



Investigation of Dense Coding with One of the Quasi-Bell States

Ardeshir Rabeie^{1,a}, Ardalan Fatahizadeh^{1,b}

¹Physics Department, Faculty of Science, Razi University, Kermanshah, Iran

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Abstract In this paper, we study the qubit-photon quantum logic utilizing a strong off-resonant coupling of a qubit and cavity. We study dense coding with one of the quasi-Bell states and propose an experimental scheme for dense coding with quasi-Bell measurement.

1 Introduction

Although dense coding is an important quantum communication protocol to communicate two classical bits of information by only transmitting a Bell state between the sender (Alice) and the receiver (Bob) [1–6], in this paper, we perform this protocol by sharing one of the quasi-Bell states. The quasi-Bell states are defined as:

$$|\phi_1\rangle = \frac{1}{\sqrt{N_1}} (|a\rangle| - a\rangle + | - a\rangle|a\rangle), \quad (1)$$

$$|\phi_2\rangle = \frac{1}{\sqrt{N_2}} (|a\rangle| - a\rangle - | - a\rangle|a\rangle), \quad (2)$$

$$|\phi_3\rangle = \frac{1}{\sqrt{N_1}} (|a\rangle|a\rangle + | - a\rangle| - a\rangle), \quad (3)$$

$$|\phi_4\rangle = \frac{1}{\sqrt{N_2}} (|a\rangle|a\rangle - | - a\rangle| - a\rangle). \quad (4)$$

where $N_1 = 2(1 + e^{-4a^2})$, $N_2 = 2(1 - e^{-4a^2})$, and $|\pm a\rangle$ are the standard coherent states [7–10]. The standard coherent

state can be represented in terms of photon number states [11]:

$$|a\rangle = \exp\left(-\frac{|\alpha|^2}{2}\right) \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle, \quad (5)$$

where a is a complex amplitude. In quantum optics, the electromagnetic field can be decomposed into modes, where the dynamics of each mode in free space is equivalent to the dynamics of a simple harmonic oscillator, with n representing the number of photons in the given mode. The coherent state is a state for the corresponding electromagnetic field mode.

In this work, we first present an interesting method to create the quasi-Bell state $|\phi_2\rangle$, then we study the dense coding by sharing this state¹.

2 Creation of the Quasi-Bell State $|\phi_2\rangle$

The qubit-photon quantum logic via strong off-resonant coupling between a qubit and cavity is modeled by the dispersive Hamiltonian [12]:

$$\frac{H}{\hbar} = \omega_q |e\rangle\langle e| + \omega_s a^\dagger a + \chi_{qs} a^\dagger a |e\rangle\langle e|, \quad (6)$$

where $\omega_{q,s}$ are the qubit and cavity, $|e\rangle$ is the excited state of the qubit, transition frequencies, a (a^\dagger) is the lowering (raising) ladder operator of the cavity resonator and, and χ_{qs} is the dispersive interaction between these modes. This interaction generates a state-dependent shift in either the qubit or cavity transition frequency. We use this conditional frequency shift to

^aEmail: rabeie@razi.ac.ir

^bEmail: ardalan_fatahizadeh@yahoo.com

¹Under conditions, the quasi-Bell state $|\phi_2\rangle$ has maximum entanglement and this is very important for sharing

produce qubit-photon entanglement with two operations: conditional cavity phase shifts and conditional qubit rotations. The conditional cavity phase shift may be represented as

$$C_\Phi = e^{i\Phi' a^\dagger a |e\rangle\langle e|} = I \otimes |0\rangle\langle 0| + e^{i\Phi' a^\dagger a} \otimes |e\rangle\langle e|, \quad (7)$$

where Φ' is the conditional phase shift induced on the cavity state and $|0\rangle$ is the ground state of the qubit. This conditional phase emerges from the free evolution of the dispersive Hamiltonian for a time τ , where $\Phi = \chi_{qs}\tau$. The conditional qubit rotation is a rotation on the qubit state for the photon number of the cavity state.

Because the qubit transition frequency strongly depends on the photon number, we can choose the specific transition in the cavity Fock state. The rotation on the qubit state for an n -photon Fock state can be ideally described as follows [12]:

$$R_{\vec{n}',\theta}^n(\theta, \eta) = |n\rangle\langle n| \otimes R_{\vec{n}',\theta} + I \sum_{m \neq n} |m\rangle\langle m| I, \quad (8)$$

where $R_n(\theta, \eta) = e^{-i\frac{\theta}{2}(\vec{\sigma} \cdot \vec{n}')} is a qubit rotation around the vector \vec{n}' with rotation angle θ , and $\vec{\sigma} = \sigma_x \hat{i} + \sigma_y \hat{j} + \sigma_z \hat{k}$, where σ_x, σ_y , and σ_z are the Pauli matrices.$

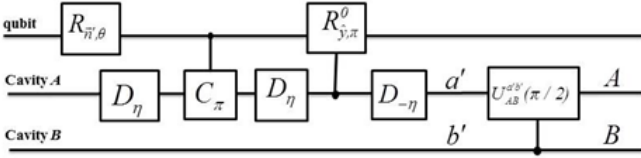


Fig. 1: A plan to create the quasi-Bell state $|\phi_2\rangle$ in several steps: qubit state preparation using a single qubit rotation $R_{\vec{n}',\theta}$; mapping the qubit to the cavity state A using conditional operations C_π and $R_{\vec{n}',\theta}^0$ along with cavity displacements D_η ; and mapping the cavity state A to the cavity state B using a 50:50 beam splitter $U_{AB}^{a'b'}(\pi/2)$.

We suggest an experimental setup as shown in Fig. 1 to create the quasi-Bell state $|\phi_2\rangle$. We start with an unentangled qubit/cavity state: $|\zeta_0\rangle = \eta \otimes (|0\rangle - |e\rangle)$, (ignore the normalization), where $|\eta\rangle$ is a coherent state. Performing a conditional cavity π -phase shift on the initialized state creates an entangled qubit-cavity state A $|\zeta_1\rangle = C_\pi |\zeta_0\rangle = |\eta, 0\rangle - |-\eta, e\rangle$. With acting cavity displacements D_η on this state, we have $|\zeta_2\rangle = D_\eta |\zeta_1\rangle = |2\eta, 0\rangle - |0, e\rangle$. We here apply a qubit π -rotation conditional on the cavity vacuum state $|0\rangle$, which produces the unentangled state: $|\zeta_3\rangle \approx R_{y,\pi}^0 |\zeta_2\rangle = (|2\eta\rangle - |0\rangle) \otimes |0\rangle$. With acting cavity displacements $D_{-\eta}$ on this state, we have $|\zeta_4\rangle = D_{-\eta} |\zeta_3\rangle = (|\eta\rangle - |-\eta\rangle) \otimes |0\rangle$. Finally, suppose

that each mode of the cavity A and cavity B is incident on the beam splitter. After passing through the beam splitter, the quasi-Bell state becomes:

$$|\zeta_5\rangle_{AB} = U_{AB}^{a'b'}(\phi') (|\eta\rangle_{a'} - |-\eta\rangle_{a'}) \otimes |0\rangle_{b'} \\ = \left(|\eta/\sqrt{2}\rangle_A |-\eta/\sqrt{2}\rangle_B - |-\eta/\sqrt{2}\rangle_A |\eta/\sqrt{2}\rangle_B \right), \quad (9)$$

where the 50:50 beam splitter for two field modes is represented as $U_{AB}^{a'b'}(\phi') = e^{-\frac{\phi'}{2}(a_a^\dagger a_{b'} - a_{a'} a_b^\dagger)}$ where a' and b' are the two field modes entering the beam splitter, A and B are the two field modes passing through the beam splitter, and ϕ is related to the transmissivity as $\theta = \cos^2(\phi/2)$. For $\alpha = \eta/\sqrt{2}$, the state $|\zeta_5\rangle_{AB}$ reduces to the quasi-Bell state $|\phi_2\rangle_{AB}$.

3 Dense Coding Process

Let us assume that the quasi-Bell state $|\phi_2\rangle_{AB}$ has been shared between Alice and Bob. Alice wants to send Bob a binary number $x \in \{00, 01, 10, 11\}$. She chooses one of the following quasi-Pauli gates:

$$I = |-\alpha\rangle\langle -\alpha| + |\alpha\rangle\langle \alpha|, \quad (10)$$

$$X = |\alpha\rangle\langle -\alpha| + |-\alpha\rangle\langle \alpha|, \quad (11)$$

$$Y = -i|\alpha\rangle\langle -\alpha| + i|-\alpha\rangle\langle \alpha|, \quad (12)$$

$$Z = |\alpha\rangle\langle \alpha| - |-\alpha\rangle\langle -\alpha|. \quad (13)$$

According to x , Alice applies the transformation on mode A . Applying the transformation for mode A means she applies an identity transformation for mode B , which Bob keeps with him. By applying these conditions, the shared mode changes as follows:

$$00 \rightarrow I \otimes I \rightarrow |\phi_2\rangle_{AB}, \quad (14)$$

$$01 \rightarrow X \otimes I \rightarrow |\phi_4\rangle_{AB}, \quad (15)$$

$$10 \rightarrow Y \otimes I \rightarrow |\phi_3\rangle_{AB}, \quad (16)$$

$$11 \rightarrow Z \otimes I \rightarrow |\phi_1\rangle_{AB}. \quad (17)$$

When α is reasonably large, after performing one of the operations described above, Alice can send her mode A to Bob

using a quantum network through some conventional physical medium. Now the two modes have been sent to Bob, and he can find the transmitted state using the quasi-Bell measurement [13]. Additionally, according to the contract between himself and Alice, Bob knows which pair of binary numbers $x \in \{00, 01, 10, 11\}$ has been sent by Alice.

We suggest an experimental setup, as shown in Fig. 2, to discriminate quasi-Bell states when α is reasonably large. To discriminate between the quasi-Bell states, we use two photodetectors and a 50:50 beam splitter for modes A and B . After passing through the beam splitter, the quasi-Bell states become:

$$|\Phi_1\rangle_{AB} \rightarrow \frac{1}{\sqrt{N_1}}|0\rangle_F|\text{even}\rangle_G,$$

$$|\Phi_2\rangle_{AB} \rightarrow \frac{1}{\sqrt{N_2}}|0\rangle_F|\text{odd}\rangle_G,$$

$$|\Phi_3\rangle_{AB} \rightarrow \frac{1}{\sqrt{N_1}}|\text{even}\rangle_F|0\rangle_G,$$

$$|\Phi_4\rangle_{AB} \rightarrow \frac{1}{\sqrt{N_2}}|\text{odd}\rangle_F|0\rangle_G,$$

where $|\text{even}\rangle = \frac{|\sqrt{2\alpha}\rangle + |-\sqrt{2\alpha}\rangle}{\sqrt{N_1}}$ and $|\text{odd}\rangle = \frac{|\sqrt{2\alpha}\rangle - |-\sqrt{2\alpha}\rangle}{\sqrt{N_2}}$ have been termed respectively the *even* and *odd* coherent states. Also the term Cat states have been applied to the even and odd coherent states. The even-odd terminology refers to the fact that the photon number distribution is non-zero only for even photon numbers in the case of the even coherent state and is non-zero only for odd photon numbers in the case of the odd coherent state.

If an odd number of photons is detected at detector L for mode F , then we know that the entangled state incident on the measurement setup was $|\Phi_4\rangle \rightarrow 01$. On the other hand, if an odd number of photons is detected at detector R for mode G , then the incident entangled state was: $|\Phi_2\rangle \rightarrow 00$. If a non-zero even number of photons is detected for mode F , the incident state was $|\Phi_3\rangle \rightarrow 10$ and if a non-zero even number is detected for mode G , it was $|\Phi_1\rangle \rightarrow 00$.

4 Conclusions

In this paper, we have studied the qubit-photon quantum logic utilizing a strong off-resonant coupling of a qubit and cavity. We have investigated the creation of the quasi-Bell state $|\Phi_2\rangle$. We have studied dense coding with one of the quasi-Bell states (as channels). Additionally, we have proposed an experimental scheme for dense coding with quasi-Bell measurement.

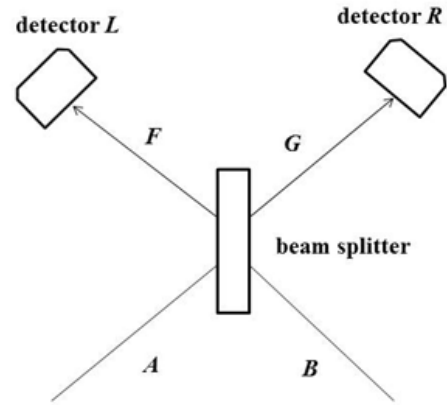


Fig. 2: Schematic diagram of the detection setup. A beam splitter separates the paths into detectors L and R , allowing for discrimination between different quasi-Bell states. Detectors L and R are positioned to observe the outputs F and G from inputs A and B .

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