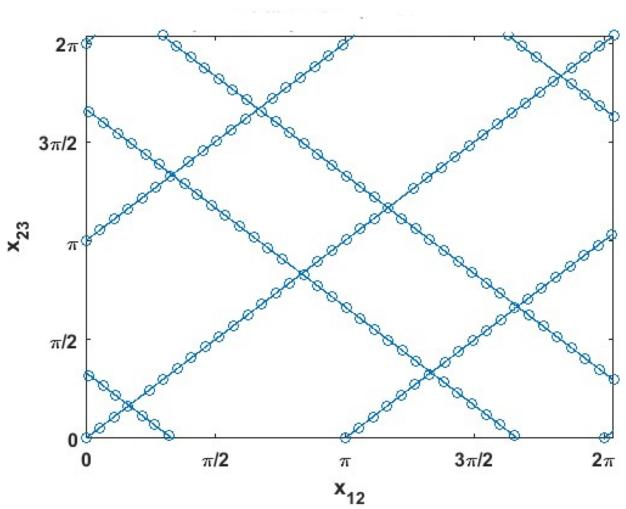

Fig. 8 The configurations with null probability for $a = 0.5$, in the case of three identical distinguishable bosons with pairwise interactions. x_{12} is the relative distance between particles labelled 1 and 2 and, x_{23} is the relative distance between particles labelled 2 and 3.

Figure 7 is a superposition of lines and multitude of small circles on these lines. Any point of these lines or circles constitutes a forbidden configuration of the system. For instance, the equation of one of these lines reads $x_{23} = x_{12}$. Therefore $x_{12} = \frac{\pi}{2}$ constitute a forbidden configuration. i.e. the angular distance $\widehat{x_1x_2} = \widehat{x_2x_3} =$ 90 degrees is one of the infinite number of forbidden configurations in this case.

7 Conclusions and outlooks

In the three-body quantum problem, an important question is to know which of the pairwise, pure three-body or full three-body interactions best fits a given physical situation. In this paper, we have a strong tool to answer this question by comparing theoretical predictions of forbidden configurations to experimental observations.

To this end, we write explicitly the wave function for three identical distinguishable particles on a ring with the so-called trigonometric interactions. We show that in all cases, there are infinite sets of forbidden configurations associated with the system's null density probability. A configuration corresponds to a given set of relative distances between particles, x_{12} and x_{23}.

These configurations are exhibited in six figures: Figures 3, 5 and 7 represent forbidden configurations for three fermions on a ring in the case of full three-body interactions, pure three-body interactions and pairwise interactions, respectively. Figures 4, 6 and 8 represent forbidden configurations for three bosons on a ring in the case of full three-body interactions, pure three-body interactions and pairwise interactions, respectively.

Given a system with, let's say, three fermions on a ring. The question is, with which kind of interactions, full three-body, pure three-body, or pairwise, should one model the system? Here is the procedure to choose the correct interactions. Experimentally, it is possible to point out some configurations with null probability. These configurations, to experimental uncertainties, will be closed to one of the figures 3, 5 or 7. Hence, this simple comparison will decide which of the three kinds of interactions prevail in this system.

We suggest the context of cold atoms ring-shaped optical lattices as an observational, experimental context [17–19].

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