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Determination of infinitesimal generators on de Sitter space-time

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Abstract In this work, we study the motion of a free particle in 1+3-de Sitter spacetime and introduce the associated symmetric group. We also present ten one-parameter subgroups of this group and derive the corresponding algebraic bases (or the Lie algebra of the de Sitter group). Finally, these subgroups introduce ten associated infinitesimal generators in 1+3-de Sitter spacetime.

1 Introduction

In general relativity, gravity is considered as the curvature of space-time, and the motion of a particle in this space is expressed by Einstein's equation [1]:

$$G_{\mu\nu} + \lambda g_{\mu\nu} = -T_{\mu\nu}\ , \tag{1}$$

where $T_{\mu\nu}$ and λ are the energy-momentum tensor and cosmological constant, respectively. From this equation, we can derive one of the Friedmann-Lemaitre-Robertson-Walker solutions for a homogeneous and isotropic universe [2]:

$$H_\circ{}^2 = \frac{\lambda}{3} + \frac{8\pi G}{3}\rho - \frac{K}{a^2}, \qquad H_\circ = \frac{\dot{a}}{a}, \tag{2}$$

where $H_\circ, G, a$ and K are respectively the Hubble parameter, Gravitational constant, scale factor and curvature of the universe. This equation can be rewritten by density parameters as:

$$\Omega_\lambda + \Omega_{other} = 1, \tag{3}$$

where Ω_λ is the density of Gravitational constant or density of dark energy[1] and Ω_{other} includes the densities of radiation (Ω_r), (cold) dark matter (Ω_c), baryonic matter (Ω_b) and curvature of space (Ω_K) [2]. Eq. (3) can be used for different epochs of cosmology [2,3]. For the current epoch, the experimental data show that approximately 69% of the universe is made up of dark energy, and the contribution of other factors[2] is 31% [4]. This means that for an approximate solution to Einstein's equation, we can consider only the cosmological constant and ignore the energy-momentum tensor on the right-hand side of Eq. (1). In other words, in the first approximation, the universe can be considered as a 1+3-de-Sitter space (i.e., the case $\lambda > 0$ and $T_{\mu\nu} = 0$ in Eq. (1)).

The 1+3-de Sitter metric is visualised as a hyperboloid embedded in five-dimensional Minkowski space. It is very important to study the symmetrical group associated with the motion of a free particle on this hyperboloid and its Lie algebra, which we discuss in this work. We also introduce all infinitesimal generators on this manifold.

2 1+3-de Sitter space-time

The de Sitter metric $g_{\mu\nu}$ is the solution of Einstein's equation for $\lambda > 0$ and $T_{\mu\nu} = 0$, i.e.

$$G_{\mu\nu} + \lambda g_{\mu\nu} = 0. \tag{4}$$

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[1] The cosmological constant can be considered as the dark energy (see chapter 6 of [2]).

[2] The Ω_{other} is given by:
$\Omega_{other} = \Omega_r + \Omega_c + \Omega_b + \Omega_K$,
where constitutes 31% of the universe (for thecurrent epoch).
The contribution of each factor is as follows [2]:
$\Omega_r \rightarrow 0.008\% \Omega_c \rightarrow 26.07\%$
$\Omega_b \rightarrow 04.90\% \Omega_K \rightarrow 0.07\%$

The 1+3-de Sitter space-time as a hyperboloid embedded in five-dimensional Minkowski space (ambient space) is defined by:

$$X \equiv \{x \in \mathbb{R}^5 | x^2 = \eta_{\gamma\beta} x^\gamma x^\beta = -H_\circ^{-2}\}, \tag{5}$$

where $\eta_{\gamma\beta} = diag(1,-1,-1,-1,-1)$ is Minkowski metric of five-dimensional space and $\gamma, \beta = 0,1,2,3,4$. The associated isometry group is group $SO_0(1,4)$, or equivalently its universal covering, i.e. the symplectic group $Sp(2, 2)$. The group $SO_0(1,4)$ is defined as:

$$SO_0(1,4) = \{\Lambda \in M_5(R) | \Lambda^{00} \geq 1, \Lambda^t \eta \Lambda = \eta\}. \tag{6}$$

This group acts on the vector x in ambient space as follows:

$$x' = \Lambda x, \tag{7}$$

where $x = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \\ x^4 \end{pmatrix}$.

But the $Sp(2,2)$ group is described by 2×2 matrices with quaternionic[3] coefficients as [5]:

$$Sp(2,2) = \left\{ g = \begin{pmatrix} ab \\ cd \end{pmatrix} \middle| det\, g = 1, g^\dagger \gamma^0 g = \gamma^0 \right\}, \tag{8}$$

where a, b, c, d belong to quaternion field $Q \simeq \mathbb{R}_+ \times SU(2)$ and $\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$ that I and 0 are respectively 2×2 unit and zero matrices. This group acts on matrix h as:

$$h' = ghg^{-1}, \tag{9}$$

where

$$h = \begin{pmatrix} x^0 I - P \\ \overline{P} - x^0 I \end{pmatrix}, g^{-1} = \begin{pmatrix} \overline{a} - \overline{c} \\ -\overline{bd} \end{pmatrix}, P = \begin{pmatrix} x^4 + ix^3 ix^1 - x^2 \\ ix^1 + x^2 x^4 - ix^3 \end{pmatrix},$$

and the bar sign means conjugate. Between two mentioned groups, there is a homomorphism as follows:

$$\Lambda^\alpha_\beta = \frac{1}{4} tr(\gamma^\alpha g \gamma_\beta g^{-1}), \tag{10}$$

Where $\gamma^4 = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \gamma^k = \begin{pmatrix} 0 & e_k \\ e_k & 0 \end{pmatrix}, k = 1,2,3,$
and $e_k = (-1)^{k+1} i\sigma_k$. The σ_k is one of the three Pauli matrices. In the following, we only use the group $Sp(2,2)$.

3 Lie algebra of de Sitter group

The study of Lie algebra and, consequently, the infinitesimal generators of a group is done by decomposing every element of that group. There are three decompositions for presentation of $Sp(2,2)$ group. The Lorentz space-time, Kartan and Ivasava decompositions [6]. Here, we use the first decomposition. Based on the Lorentz space-time decomposition, each element of $Sp(2,2)$ group is given as [5]:

$$g = jl \in Sp(2,2), \tag{11}$$

$$j = \begin{pmatrix} \eta & 0 \\ 0 & \bar{\eta} \end{pmatrix} \begin{pmatrix} cosh(\frac{\psi}{2}) & sinh(\frac{\psi}{2}) \\ sinh(\frac{\psi}{2}) & cosh(\frac{\psi}{2}) \end{pmatrix}, \tag{12}$$

$$l = \begin{pmatrix} \xi & 0 \\ 0 & \xi \end{pmatrix} \begin{pmatrix} cosh(\frac{\varphi}{2}) & \hat{u}\, sinh(\frac{\varphi}{2}) \\ -\hat{u}\, sinh(\frac{\varphi}{2}) & cosh(\frac{\varphi}{2}) \end{pmatrix}, \tag{13}$$

where $\varphi, \psi \in \mathbb{R}$, $\eta, \xi \in SU(2)$ and $\hat{u}$ is a pure quaternion i.e., $\bar{\hat{u}} = -\hat{u} \in SU(2)$. The factors j and l belong to the set of space-time translation and the homogeneous Lorentz subgroup (including the Lorentz boost and spatial rotation), respectively. This shows that the group $Sp(2,2)$ has ten parameters (three for space translation, one for time translation, three for spatial rotation and three for Lorentz boost). Therefore, we have ten one-parameter subgroup as:

$$S_i = \begin{pmatrix} e^{e_i t} & 0 \\ 0 & e^{-e_i t} \end{pmatrix}, \tag{14}$$

$$\Theta = \begin{pmatrix} cosh(\frac{t}{2}) & sinh(\frac{t}{2}) \\ sinh(\frac{t}{2}) & cosh(\frac{t}{2}) \end{pmatrix}, \tag{15}$$

$$R_i = \begin{pmatrix} e^{e_i t} & 0 \\ 0 & e^{e_i t} \end{pmatrix}, \tag{16}$$

[3] Every quaternion is given by:
$\boldsymbol{\zeta} = (\boldsymbol{\zeta_0}, \vec{\boldsymbol{\zeta}}) = \boldsymbol{\zeta_0} + \boldsymbol{\zeta_1 e_1} + \boldsymbol{\zeta_2 e_2} + \boldsymbol{\zeta_3 e_3}$,
where
$e_i e_j = \varepsilon_{ijk} e_k i, j, k = 1,2,3,$
$e_j e_j = -1.$

$$B_i = \begin{pmatrix} cosh(\frac{t}{2}) & e_i\, sinh(\frac{t}{2}) \\ -e_i\, sinh(\frac{t}{2}) & cosh(\frac{t}{2}) \end{pmatrix}, \qquad (17)$$

and ten corresponding algebraic bases:

$$X_{i4} = \frac{1}{2}\begin{pmatrix} e_i & 0 \\ 0 & -e_i \end{pmatrix}, \qquad (18)$$

$$X_{04} = \frac{1}{2}\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \qquad (19)$$

$$X_{ij} = \frac{1}{2}\varepsilon_{ijk}\begin{pmatrix} e_k & 0 \\ 0 & e_k \end{pmatrix}, \qquad (20)$$

$$X_{0i} = \frac{1}{2}\begin{pmatrix} 0 & e_i \\ -e_i & 0 \end{pmatrix}, \qquad (21)$$

Where t is a parameter, and $i, j, k = 1,2,3$.

4 Infinitesimal generators of de Sitter group

For determination of the infinitesimal generators of de Sitter group (or equivalently on hyperboloid manifold), we have to first introduce the generator (operator) $\ell^{\tau}(g)$. This operator acts on complex function $F(\zeta) \in \mathbb{C}$ on Hilbert subspace $H = L_c^2(SU(2))$ as [7]:

$$\ell^{\tau}(g)F(\zeta) = (\chi(\zeta,g))^{-2\tau}F(g^{-1}.\zeta) = (\chi(\zeta,g))^{-2\tau}F((a'\zeta + b')(c'\zeta + d')^{-1}), \quad (22)$$

where $\chi(\zeta, g) = det(\, c'\zeta + d')$, $\zeta \in SU(2)$, and

$$g^{-1} = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}.$$

Indeed, the mapping $\zeta \to g^{-1}.\zeta = (a'\zeta + b')(c'\zeta + d')^{-1}$ is a homomorphism mapping from $SU(2)$ group to itself. The representation of (22) is a linear representation. This representation is the unitary representation of the principal series if and only if

$$\tau = \frac{3}{2} + i\nu, \quad \nu \in \mathbb{R}. \qquad (23)$$

Now we construct the representation of each one parameter subgroup of $Sp(2,2)$. For this, we use the general coordinate for introducing $\zeta \in SU(2) \sim S^3$:

$$\begin{cases} \zeta_1 = sin\,\alpha\, sin\,\theta\, cos\,\phi \\ \zeta_2 = sin\,\alpha\, sin\,\theta\, sin\,\phi \\ \zeta_3 = sin\,\alpha\, cos\,\theta \\ \zeta_0 = cos\,\alpha \end{cases} \qquad (24)$$

where $0 \le \alpha, \theta \le \pi$ and $0 \le \phi < 2\pi$. For every one-parameter subgroup and $(\alpha', \theta', \phi') = g(t).(\alpha, \theta, \phi)$, we can introduce the infinitesimal generator Υ as:

$$\frac{\partial[\ell^{\tau}(g)F(\alpha,\theta,\phi)}{\partial t}\bigg|_{t=0} = \frac{\partial[(\chi(\zeta,g))^{-2\tau}]}{\partial t}\bigg|_{t=0} F(\alpha,\theta,\phi) + (\chi(\zeta,g))^{-2\tau}|_{t=0}\frac{\partial F(\alpha',\theta',\phi')}{\partial t}\bigg|_{t=0}$$

$$= \left[\frac{\partial[(\chi(\zeta,g))^{-2\tau}]}{\partial t}\bigg|_{t=0} + \frac{\partial\alpha'}{\partial t}\bigg|_{t=0}\frac{\partial}{\partial\alpha} + \frac{\partial\theta'}{\partial t}\bigg|_{t=0}\frac{\partial}{\partial\theta} + \frac{\partial\phi'}{\partial t}\bigg|_{t=0}\frac{\partial}{\partial\phi}\right]F(\alpha,\theta,\phi)$$

$$= -i\Upsilon F(\alpha,\theta,\phi)$$

$$\Rightarrow \Upsilon = i\left(\frac{\partial[(\chi(\zeta,g))^{-2\tau}]}{\partial t}\bigg|_{t=0} + \frac{\partial\alpha'}{\partial t}\bigg|_{t=0}\frac{\partial}{\partial\alpha} + \frac{\partial\theta'}{\partial t}\bigg|_{t=0}\frac{\partial}{\partial\theta} + \frac{\partial\phi'}{\partial t}\bigg|_{t=0}\frac{\partial}{\partial\phi}\right). \qquad (25)$$

From this later equation and Eqs (14) - (17), we can obtain ten infinitesimal generators on 1+3-de Sitter space.

Calculations of one of the generators (calculations of other similar generators) are brought in Appendix A.

I) Infinitesimal generators of space translation (P)

We derive the infinitesimal generators of space translation from Eq. (14) and (25). This generator is the momentum generator that has vector properties

$$P_1 = -i\left(sin\,\theta\, cos\,\phi\frac{\partial}{\partial\alpha} + cot\,\alpha\, cos\,\theta\, cos\,\phi\frac{\partial}{\partial\theta} - \frac{cot\,\alpha\, sin\,\phi}{sin\,\theta}\frac{\partial}{\partial\phi}\right),$$

$$P_2 = -i\left(sin\,\theta\, sin\,\phi\frac{\partial}{\partial\alpha} + cot\,\alpha\, cos\,\theta\, sin\,\phi\frac{\partial}{\partial\theta} + \frac{cot\,\alpha\, cos\,\phi}{sin\,\theta}\frac{\partial}{\partial\phi}\right),$$

$$P_3 = -i\left(cos\,\theta\frac{\partial}{\partial\alpha} - cot\,\alpha\, sin\,\theta\frac{\partial}{\partial\theta}\right).$$

II) Infinitesimal generator of time translation (T)

This generator has the scalar properties (see appendix)

$$T = i\left(\tau \cos\alpha + \sin\alpha \frac{\partial}{\partial\alpha}\right).$$

III) Infinitesimal generators of space rotation (J)

We can obtain this generator from Eqs (16) and (25). It is the angular momentum generator that has the vector properties

$$J_1 = i\left(\sin\phi \frac{\partial}{\partial\theta} + \cot\theta\cos\phi \frac{\partial}{\partial\phi}\right),$$

$$J_2 = i\left(-\cos\phi \frac{\partial}{\partial\theta} + \cot\theta\sin\phi \frac{\partial}{\partial\phi}\right),$$

$$J_3 = i\left(-\frac{\partial}{\partial\phi}\right).$$

IV) Infinitesimal generators of Lorentz boosts (K)

This generator has the vector properties obtained from Eqs (17) and (25)

$$K_1 = i\left(\tau\sin\alpha\sin\theta\cos\phi - \cos\alpha\sin\theta\cos\phi\frac{\partial}{\partial\alpha} - \frac{\cos\theta\cos\phi}{\sin\alpha}\frac{\partial}{\partial\theta} + \frac{\sin\phi}{\sin\alpha\sin\theta}\frac{\partial}{\partial\phi}\right),$$

$$K_2 = i\left(\tau\sin\alpha\sin\theta\sin\phi - \cos\alpha\sin\theta\sin\phi\frac{\partial}{\partial\alpha} - \frac{\cos\theta\sin\phi}{\sin\alpha}\frac{\partial}{\partial\theta} - \frac{\cos\phi}{\sin\alpha\sin\theta}\frac{\partial}{\partial\phi}\right),$$

$$K_3 = i\left(\tau\sin\alpha\cos\theta - \cos\alpha\cos\theta\frac{\partial}{\partial\alpha} + \frac{\sin\theta}{\sin\alpha}\frac{\partial}{\partial\theta}\right).$$

5 Commutation relations of infinitesimal generators

In the previous section, we introduced all of the ten infinitesimal generators on 1+3-de Sitter space. We can directly obtain the associated commutation relations as:

$$[J_i, T] = 0, \quad (26)$$

$$[J_i, J_j] = i\varepsilon_{ijk}J_k, \quad (27)$$

$$[J_i, P_j] = i\varepsilon_{ijk}P_k, \quad (28)$$

$$[J_i, K_j] = i\varepsilon_{ijk}K_k, \quad (29)$$

$$[P_i, P_j] = i\varepsilon_{ijk}J_k, \quad (30)$$

$$[P_i, K_j] = -i\delta_{ij}T, \quad (31)$$

$$[K_i, K_j] = -i\varepsilon_{ijk}J_k, \quad (32)$$

$$[T, P_j] = -iK_j, \quad (33)$$

$$[T, K_j] = -iP_j. \quad (34)$$

To verify the above relations, we profit the Bacry-Levey-leblond (BL) method. Based on BL method, the effect of parity and time reversal operators (Π and Θ) on infinitesimal generators is given by [8]:

$$\Pi: \{T \to T, P \to -P, J \to J, K \to -K\}, \quad (35)$$

$$\Theta: \{T \to -T, P \to P, J \to J, K \to -K\}. \quad (36)$$

If we apply parity and time reversal operators on the left side of Eqs (26) to (34), we can guess the right side of these equations.
For example, we consider the left side of Eq. (32), i.e. $[K_i, K_j]$. From relations (35) and (36) we have

$$\Pi[K_i, K_j] = [\Pi K_i, \Pi K_j] = [-K_i, -K_j] = [K_i, K_j],$$

$$\Theta[K_i, K_j] = [\Theta K_i, \Theta K_j] = [-K_i, -K_j] = [K_i, K_j].$$

In other words, the relation $[K_i, K_j]$ does not change under parity and time reversal operators. According to relations (35) and (36), the only generator not changing under parity and time reversal operators is the generator J. This means that the right side of the relation $[K_i, K_j]$ must be a component of the generator J up to a constant factor (see Eq. (32)).

6 Conclusion

In this work, we studied the ten-parameter de Sitter group. Using the Lorentz space-time decomposition, we introduced ten one-parameter subgroups (or equivalently ten algebraic bases) and ten infinitesimal generators on 1+3-de Sitter space-time. Then, we obtained the commutation relations between these infinitesimal generators. Finally, using the Bacry-Levey-Leblond method, we verified the correctness of one of the commutation relations.

Appendix A: Calculation of infinitesimal generator of time translation

From Eq. (15) we obtain:

$$g^{-1} = \Theta^{-1} = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} cosh(\frac{t}{2}) & -sinh(\frac{t}{2}) \\ -sinh(\frac{t}{2}) & cosh(\frac{t}{2}) \end{pmatrix}. \quad (A.1)$$

By choosing ζ as a quaternion, i.e.

$$\zeta = (\zeta_0, \vec{\zeta}) = \zeta_0 + \zeta_1 e_1 + \zeta_2 e_2 + \zeta_3 e_3, \quad (A.2)$$

where $\zeta_0, \zeta_1, \zeta_2, \zeta_3$ are given in Eq. (24), we can write:

$$\chi(\zeta, g) = \chi(\zeta, \Theta) = det(c'\zeta + d') = (cosh\, t - sinh\, t \cos\alpha)^{\frac{1}{2}}, \quad (A.3)$$

$$g^{-1}.\zeta = \Theta^{-1}.\zeta = (a'\zeta + b')(c'\zeta + d')^{-1} = \frac{(cosh\, t \cos\alpha - sinh\, t\, , \vec{\zeta})}{cosh\, t - sinh\, t \cos\alpha} = (\zeta_0', \vec{\zeta}'). \quad (A.4)$$

From the latter equation, we obtain:

$$\zeta_0' = \cos\alpha' = \frac{cosh\, t \cos\alpha - sinh\, t}{cosh\, t - sinh\, t \cos\alpha}, \quad (A.5)$$

$$\zeta_1' = \sin\alpha' \sin\theta' \cos\phi' = \frac{\sin\alpha \sin\theta \cos\phi}{cosh\, t - sinh\, t \cos\alpha}, \quad (A6)$$

$$\zeta_2' = \sin\alpha' \sin\theta' \sin\phi' = \frac{\sin\alpha \sin\theta \sin\phi}{cosh\, t - sinh\, t \cos\alpha}, \quad (A.7)$$

$$\zeta_3' = \sin\alpha' \cos\theta' = \frac{\sin\alpha \cos\theta}{cosh\, t - sinh\, t \cos\alpha}. \quad (A.8)$$

Also, from these four equations and Eq. (A.3) we can derive:

$$\left.\frac{\partial\alpha'}{\partial t}\right|_{t=0} = \sin\alpha, \quad (A.9)$$

$$\left.\frac{\partial\theta'}{\partial t}\right|_{t=0} = 0, \quad (A.10)$$

$$\left.\frac{\partial\phi'}{\partial t}\right|_{t=0} = 0, \quad (A.11)$$

$$\left.\frac{\partial[(\chi(\zeta, g))^{-2\tau}]}{\partial t}\right|_{t=0} = \left.\frac{\partial[(cosh\, t - sinh\, t \cos\alpha)^{-\tau}]}{\partial t}\right|_{t=0} = \tau \cos\alpha. \quad (A.12)$$

Finally, from Eq. (25), we derive the infinitesimal generator of time translation as follows:

$$T = i\left(\tau \cos\alpha + \sin\alpha \frac{\partial}{\partial\alpha}\right).$$

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