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Explorations on elementary mathematics and physics: Some relations on Bessel functions

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Abstract Relations and identities on special functions are traditionally emerging from investigations on mathematical analysis. However, some special functions such as Bessel appear in various physical problems, a way in which some of their properties can be analyzed. There are plenty of identities on special functions which have been derived in such a way, especially some elementary problems in basic physics that play interesting roles in this. A uniformly charged electrostatic disk is such a problem that it appears unexpectedly in some cases. We study it more here and find some relations on Bessel functions.

1 Introduction

Special functions have an important place in all investigations of applied physics and mathematics. Topological properties, zeros, convergence, asymptotic expansions and many other properties are still open problems of some of these functions. Identities on special functions have also been studied a lot [1, 2]. Naturally, investigations of specific functions are performed in mathematical analysis. However, historically, many mathematical theories have arisen in theoretical physics. Today, in terms of formalism, it is difficult to distinguish between theoretical physics and some branches of new mathematics. For this reason, specific investigations in theoretical physics have become an approach to discovering and creating some mathematical theories, and the scientific community has noticed such closeness more than before. If we want to find simple examples to express such a relationship, we can find many examples of a genetic relationship between theoretical physics and today's mathematics even in preliminary problems. There are many elementary examples and the author presented an example of it in the previous conferences [3]. In this article, following our results in recent conferences [3, 4], we report the main results presented there on the problems of the electric potential of uniformly charged rings and discs. The idea is that we solve the problem with two methods, one of which is inspired by physics (direct method) and the other is the standard method of solving a partial differential equation. By equating the answers in two methods, we reach interesting identities. This operation can show us what mathematical physics is practically.

2 Uniformly charged ring

In this section, we first find the electric potential of the ring using the direct method of integration over elements. In subsection B, we show that the Method of Separation of variables (MSV) is also capable of giving the correct answer.

2.1 Direct method

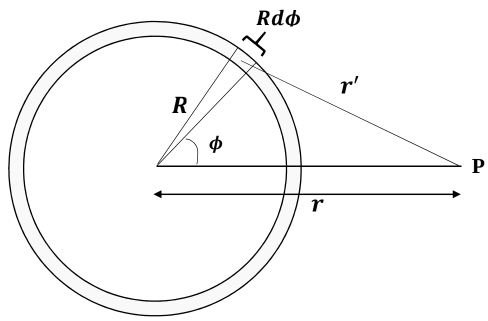

Figure 1: line element to find electric potential at the plane of a uniformly charged ring

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Here we first calculate the electric potential of the uniformly charged ring at a typical point P with the aid of direct integration over elements. The point P might be inside or outside of the ring. Our setting is according to Fig.1. Thus, we see that

$$dV = \frac{kdq}{r'} = k\frac{\lambda_0 R d\phi}{r'}, \tag{1}$$

$$r' = (R^2 + r^2 - 2Rr\cos\phi)^{\frac{1}{2}}, \tag{2}$$

where λ_0 is charge density and $k = \frac{1}{4\pi\varepsilon_0}$.

2.1.1 The case $t = \frac{R}{r} < 1$

In this case, the point P is outside the ring and it founds that

$$\begin{aligned} V_{out} &= \frac{\lambda_0 R}{4\pi\varepsilon_0}\int_0^{2\pi}\frac{d\phi}{(R^2+r^2-2Rr\cos\phi)^{\frac{1}{2}}} \\ &= \frac{\lambda_0}{4\pi\varepsilon_0}\int_0^{2\pi}\frac{t d\phi}{(t^2+1-2t\cos\phi)^{\frac{1}{2}}} \\ &= \frac{\lambda_0}{4\pi\varepsilon_0}\int_0^{2\pi} t\sum_{n=0}^{\infty} t^n P_n(\cos\phi)d\phi \\ &= \frac{\lambda_0}{4\pi\varepsilon_0}\sum_{n=0}^{\infty} t^{n+1}\int_0^{2\pi} P_n(\cos\phi)d\phi, \end{aligned} \tag{3}$$

where we have used the following generating function for Legendre polynomials

$$\frac{1}{(1+t^2-2t\cos\phi)^{\frac{1}{2}}} = \sum_{n=0}^{\infty} t^n P_n(\cos\phi)d\phi. \tag{4}$$

Using the identity (7.221(3) in [2])

$$\int_0^{2\pi} P_{2m}(cos\phi)d\phi = 2\pi\binom{-\frac{1}{2}}{m}^2, \tag{5}$$

the potential is found to be

$$V_{out}(r) = \frac{\lambda_0}{2\varepsilon_0}\sum_{m=0}^{\infty}\binom{-\frac{1}{2}}{m}^2 (\frac{R}{r})^{2m+1}. \tag{6}$$

Note that

$$\binom{x}{m} = \frac{x(x-1)...(x-(m-1))}{m!}, \qquad x\in\mathbb{R}. \tag{7}$$

Therefore

$$\begin{aligned} \binom{-\frac{1}{2}}{m} &= \frac{-\frac{1}{2}\cdot-\frac{3}{2}\cdot\dots\cdot(-\frac{1}{2}-m+1)}{m!} \\ &= (-1)^m\frac{1}{2^m}\frac{1\cdot3\cdot5\cdot\dots\cdot(2m-1)}{m!} \\ &= \frac{(-1)^m}{2^m m!}\cdot\frac{1\cdot2\cdot3\cdot4\cdot\dots\cdot(2m)}{2\cdot4\cdot\dots\cdot2m} \\ &= \frac{(-1)^m(2m)!}{2^{2m}(m!)^2}. \end{aligned} \tag{8}$$

2.1.2 The case $r < R$

The process as previous section shows that

$$\begin{aligned} V_{in} &= \frac{\lambda_0 R}{4\pi\varepsilon_0}\int_0^{2\pi}\frac{d\phi}{R\left(1+\frac{r^2}{R^2}-2\frac{r}{R}\cos\phi\right)^{\frac{1}{2}}} \\ &= \frac{\lambda_0}{4\pi\varepsilon_0}\int_0^{2\pi} d\phi\sum_{n=0}^{\infty}\left(\frac{r}{R}\right)^n P_n(\cos\phi) \\ &= \frac{\lambda_0}{2\varepsilon_0}\sum_{m=0}^{\infty}\binom{-\frac{1}{2}}{m}^2\left(\frac{r}{R}\right)^{2m}. \end{aligned} \tag{9}$$

where we have used the relation [2]

$$0 = \int_0^{2\pi} P_{2m+1}(\cos\phi)d\phi, \tag{10}$$

and Eq. (5) again.

2.2 Method of Separation of variables(MSV)

We saw in section II that the MSV did not give the correct answer for the electric potential in the plan of a *uniformly* charged ring. Now we show that the MSV gives the right answer. The idea consists of finding a solution to the three dimensional Laplace equation using the MSV and reducing the final answer to two dimensions by putting $\theta \to \pi/2$. Our setting is according to Fig. 2.

2.2.1 The case $r > R$

Using the MSV, it can be shown [?] that the 3-dimensional Laplace equation ends up with the following solution in the region $r > R$ for the ring depicted in Fig. 2.

$$V_{out}(r,\theta,\phi) = \sum_{m=0}^{\infty} A_m r^{-m-1} P_m(\cos\theta). \tag{11}$$

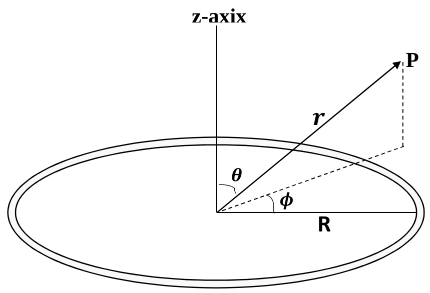

Figure 2: finding electric potential of a uniformly charged disk in a three dimensional coordinate system

Note that we have azimuthal symmetry around $z-axis$ hence (11) is independent of ϕ. Therefor, at a point on the $z-axis$, i.e. $\theta = 0 \; or \; r = z$, we have

$$V_{out}(r,0) = \sum_{m=0}^{\infty} A_m r^{-m-1} P_m(0)$$
$$\sum_{m=0}^{\infty} A_m z^{-m-1} \equiv V_{out}(z). \tag{12}$$

in which we have used $P_m(0) = 1$. On the other hand, we see from Figure 2 that the electric potential at a some point on the $z-axis$ is simply

$$V_{out}(z) = k\frac{Q}{(z^2+R^2)^{\frac{1}{2}}} = \frac{1}{4\pi\varepsilon_0}\frac{2\pi R\lambda_0}{(z^2+R^2)^{\frac{1}{2}}}$$
$$= \frac{\lambda_0 R}{2\varepsilon_0}\frac{1}{z}\left(1+\left(\frac{R}{z}\right)^2\right)^{-\frac{1}{2}}. \tag{13}$$

Using extended binomial theorem we expand Eq. (13) as follows:

$$(a+b)^x = \sum_{n=0}^{\infty}\binom{x}{n} a^n b^{x-n},$$
$$\rightarrow (1+(\frac{R}{z})^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}((\frac{R}{z})^2)^n(1)^{-\frac{1}{2}-n}$$
$$= \sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}(\frac{R}{z})^{2n}. \tag{14}$$

So we find

$$V(z)_{out} = \frac{\lambda_0}{2\varepsilon_0}\sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}(\frac{R}{z})^{2n+1}. \tag{15}$$

A comparison between Eq. (12) and Eq.(15) shows

$$A_{2n} = \binom{-\frac{1}{2}}{n}\frac{\lambda_0}{2\varepsilon_0}R^{2n+1}, \qquad A_{2n+1} = 0. \tag{16}$$

Thus, Eq. (11) reads

$$V_{out}(r,\theta) = \frac{\lambda_0}{2\varepsilon_0}\sum_{n=0}^{\infty}.\binom{-\frac{1}{2}}{n}\frac{R^{2n+1}}{r^{2n+1}}P_{2n}(\cos\theta). \tag{17}$$

Finally, the electric potential in the plane of a uniformly charged ring in the region $r > R$ is found by setting $\theta = \frac{\pi}{2}$ in Eq. (17) and the result is

$$V_{out}(r,\frac{\pi}{2}) = V_{out}(r) = \frac{\lambda_0}{2\varepsilon_0}\sum_{n=0}^{\infty},\binom{-\frac{1}{2}}{n}^2\frac{R^{2n+1}}{r^{2n+1}}, \tag{18}$$

in which use is made of $P_{2n}(0) = 1$. This is exactly what we found in Eq. (6) using the direct method.

2.2.2 The case $r < R$

A process similar to previous section can be implemented here. For the region $R < r$, the MSV will results in [**?**]

$$V_{in,ring,MSV}(r,\theta) = \sum_{n=0}^{\infty} B_n r^n P_n(\cos\theta), \tag{19}$$

To find B_n we seek again the potential at $z-axix$. From Eq. (18) it should be

$$V_{in}(r,0) = \sum_{n=0}^{\infty} B_n r^n P_n(1) = \sum_{m=0}^{\infty} B_{2m} z^{2m} = V_{in}(z). \tag{20}$$

By rearranging Eq. (12) we have

$$V_{in}(z) = \frac{\lambda_0 R}{2\varepsilon_0}\frac{1}{\sqrt{R^2+z^2}} = \frac{\lambda_0}{2\varepsilon_0}\left(1+\left(\frac{z}{R}\right)^2\right)^{-\frac{1}{2}}$$
$$= \frac{\lambda_0}{2\varepsilon_0}\sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}\left(\frac{z}{R}\right)^{2n}, \tag{21}$$

in which we have used the extended binomial expansion formulae again. So, by comparing Eqs. (19) and (20) we see that

$$B_{2m} = \frac{\lambda_0}{2\varepsilon_0}\binom{-\frac{1}{2}}{m}\frac{1}{R^{2m}}, \tag{22}$$

and

$$V_{in}(r,\theta) = \frac{\lambda_0}{2\varepsilon_0} \sum_{m=0}^{\infty} \binom{-\frac{1}{2}}{m} (\frac{r}{R})^{2m} P_{2m}(cos\theta). \quad (23)$$

Finally, the electric potential in the plane of a uniformly charged ring in the region $r < R$ is found by setting $\theta = \frac{\pi}{2}$ in Eq. (22) and the result is

$$V_{in}(r) = \frac{\lambda_0}{2\varepsilon_0} \sum_{m=0}^{\infty} \binom{-\frac{1}{2}}{m}^2 (\frac{r}{R})^{2m}. \quad (24)$$

This is the previous result found in (9) using the direct method.

3 Uniformly charged disk

Consider a uniformly charged disk with the following set up

3.1 Direct method using Coulomb's law

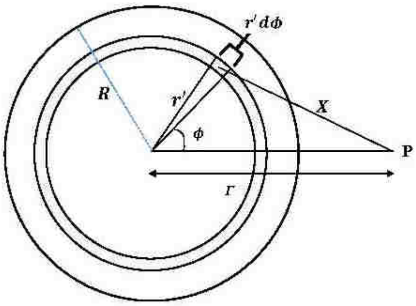

Figure 3: Line element to find electric potential at the plane of a uniformly charged disk

In this section, we first find the electric potential of the ring using the direct method of integration over elements, i.e. using Coulomb's law. In the next subsection, we show that the method of separation of variables is used to solve the same problem. Note that we will use the cylindrical coordinates from now on. The resulting electric potential at the arbitrary point P is as follows:

$$\begin{aligned} V_{out}(r) &= \frac{1}{4\pi\varepsilon_0} \int \frac{dq}{|\mathbf{X}|} \\ &= \frac{\sigma_0}{4\pi\varepsilon_0} \int \frac{r' dr' d\phi'}{(r'^2 + r^2 - 2rr' \cos\phi')^{\frac{1}{2}}}, \end{aligned} \quad (25)$$

where (r', ϕ') describes a surface element on the disk. With the Taylor expansion of the integral and through several steps of easy integration, the potential of a point on the disk at a distance $r < R$ from the center of the disk is equal to [7,8]:

$$V_{out}(r) = \frac{\sigma_0 R}{2\varepsilon_0} \sum_{m=0}^{\infty} \frac{1}{2m+2} \binom{-\frac{1}{2}}{m}^2 (\frac{R}{r})^{2m+1}, \quad (26)$$

3.2 Solution to Laplace equation

In this section, we first find the solution to Laplace's equation at an arbitrary point outside the disk (and outside its plane) with the help of cylindrical coordinates (ρ, ϕ, z) and then find the solution by setting $z = 0$, i.e. we convert the solution to two-dimensional polar coordinates. In this way, we find the potential value on the disc plane at $r < R$. Since there is an electric charge in the region $r < R$ on the disk, we should solve Poisson's equation instead of Laplace's. We are not worried that because the potential is a continuous function and at the boundary of the electric charge (here on the disk) the solution value of Poisson's and Laplace's equation should be the same. It should be noted that using this property is only possible in two dimensions where we have surface charge. The potential at an arbitrary point in cylindrical coordinates is assumed as $V(\rho, \phi, z)$. The Poisson's equation in cylindrical coordinates is given by

$$\frac{\partial^2}{\partial^2 \rho} V + \frac{\partial}{\partial \rho} V + \frac{1}{\rho^2} \frac{\partial^2}{\partial^2 \phi^2} V + \frac{\partial^2}{\partial^2 z} V = 0. \quad (27)$$

Since the solution should be independent of ϕ we assume the separation of the variables of the form

$$V(r,\theta) = R(\rho) Z(z). \quad (28)$$

Substituting Eq. (28) back into Eq. (27) we arrive to

$$\frac{1}{\rho} \frac{d}{d\rho} \left(\rho \frac{dR}{d\rho} \right) + k^2 R(\rho) = 0, \quad (29a)$$

$$\frac{d^2}{d\rho^2} Z(z) - k^2 Z(z) = 0, \quad (29b)$$

where k is some constant. After solving Eqs. (29a) and (29b) we find

$$Z(z) = e^{-kz} \quad \text{or} \quad e^{kz}, \quad (30a)$$

$$R(\rho) = J_0(k\rho) \quad \text{or} \quad R_0(k\rho), \quad (30a)$$

in which $J_0(k\rho)$ and $R_0(k\rho)$ are Bessel functions of the first and second kind. We know that our problem is internal, hence, e^{kz} and $R_0(k\rho)$ should be discarded. The 0th-order Bessel function $Y_0(k\rho)$ is not present in the answer because the answer will be divergent at $\rho = 0$. Thus, the final solution is given by

$$V(\rho,z) = \int_0^\infty B(k)J_0(k\rho)e^{-kz}dk. \tag{31}$$

We do not have any constraints or boundary conditions on the parameter k and this is why we have integrated it in Eq. (31). Note that Eq. (7) actually states that the potential $V(\rho,z)$ is a Laplace transform:

$$V(\rho,z) = \mathscr{L}(B(k)J_0(\rho k))|_{s=z}. \tag{32}$$

To find $B(k)$, it is sufficient to try a particular case of answers in Eq. (7) or Eq. (8). If we consider the field point to be on the z axis, we know from basic physics that the potential is given by

$$V(0,z) = \frac{\sigma_0}{2\varepsilon_0}(\sqrt{a^2+z^2}-z). \tag{33}$$

On the other hand, from Eq. (31) we have

$$V(0,z) = \int_0^\infty B(k)e^{-kz}dk, \tag{34}$$

in which we have used $J_0(0) = 1$. Again, Eq. (34) shows that $V(0,z)$ is also the Laplace transform of $B(k)$ with respect to the variable k, i.e.

$$B(k) = \mathscr{L}^{-1}(\frac{\sigma_0}{2\varepsilon_0}(\sqrt{a^2+s^2}-s)). \tag{35}$$

From relation 12.13(116) in [2] we have

$$\mathscr{L}\left(\frac{J_k(ax)}{x}\right) = \frac{1}{ka^k}\left[\sqrt{a^2+s^2}-s\right]^k, \quad k > -1, \quad \mathrm{Re}\, s > |\mathrm{Im}\, a|. \tag{36}$$

from which and Eq. (35) we see that

$$B(k) = \frac{\sigma_0 a}{2\varepsilon_0}\frac{J_1(ak)}{k}. \tag{37}$$

Therefore, the potential relation, Eq. (31) will be as follows:

$$V(\rho,z) = \frac{\sigma_0 a}{2\varepsilon_0}\int_0^\infty \frac{J_1(ak)}{k}J_0(k\rho)e^{-kz}dk. \tag{38}$$

This is also a Laplace transform of the form

$$V(\rho,z) = \frac{\sigma_0 a}{2\varepsilon_0}\mathscr{L}(\frac{J_1(ak)}{k}J_0(k\rho)). \tag{39}$$

Now we investigate some well-known cases.

3.2.1 Electric potential at the center of the disk

The potential at the center of the disc is as follows [7]:

$$V(0,0) = \frac{\sigma_0 a}{2\varepsilon_0}. \tag{40}$$

By Eqs. (37) and (35) we find

$$\int_0^\infty \frac{J_1(u)}{u}du = 1. \tag{41}$$

3.2.2 Electric potential at the edge of the disk

The potential due to a uniformly charged disk at its edge with a simple integration is as follows [5]:

$$V(0,0) = \frac{\sigma_0 a}{\pi\varepsilon_0}. \tag{42}$$

In addition, by putting $\rho = z = 0$ in Eq. (40) we find

$$V(a,o) = \frac{\sigma_0 a}{2\varepsilon_0}\int_0^\infty \frac{J_1(ak)}{k}J_0(k\rho)dk. \tag{43}$$

Using Eqs. (42) and (43) we find

$$\int_0^\infty \frac{J_1(ak)}{k}J_0(ak)dk = \frac{2}{\pi}. \tag{44}$$

3.2.3 Electric potential at $z = 0$ and $\rho < a$ on the disk

In this case, using Eq. (38) we have

$$V(\rho,0) = \frac{\sigma_0 a}{2\varepsilon_0}\int_0^\infty \frac{J_1(ak)}{k}J_0(k\rho)dk. \tag{45}$$

Eqs. (45) and (26) should be equal. Thus we find

$$\int_0^\infty \frac{J_1(ak)}{k}J_0(k\rho)dk = -\Sigma_{m=0}^\infty \tfrac{1}{2m-1}\binom{-\frac{1}{2}}{m}^2(\tfrac{\rho}{a})^{2m}. \tag{46}$$

4 Conclusion

In this article, the usual methods for special mathematical physics functions were used preliminarily, and the relationships that may be difficult to obtain with pure mathematical methods were obtained with less calculation. Eqs. (41), (44) and (46) are the result of solving a problem by two methods and equating the answers. To find similar relations [7] for Bessel functions of the second type, a problem in the region $r > a$ can be used. In general, the charged disc has been the source of many interesting relationships in mathematics and physics, some of which we have presented in the relevant literature and the references section.

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