



# Concept of weak derivative by resorting to physical examples

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**Abstract** In this article, we present the concept of weak derivatives with respect to the potential of a charge distribution. Furthermore, weak derivatives can define the correct self-adjoint momentum operators in quantum mechanics.

## 1 Introduction

The concept of weak derivative can be considered a generalization of the classical derivative. In other words, functions that are not differentiable but are capable of integration can have weak derivatives [1].

Most studies on weak derivatives are associated with a subspace of Hilbert space, such as Sobolev space [1-5]. This coincidence could be more pleasant for popular physicists unfamiliar with functional analysis, so we try to address this topic with a newer and more impressive approach.

In ref. [4], it has been discussed how to define weak and strong derivatives in Orlicz space (Orlicz space is a special subspace of Banach space; if we define Banach space by  $L^p$ , the Orlicz space is determined by  $p = 1$ ). The relationship between perturbation analysis and weak derivative is analyzed in ref [5].

We didn't find any paper that directly indicates the exact application of the weak derivative in physics. However, paper [6] endeavours to indicate its contribution to understanding mathematical models. The weak derivative also motivates another quite popular concept of solutions, the so-called weak solutions [6, 7].

The definition of the weak derivative can be stated as follows: Let  $f$  be a function in the measurable space (A real or complex-valued measurable function on the real line with respect to Lebesgue measure is an element of  $L^1$  if:  $\int_{-\infty}^{\infty} |f(x)| dx < \infty$ ), we call the function  $f$  in  $L^1([a, b])$  the weak derivative of  $f$  if

$$\int f(x)g(x)dx = \int f'(x)g(x)dx, \quad (1)$$

In equation (1),  $f'(x)$  is the weak derivative of  $f(x)$  and the function  $g(x)$  is called test function. These test functions are smooth and have bounded support. Note that, the test function is infinitely differentiable and  $g \in C_c^\infty$ .

The most common examples of functions with a weak derivative are the absolute value function  $f(x) = |x|$ , whose weak derivative is the sign function, and the Heaviside step function, whose derivative is the delta function.

## 2 Concept of weak derivative

Unlike the classical concept of derivative, which has a geometric concept, the weak derivative has no geometric concept. However, this can be described in relation to the concept of distribution.

First, note that one way to identify the function  $f$  with a given domain is to know the value of  $f(x)$  for each  $x$  in the domain of the function  $f$ . But there is another way. Knowledge of the effect of function  $f$  on test functions. For example, in

$$(f, g) = \int f(x)g(x)dx. \quad (2)$$

It is possible to find the effect of  $f'$  in the situation where  $f'$  exists or does not exist in the usual way, in other words  $(f', g) = -(f, g')$ . Now, for a physical understanding, let's assume that  $f(x)$  is the potential of a point charge which we know is inversely proportional to the distance

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$$\int V(x)g(x)dx = \int \frac{1}{x}g(x)dx. \quad (3)$$

The potential function has no weak derivative, but if we consider the potential function of the linear charge distribution to be like an infinite bar, that is

$$\int V(x)g(x)dx = \int \ln(x)g(x)dx. \quad (4)$$

In this case, we will see a weak derivative and the electric field can be investigated. First, consider the following example.

### Example 1

Suppose the function  $f(x)$  is defined by

$$f(x) = \begin{cases} \frac{1}{x} & x > 0 \\ 0 & x < 0 \end{cases}, \quad (5)$$

Let's now look at the following test function, which is smooth and has bounded support.

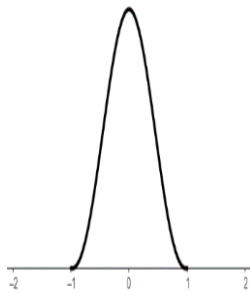


Figure 1. The desired test function is a smooth function that is differentiable, usually several times.

Then, with respect to the definition (1), we can write

$$\int_0^{\infty} \left(\frac{1}{x}\right)g(x)dx = \frac{g(\infty)}{\infty} - \frac{g(0)}{0} + \int_0^{\infty} \left(\frac{1}{x^2}\right)g(x)dx. \quad (6)$$

In equation (6), if the value of  $g(0)$  is equal to zero, then there exists a weak derivative of the function (5), but otherwise it is not equal to zero, there will be no weak derivative. According to the definition and representation of the mentioned test function, the value of  $g(0)$  can not be zero, so the weak derivative of the function  $f(x)$  does not exist.

### Example 2

Consider the function  $h(x)$  with the following definition

$$f(x) = \begin{cases} \ln(x) & x > 0 \\ 0 & x < 0 \end{cases}. \quad (7)$$

We consider the test function again, as in the previous example. Here we will see that there is  $\int h(x)g(x)dx$  for any test function  $g$ .

To evaluate the integral, we can write

$$\begin{aligned} \int h(x)g'(x)dx &= \int_0^{\infty} (\ln(x))g'(x)dx \\ &= \lim_{s \rightarrow 0^+} \ln(s)g(s) + \int_s^{\infty} \left(\frac{1}{x}\right)g(x)dx. \end{aligned} \quad (8)$$

Although each of the above expressions tends to infinity independently in the limit  $s \rightarrow \infty$ , but their sum tends to a certain value (if  $g(0) \neq 0$ ). Therefore, there exists a weak derivative of  $h(x)$ .

## 3 The electrical potential of an infinite line of charge

In this section, we obtain the electric field due to the weak derivative of an infinite uniform charge line. Before that, we will discuss an important property of the delta function.

From mathematical physics textbooks, the delta function has many properties. However, one advanced but notable characteristic associated with this function is related to the equation, which states that it  $x\delta(x)$  is identically zero. As a result, we may add any finite multiple of this zero to one side of the equation.

$$A(x) = B(x) = B(x) + c x \delta(x), \quad (9)$$

where  $c$  is an arbitrary finite constant. However, if we divide both sides by  $x$ , the addition  $c \delta(x)$  is no longer to zero at  $x = 0$ . Then

$$\frac{A(x)}{x} = \frac{B(x)}{x} + c \delta(x), \quad (10)$$

is not necessarily true for arbitrary values of  $c$ . For example, consider

$$x \frac{d \ln(x)}{dx} = 1 = 1 + c x \delta(x), \quad (11)$$

is valid for any constant  $c$ . However,

$$\frac{d \ln(x)}{dx} = \frac{1}{x} + c \delta(x). \quad (12)$$

This is true for any special values of  $c$ . To determine these values, we need to examine the behavior of these functions in the neighborhood of  $x = 0$  by integrating both sides of the expression (12) from  $-\varepsilon$  to  $\varepsilon$ . Thus  $c = i(2n + 1)\pi$ , where  $n$  is any integer.

An equation similar to Eq. (12) can be seen in [8]. As this reference shows, this equation is not just a mathematical curiosity. It plays an important role in quantum mechanics. Let us now return to the potential of an infinite line of charge. We know that the potential of an infinite bar with charge density  $\lambda$  at vertical distance  $y$  is equal to

$$V = \frac{\lambda}{2\pi\varepsilon_0} \ln(y) = \begin{cases} \frac{\lambda}{2\pi\varepsilon_0} \ln(y) & y > 0 \\ 0 & y < 0 \end{cases}. \quad (13)$$

Thus, with respect to the Eq. (12), the electric field can be written

$$E = \frac{-\lambda}{2\pi\varepsilon_0} \left( \frac{1}{y} + i(2n + 1)\pi\delta(y) \right). \quad (14)$$

Eq. (14) shows that despite the singularity, we can obtain an identified electric field at the location of the electric charge.

Note that although the weak derivative of Eq. (13) does exist, the electric field of (14) is not exactly its weak derivative.

#### 4 The Weak derivative in quantum mechanics

The momentum operator in quantum mechanics is an unbounded operator, and its domain in  $L^2(\mathbb{R}, dx)$  is equal to

$$D(P) = \{\varphi(x) \in L^2(\mathbb{R}, dx) | \varphi'(x) \in L^2(\mathbb{R}, dx)\}. \quad (15)$$

Eq. (15) states that the domain of momentum operators consisting of square-integrable functions must be differentiable, and their derivatives must lie in Hilbert space. However, there are some examples, such as

$$f(x) = \frac{1}{1+x^2}, \quad (16)$$

which despite being square-integrable but its derivative is not in  $L^2(\mathbb{R}, dx)$ . Also, there is another problem.

For functions such as  $f(x)$  and  $g(x)$  belonging to the domain of the momentum operator, we can write:

$$(g, Pf) - (Pg, f) = [-ig^*f]_{-\infty}^{+\infty}. \quad (17)$$

Normally the functions  $f$  and  $g$  are considered to be zero in the  $\lim x \rightarrow \pm\infty$ , but the reality is that not all square-integrable functions vanish or even tend to reach a finite value at infinity. We can see plenty of examples in [9, 10] to confirm this point.

Thus, in general, the momentum operator is not self-adjoint with its common definition based on the concept of classical derivative [10]. For this reason, this operator is defined based on the weak derivative [11]. In this case, the domain of the momentum operator coincides with the Sobolev space  $H^1(\mathbb{R})$  [11]. This is very impressive, every function in Sobolev space tends to zero at infinity [12]. Therefore, the correct domain of the momentum operators in quantum mechanics can be as follows

$$D(P) = \{\varphi(x) \in L^2(\mathbb{R}, dx) | \varphi'(x) \in L^2(\mathbb{R}, dx)\}, \quad (18)$$

where  $\varphi'(x)$  is the weak derivative of the function  $\varphi(x)$ .

All of the mentioned discussions in this section have been examined for the generalized momentum operators in ref. [13].

#### 5 Conclusion

Although the weak derivative has no geometric description, as we have seen, it is possible to express concepts related to distribution functions. We have chosen two examples related to the potential of the charge point and the linear distribution of charge. The weak derivative of the first example does not exist in the singularity but in the second example.

In standard quantum mechanics, momentum operators are generally not self-adjoint. This subject led mathematical physicists to propose subspaces such as Sobolev space. In this space, the weak derivative determines the momentum operators with a new domain. As discussed in the last section, the discussion about the self-adjointness of momentum operators moved on to the generalized momentum operators. For these operators, see the ref. [14] and the references therein.

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