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A mathematical procedure for work of the friction force on the arbitrary path

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Abstract In this paper, we have obtained the work of friction force on arbitrary paths. Our results indicate that the The work of the friction force depends highly on the path equation. Our explicit calculations show that the procedure and related examples are not simple.

1 Introduction

In elementary classical mechanics, the work done by friction is introduced as the change of mechanical energy of the system. Whenever the energy of the system is not conserved, we can write:

$$W' = \varDelta E = \varDelta U + \varDelta K \tag{1}$$

In this equation, W' is the work done by friction, $\varDelta E$ is the change in mechanical energy, and $\varDelta U$ and $\varDelta K$ are the changes in potential and kinetic energy, respectively. Obviously, Eq. (1) shows that the work done by friction consists of two parts: the change in potential energy, which is path-independent if the force is conservative, and the change in kinetic energy, which is path-dependent. If we want to calculate the work done by friction directly without using Eq. (1), then we can write the following:

$$W' = \int \overrightarrow{\mathbf{F}} \cdot \overrightarrow{d\mathbf{l}}. \tag{2}$$

In this procedure, it is not important to know the speed values at the start and end points of the path. However, the equation of the path is what is very important. This procedure with the mentioned purposes has been reported in [1]. However, we indicate that this paper [1] did not lead to the correct results.

A different article [3] has shown that the direct equation of work of friction on a particular curved path leads to Eq. (1). This paper [3] somehow concludes the equivalency between two equations (1) and (2).

In some textbook exercises, we can see that the action of dissipative forces leads to entering the domain of thermodynamics [2]. Indeed, the work done by frictional forces has often been calculated incorrectly [4]. The key to a correct treatment lies in distinguishing between the energy equation and the center of mass equation [4–6]. This subject leads to an informal concept called "Pseudowork" [7]. In any case, this concept is not considered in our work. Section 2 begins with a critique of reference [1] and continues with direct calculations of the work done by the friction force. An example is given in Section 3 to test the ultimate relationship for the work of friction.

2 Calculation of the Work of Friction

Now consider a particle sliding on a curved path with the equation $y = f(x)$, under the action of gravity. The final equation in Ref. [1] is given as:

$$W' = \frac{1}{2}\mu mg \int \frac{1 + [f'(x)]^2}{2 + [f'(x)]^2 - \mu f'(x)} dx, \tag{3}$$

where m is the mass of the particle and μ is the coefficient of friction.

We believe that Eq.(3) is not correct. At the very least, it does not give the correct result in limiting cases. For example, consider the case of an inclined plane. In this case, $f'(x) = \tan\theta$, where θ is constant. Substituting this into Eq. (3) from [1] gives:

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$$W' = 2\mu mg \tan\theta 2 \frac{1+\tan^2\theta}{2+\tan^2\theta - \mu\tan\theta}\Delta x. \tag{4}$$

With respect to $\Delta x = (\Delta l)\cos\theta$, we can write:

$$W' = (\mu mg\cos\theta)(\Delta l)\frac{2}{1+\cos^2\theta - \mu\sin\theta\cos\theta}\Delta x. \tag{5}$$

But for an inclined plane, from basic classical mechanics we know that $N = mg\cos\theta$ and $W' = (\mu mg\cos\theta)(\Delta l)$ which is not the same as Eq. (5).

In fact, the definition of speed in the mentioned article is problematic. Specifically, the equations:

$$v = \frac{dl}{dt}, \quad \text{and} \quad v \neq \frac{dl}{dx}.$$

Newton's law, projected along the path and normal to the path give the following equations:

$$\frac{1}{g}\frac{dv}{dt} = -\sin\theta - \mu n, \tag{6}$$

$$\frac{v^2}{gR} = n - \cos\theta, \tag{7}$$

where R is the curvature radius of the path, given by $R = \frac{dl}{d\theta}, \quad n = \frac{N}{mg}$. By using

$$\frac{dv}{dt} = \frac{dv}{dl}\frac{dl}{dt} = v\frac{dv}{dl}, \tag{8}$$

and defining:

$$\psi = \frac{v^2}{g} = R(n - \cos\theta), \tag{9}$$

we obtain:

$$\frac{d\psi}{dl} = -2(\sin\theta + \mu n). \tag{10}$$

So:

$$\frac{d\psi}{d\theta} = -2R(\sin\theta + \mu n) = -2\mu\psi - 2R(\mu\cos\theta + \sin\theta). \tag{11}$$

Integrating this equation results in:

$$\psi(\theta) = \exp\left(-2\mu(\theta-\theta_1)\right)\left\{\psi(\theta_1) - 2\int_{\theta_1}^{\theta} d\varphi \right.$$
$$\left. (\mu\cos\varphi + \sin\varphi)R(\varphi)\exp\left[2\mu(\varphi-\theta_1)\right]\right\}. \tag{12}$$

Assuming the particle starts from rest, we conclude:

$$\psi(\theta) = -2\int_{\theta_1}^{\theta}(\mu\cos\varphi + \sin\varphi)$$
$$\times R(\varphi)\exp\left(2\mu(\varphi-\theta)\right)d\varphi. \tag{13}$$

Therfore

$$n(\theta) = \cos\theta - \frac{2}{R(\theta)}\int_{\theta_1}^{\theta}(\mu\cos\varphi + \sin\varphi)$$
$$\times R(\varphi)\exp\left[2\mu(\varphi-\theta)\right]d\varphi. \tag{14}$$

Finally, W' or the work of friction is given by:

$$W' = \mu mg\int_{\theta_1}^{\theta_2} R(\theta)n(\theta)d\theta. \tag{15}$$

3 Example

Consider a trajectory with the following equation:

$$f(x) = \sqrt{x}. \tag{16}$$

and:

$$\tan\theta = -\frac{1}{2\sqrt{x}}, \quad \text{and} \quad \int_0^1 d\xi\sqrt{1+\frac{1}{4\xi}}. \tag{17}$$

Defining the variable α as $\sinh(\alpha) = \frac{1}{2\sqrt{x}}$, we get

$$\frac{d\theta}{d\alpha} = \frac{1}{\cosh\alpha}, \quad \frac{dl}{d\alpha} = \frac{\cosh^2 h\alpha}{2}, \quad R = \frac{\cosh^3 h\alpha}{2}. \tag{18}$$

We can rewrite Eqs (14) and (15) in terms of α instead of θ:

$$\theta = q(\alpha), \tag{19}$$

where:

$$g(\alpha) = -\frac{\pi}{2} + \int_0^\alpha \frac{d\beta}{\cosh\beta} = -\frac{\pi}{2} + \tan^{-1}(\sinh\alpha), \tag{20}$$

and

$$R\left[\rho(\alpha)\right] =: \rho(\alpha), \quad n\left[\rho(\alpha)\right] =: \nu(\alpha). \tag{6}$$

These latter equations give $R(\alpha)$ and $n(\alpha)$ in terms of α instead of θ. Additionally, defining $\lambda(\alpha)$ as:

$$\lambda(\alpha) = \rho(\alpha)q'(\alpha) = \frac{\cosh^2 h\alpha}{2}, \tag{22}$$

we arrive at:

$$\nu(\alpha) = \cos\left[q(\alpha)\right] - \frac{2}{\rho(\alpha)}\int_{\alpha_1}^{\alpha}\Big\{\mu\cos\left[q(\beta)\right] + \sin\left[q(\beta)\right]\Big\} \times \exp\Big\{\mu\left[q(\beta) - q(\alpha)\right]\Big\}\lambda(\beta)d\beta. \tag{23}$$

Finally, the work of friction is:

$$W' = \mu mg\int_{\alpha_1}^{\alpha} d\beta\lambda(\beta)\nu(\beta). \tag{24}$$

4 Conclusion

We have investigated, in a general form, the direct calculation of the work done by the friction force. As observed, explicit calculations of the integrals may not be straightforward. However, we invite the reader to examine the same procedure for a given path $f(x) = -x^2$, because the radius of curvature of these two functions is identical. Comparing the work of friction on these two paths and analyzing the speed of objects sliding along these paths at the end of the trajectories is highly desirable.

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