TTMP vol 1(4), 2024
doi: 10.30511/TTMP.2024.2039661.1039

Transactions in Theoretical and Mathematical Physics

# Einstein manifolds from a mathematical and physical point of view

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Received: 26 August 2024 / Accepted: 17 October 2024 / Published: 03 November 2024

**Abstract** In this paper, we introduce Einstein's manifold theory from a mathematical and physical viewpoint. We present some obstructions for a manifold to be equipped with the structure of an Einstein manifold. We discuss the Hamiltonian approach to Einstein manifold theory and the chirality of the Lie algebra of an Einstein Lie group. We also discuss Einstein's structure on non-compact manifolds and present some partial results on Einstein's structure on the tangent bundle.

## 1 Introduction

The vacuum Einstein field equation is the main motivation for considering Einstein manifolds. The Einstein field equation is

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}, \tag{1}$$

where $G_{\mu\nu}$ is the Einstein tensor, $g_{\mu\nu}$ is the metric tensor, $T_{\mu\nu}$ is the stress-energy tensor, $\kappa$ is the Einstein gravitational constant, and $\Lambda$ is the cosmological constant. In the vacuum case, the equation becomes:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 0. \tag{2}$$

In the mathematical setting, the Einstein tensor is the Ricci tensor of a certain Riemannian metric, motivating the definition of an Einstein manifold, a manifold whose Ricci tensor is a constant multiple of the metric tenso, namely $Ric = \kappa g$ where $g$ is the metric tensor and $Ric$ is the Ricci tensor. A particular case is when $\kappa = 0$, which defines a Ricci-flat manifold.

The scalar curvature of an Einstein manifold is constant. The theory is deeply involved with problems in Riemannian geometry that concern objects with constant Ricci curvature.

## 2 Preliminaries

In this section, we provide the prerequisites and notations necessary to introduce the theory of Einstein manifolds. Let $(M, g)$ be a Riemannian manifold with the Levi-Civita connection $\nabla$. The curvature tensor is defined as

$$R(X,Y)Z = (\nabla_X \circ \nabla_Y - \nabla_Y \circ \nabla_X - \nabla_{[X,Y]})Z. \tag{3}$$

The Ricci tensor $R$ is a 2-linear map $Ric$ defined as

$$Ric(X,Z) = \text{trace}(Y \mapsto R(X,Y)Z), \tag{4}$$

which is a symmetric tensor.

**Proposition 1** The scalar curvature is the trace of $Ric$ with respect to the metric tensor $g$.

*Proof* Let $(M, g)$ be a $n$ dimensional Einstein manifold. The scalar curvature is $scal = g^{ij} R_{ij}$ where $R_{ij}$ are Ricci coefficients and $g^{ij}$ are coefficient of inverse metric $g^{-1}$. From the equation $Ric = \lambda g$ we get $scalar(M) = n\lambda$

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[a]This research was in part supported by a grant from IPM (1403550049).

**Proposition 2** Let $(M, g)$ be a Riemannian manifold with constant sectional curvature then $M$ is an Einstein manifold.

*Proof* Every manifold of constant sectional curvature is locally isometric to either Euclidean space, the hyperbolic space or the round sphere. The property of being Einstein is a local isometric property. So every manifold of constant curvature is an Einstein manifold since each of the above 3 mentioned spaces is an Einstein manifold.

**Remark** There are Einstein manifolds whose sectional curvature is not constant. The complex projective space $\mathbb{C}P^n$ is an example of Einstein manifolds whose sectional curvature is not constant. However, the holomorphic sectional curvature of the complex projective space is constant.

For a compact manifold $M$ the Euler characteristic $\chi(M)$ can be defined n several equivalent forms; First definition is based on the self ntersection number of $M$ as a submanifold of $TM$.

An equivalent definition is in terms of the cell structure of a manifold which admit a $CW$ complex structure. Note that every compact manifold is homotopic equivalent to a $CW$ complex. For any such manifold the Euler characteristic is defined as $k_0 - k_1 + k_2 - k - 3 + \ldots \pm k_n$ where $k_i$ is the number of cells of dimension $i$. A more general definition is in terms of Betti numbers. Namely the Euler characteristic $\chi(M) = \sum_i (-1)^i b_i$ where $b_i$ is the rank of the ith singular homology group $H_i(M)$. All of these equivalent definitions are indeed equivalent to the Euler characteristic of a triangulated manifold. From a dynamical point of view the definition we mentioned in terms of self-intersection number is identical to the sum of the index of singularities of a generic vector field on $M$

The signature of a Riemannian manifold $M$ of dimension $4k$ is defined as follows: We define a symmetric 2-form $S : H^{2k}(M) \times H^{2k}(M) \to \mathbb{R}$ via $(\alpha, \beta) \to \int_M \alpha \wedge \beta$. This symmetric form can be represented by a symmetric matrix $M$. Let $n_+, n_-$ be the number of positive and negative eigenvalues of $M$ respectively. Then the signature is defined as $\sigma(M) = n_+ - n_-$. The signature is a bordism invariant namely if a compact manifold is the boundary of a compact orientable manifold $N$ the $Sign(M) = 0$ so we get that the signature of $S^{4k} = 0$. Moreover, we conclude that the projective space $\mathbb{C}P^{2k}$ can not be the boundary of any manifold since the signature of all $\mathbb{C}P^{2k}$ is equal to 1.

**Proposition** The Euler characteristic $\chi(M)$ and the signature $\sigma(M)$ have the same parity namely $\chi(M) \equiv \sigma(M) \pmod 2$

**Remark** A common property of Euler characteristic and signature is that both quantities are multiplicative.

**A Fredholm index interpretation of signature of manifolds**. The famous Hirzebruch signature theorem presents an index theoretical interpretation for the signature of a smooth manifold. To formulate this interpretation we need to define the signature operator acting on a certain space of differential form. Let $(M, g)$ be a Riemannian manifold of dimension $2l$ with Hodge star operator $\star$ and the Dirac operator $d + d^*$. For every given $p$ we consider the exterior differential $d : \omega^p(M) \to \omega^{p+1}(M)$ and $d^* : \Omega^{p+1}(M) \to \omega^p(M)$. To these data we assign an involution $\tau(\alpha) = i^{p(p-1)+l}\alpha$. This involution anti commutes with the direct operator $d+d^*$. the involution $\tau$ has two eigenspaces $\Omega_+, \Omega_-$ corresponds to eigenvalues $\pm 1$ of the involution $\tau$. This implies that the Dirac operator $d+d^*$ maps $\Omega_+$ to $\Omega_-$ and vice versa. So we can represent the Dirac operator in the matrix form $d+d^* = \begin{pmatrix} 0 & D_+ \\ D_- & 0 \end{pmatrix}$ where $D : \Omega_+ \to \Omega_-$.

**Hirzebruch Signature Theorem** $\sigma(M) = Ind(D)$. A Lie algebra $L$ is called a chiral algebra if every lie algebra automorphism is an orientation preserving linear map.

**Remark** We should not confuse this terminology with the similar name terminology chiral algebra in algebraic geometry and $D$ module theory mentioned in [2] and [3].

Example of a chiral Lie algebra is $\mathbb{R}^3$ with the cross-product operation. In the next section we address the existence of an Einstein structure on Lie groups and pose the question of possible relation to chirality of the associated lie algebra. In the paper, we would focus on possible relation of chirality of Lie algebras and Einstein structure on Lie groups. But the classification of all finite dimensional chiral algebra is another problem which can be studied independently.

## 3 Einstein manifolds

In this section we provide some obstructions on a manifold to have an Einstein structure.

**Hitchin Thorpe Inequality** Every compact 4 dimensional Einstein manifold satisfies the Einstein Thorpe inequality $\chi(M) \geq \frac{3}{2}|\sigma(M)|$.

*Proof* The idea of proof is based on computation of the Euler characteristic and signature of manifold in terms of entries of the matrix of the curvature operator. The curvature operator $T$ defined on $\Lambda^2 TM$ satisfies $< T(x \wedge y), z \wedge w >= R(x, y, z, w)$. Existence of Einstein metrics enable use to represent the curvature operator in the $6 \times 6$ matrix $I_2 \otimes A + J_2 \otimes B$ where $I_2, J_2$ are the identity and almost complex 2- matrix respectively and $A, B$ are diagonal matrices $A = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix}$ and $B = \begin{pmatrix} b_1 & 0 & 0 \\ 0 & b_2 & 0 \\ 0 & 0 & b_3 \end{pmatrix}$. Then as a consequence of Chern weil theory we have

$$\chi(M) = \frac{1}{4\pi^2} \int_M \left( \sum a_i^2 + \sum b_i^2 \right), \tag{5}$$

and

$$\sigma(M) = \frac{1}{3\pi^2} \int_M (a_1 b_1 + a_2 b_2 + a_3 b_3). \tag{6}$$

This obviously implies that $\chi(M) \geq 3/2|\sigma(M)|$ because $\sum a_i^2 + \sum b_i^2 = \sum (a_i - b_i)^2 + 2 \sum a_i b_i$.

**Remark** Note that the Hitchin-Thorpe inequality is a necessary condition for the existence of an Einstein structure on a manifold. But it is not a sufficient condition condition. For a counterexample of manifolds which satisfy this inequality but does not admit any Einstein structure see [5] and [7].

**Example** Every compact manifold whose Euler characteristic is negative can not be an Einstein manifold.

## 4 Summery and discussion

Einstein manifolds can be seen as critical points of the Hilbert functionals. For every Riemannian manifold $(M, g)$ we define the Hilbert functional $\int_M scaldv - g$ the integral of scalar curvature with respect to the volume form associated to the metric $g$. This functional is defined on the space of all Riemannian metrics on a given manifold. The Hilbert functional is invariant under the action of all diffeomorphisms. So this functional can be studied in the equivalent class of all Riemannian metrics up to isometric. Recall that two metrics on a manifold are equivalent if there is a diffeomorphism on the manifold which pulls back one metric to another one. Consideration of such a functional generates some researches on study some other kind of Hilbert like functionals. For example in [1] it is proved that the critical points of the functional $\int_m |Ric_g|^2 vol_g$ are flat metric provided they have non negative scalar curvature.

The Ricci flow structure of an Einstein manifold has a simple formulation. Recall that a Ricci flow is a 1 parameter of Riemannian metrics $g_t$ which satisfy $\frac{d}{dt} g_t = -2Ricg_t$. These flows are very important framework in Perleman's approach to the Poincare conjecture see [6]. Now if $(M, g)$ is an Einstein manifold then the Ricci flow can be produced as $g(t) = (1 - 2\lambda t)g$ for $|t| < \frac{1}{2|\lambda|}$ where $\lambda$ is the Einstein constant with $Ric_g = \lambda g$.

The instantons interpretation of 4 dimensional Einstein manifolds are presented in [8]. The Einstein manifolds can be viewed as a problem in Hamiltonian dynamics. We explain how one can investigate existence of an Einstein structure via Hamiltonian vector field:

Let $(M, g)$ be a Riemannian manifold. So $TM$ has a natural structure of a symplectic manifold. The zero section is denoted by $Z$. We define a Hamiltonian on $T^0 M = TM \setminus Z$, via

$$H = \frac{Ric(V, V)}{|V|^2}. \tag{7}$$

For Einstein manifold this produce the trivial Hamiltonian dynamics(All points are singularity). But what about general case? What can be said about the critical points of this Hamiltonian?

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