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Green's function for a Casimir problem in curved spacetime

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Abstract Obtaining Green's function for problems in curved spacetime is quite important. An important reason is based on the fact that the energy-momentum tensor can be directly calculated from Green's function. The author has found the energy-momentum tensor of the Casimir effect of two plates for a general static spacetime. The process of finding Green's function is discussed here through an introductory manner.

1 Introduction

In curved spacetime, due to the lack of global symmetries, the energy-momentum tensor is very important as a local quantity. Although in some space-times with maximum symmetry such as de Sitter, certain interpretations can be placed on physical quantities, finally, except for some definitions related to the measurement of the number of particles, temperature and the energy-momentum tensor itself, we don't have any other tools to investigate them [1]. Meanwhile, approximate methods in weak spacetimes, as examples close to flat space, are very important. In general, Green's functions are directly, and more easily, related to the energy-momentum tensor through the method of separation of points and reduce the calculations significantly [2]. For weak spacetimes, it is easier to explicitly calculate the Green's function [3] and, in this article, we intend to find an example of it for Casimir plates in a weak spacetime by using elementary methods and without using iterative methods frequently used in the literature. In a way, it can be developed for the strong field as well, which we will mention in the future.

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2 History of Green's function calculation in the subject

The Green's function for a field theory in the presence of a material boundary is less explicitly calculated. Ref. [4] lists some of these efforts. In general, the attempts made to explicitly calculate the energy-momentum tensor in the Casimir effect are very few. On the other hand, many of implicit results have been investigated [5] and until the author's researches, only in weak spacetime.

$$ds^2 = (1+2\frac{g}{c^2}z)dt^2 - dx^2 - dy^2 - dz^2 \tag{1}$$

sufficient investigations had been done. If $\frac{g}{c^2} \ll 1$, this spacetime will be nothing but accelerated Rindler coordinates in flat spacetime [6]. The metric in accelerated Rindler coordinates can be expressed as

$$ds^2 = (1+\frac{g}{c^2}z)^2dt^2 - dx^2 - dy^2 - dz^2 \tag{2}$$

whose first-order approximation gives (1). It can be said that the momentum-energy tensor for the two parallel plates Casimir effect has not been calculated so far in curved spacetime. The author has completely solved such a problem for an arbitrary static space-time up to the second-order approximation, which is currently under review [7]. In this article, we will discuss in more detail the part related to how to find the Green's function for it.

3 Some basic theorems

Although the following theorems are available in the basic textbooks, their application can create ambiguities. In this section, we emphasize their main aspects. The proof of the following theorems are in ref. [8]

Theorem 1:
The equation

$$a_0(x)y^{''}(x)+a_1(x)y^{'}(x)+a_2(x)y(x)=0, \tag{3}$$

on the closed interval $[a,b]$ under boundary conditions

$$\alpha_1 y(a)=\alpha_2 y^{'}(a),\ \ \beta_1 y(b)=\beta_2 y^{'}(b), \tag{4}$$

where α_1, α_2, β_1 and β_2 are fixed numbers. We consider the function $g(x,s)$ to be a Green's function for the above differential equation with the following conditions [5]:

a) The function $g(x,s)$ solves Eq. (1) for $s\neq x$ in the intervals $a\leq x<s$ and $s<x\leq b$.

b) The function $g(x,s)$ in the interval $a\leq s\leq b$ satisfies the boundary conditions (4).

c) $g(x,s)$ is a continuous function with respect to x and s.

d) The function $\frac{\partial g}{\partial x}$ is continuous in the interval $a\leq x\leq b$, but in $x=s$ has a jump as follows:

$$\frac{\partial g}{\partial x}|_{x=s^+}-\frac{\partial g}{\partial x}|_{x=s^-}=-\frac{1}{a_0(x)}, \tag{5}$$

Theorem 2:
Green's function of Eq.(3) under boundary conditions (4) can be found as follows:

$$G(z,\acute{z})=\begin{cases}\dfrac{Y_1(z)Y_2(\acute{z})}{W(\acute{z})p_0(\acute{z})} & z<\acute{z}, \quad (6a)\\[2ex] \dfrac{Y_1(\acute{z})Y_2(z)}{W(\acute{z})p_0(\acute{z})} & \acute{z}<z, \quad (6b)\end{cases}$$

Theorem 3:
The unique answer of the inhomogeneous differential equation

$$a_0(x)y^{''}(x)+a_1(x)y^{'}(x)+a_2(x)y(x)=f(x), \tag{7}$$

in the closed interval [a,b] and under conditions (7) is given by

$$y(x)=-\int_a^b g(x,x^{'})f(x)dx, \tag{8}$$

where $g(x,x^{'})$ is the Green's function of the homogeneous Eq. (8) and $w(s)$ is the Wronskian of two independent solutions $y_1(x)$ and $y_2(x)$ that satisfy the following conditions:

$$\alpha_1 y_2(a)=\alpha_2 y_2^{'}(a),\ \ \ \beta_1 y_1(b)=\beta_2 y_1^{'}(b), \tag{9}$$

Theorem 4:
If the operator L_x is self-adjoint, then in the differential equation

$$L_x g(x,x^{'})=\delta(x-x^{'}), \tag{10}$$

The function $g(x,x^{'})$ is the same Green's function of Theorem 1 and will be symmetric.

Now we use the above theorems to find the Green's function of the Klein-Gordon equation.

4 Green's function for Casimir effect of parallel plates

Let's assume that two parallel neutral conducting plates are placed near each other and the assembly is placed in a gravitational field. The distance between the plates is a and the metric describes the gravitational field as

$$\begin{aligned} ds^2=&(1+2\gamma_0+2\lambda_0 z)dt^2\\ &-(1+2\gamma_1+2\lambda_1 z)(dx^2+dy^2+dz^2), \end{aligned} \tag{11}$$

where $\lambda_0 z,\lambda_1 z\ll 1$ and $\gamma_0,\gamma_1\ll 1$. The quantum field in the space between the plates is a scalar field satisfying the Klein-Gordon equation

$$\Box G_F(x,x^{'})=-\frac{1}{\sqrt{-g}}\delta(x,x^{'}), \tag{12}$$

where

$$\Box=\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu)G_F(x,x^{'}), \tag{13}$$

is the Green's function, which is the Feynman propagator. Looking at (11) and (13), we can write the Klein-Gordon equation as follows:

$$\begin{aligned}&\sqrt{-g}g^{11}\partial_z^2 G_F+\partial_z(\sqrt{-g}g^{11})\partial_z G_F\\ &-\sqrt{-g}g^{11}(k_\perp^2+\frac{g_{11}}{g_{00}}\omega^2)G_F=0,\ \ x\neq x^{'},\end{aligned} \tag{14}$$

The condition for the above equation to be self-adjoint is that the coefficient behind $\partial_z G_F$ i.e. $\partial_z(\sqrt{-g}g^{11})$ is the derivative of the coefficient behind $\partial_z^2 G_F$ i.e. $\sqrt{-g}g^{11}$ which is true for the Klein-Gordon equation. Therefore, the obtained Green's function must be symmetrical, and when checking the correctness of the calculations, the symmetry of the obtained Green's function can always be kept in mind. The relation (14) above can be expanded for the metric (11) and by keeping the second order terms in terms of the parameters $\lambda_0 z,\lambda_1 z\ll 1$ and $\gamma_0,\gamma_1\ll 1$ as

$$y^{''}(z)+(\lambda_0+\lambda_1)y^{'}(z)+(a+bz)y(z)=0, \tag{15}$$

where in it we have

$$\begin{aligned}&a=-2B\omega^2, b=(1-2A)\omega^2-k_\perp^2,\\ &A=\gamma_0-\gamma_1,\ B=\lambda_0-\lambda_1,\ \lambda\equiv\lambda_1+\lambda_0.\end{aligned} \tag{16}$$

We note that we have considered the function G_F as the Fourier transform of the function $g_F(z,z^{'})$. The independent answers of Eq. (15) are the same functions $y_1(x)$ and $y_2(x)$ in Theorem 2. These answers have already been found for another purpose in reference [6] and are as follows:

$$Y(z)=D_0(1-(\frac{\lambda}{2}+\frac{a}{4b})z)\sin(\sqrt{b}z(1+\frac{a}{4b}z)+\Theta_0), \tag{17}$$

By setting Θ_0 through applying the boundary condition (Dirichlet), two independent linear responses can be extracted from it. After doing that, we reach the following results for the Dirichlet boundary condition:

$$y_1(z) = (1-(\frac{\lambda}{2}+\frac{a}{4b})z)\sin\sqrt{b}(z+\frac{a}{4b}z^2) \qquad z<z',$$
$$y_2(z) = (1-(\frac{\lambda}{2}+\frac{a}{4b})z)\sin\sqrt{b}((z-l) + \frac{a}{4b}(z^2-l^2)), \tag{18}$$

As can be seen, according to theorem 3 (see Eq. (9)), the response y_1 at $z=0$ and the response y_2 at $z=l$ become zero. The Wronskian of these two answers is approximated to the second order as follows [4]:

$$W(z') = (1-\lambda z')\sqrt{b}\sin(\sqrt{b}(l+\frac{a}{4b}l^2)). \tag{19}$$

Putting the above items aside and using Theorem 2, we get after some calculation for $z<z'$,

$$g_F(z,z') = \frac{1-\gamma_0-\gamma_1-\lambda(z+z')}{2\sqrt{b}\sin(\sqrt{b}(l+\frac{a}{4b}l^2))}(\cos(\sqrt{b}\alpha)-\cos(\sqrt{b})\beta) + \frac{a}{4\sqrt{b}}(z^2-z'^2+l^2)\sin(\sqrt{b}\beta) - \frac{a}{4\sqrt{b}}(z^2+z'^2-l^2)\sin(\sqrt{b}\alpha),$$

and for $z>z'$ we find

$$g_F(z,z') = \frac{1-\gamma_0-\gamma_1-\lambda(z+z')}{2\sqrt{b}\sin(\sqrt{b}(l+\frac{a}{4b}l^2))}(\cos(\sqrt{b}\alpha) - \cos(\sqrt{b})(\Delta z-l)) + \frac{a}{4\sqrt{b}}(z^2-z'^2-l^2)\sin(\sqrt{b}(\Delta z-l)) - \frac{a}{4\sqrt{b}}(z^2+z'^2-l^2)\sin(\sqrt{b}\alpha)), \tag{20}$$

where in it $\alpha = z+z'-l, \beta = z-z'+l = \Delta z+l$. Acoording to theorem 4 the Green function should be symmetric which can be seen from Eqs. (4) and (20). In fact, if one impose the interchange $z \leftrightarrow z'$ to Eq. (4), then Eq. (20) is obtained and vice versa. The point that should be noted is that Eq. (15) is not self-adjoint. This issue is not important because this equation, which is a first-order approximation of the Klein-Gordon equation, has the same answers as Eq. (14).

5 Conclusion

Using preliminary methods, we found the Green's function for the Casimir effect of two parallel conducting plates in a static field described by (11) and showed that it satisfies the Dirichlet boundary conditions on the scalar field in which the Casimir plates are immersed. The metric (11) is actually the extension of an arbitrary static metric in the space between the plates provided that the distance between the plates (denoted by a) is a small value. This distance due to the Casimir effect is usually small and the mentioned condition is fulfilled. On the other hand, if the gravity field is weak by itself, there is no need for a to be small, and the metric (11) can still be considered as a representative of an arbitrary static field. The obtained Green's function in (4) and (20) is very important for calculating the energy-momentum tensor of the problem.

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