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Bound state current density of an isolated vortex line in a superconductor

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Abstract We investigate the vortex lines in a heterogeneous superconducting system. Using Bogoliubov's equations, we calculate the equations governing the vortex line structure in this system. Following that, we use the Wentzel-Kramers-Brillouin (WKB) approximation and numerically calculate the bound state current of the isolated vortex line. Finally, we study the behavior of current in terms of temperature.

1 Introduction

Vortex lines do not exist in a homogeneous microscopic system because they are unstable in a magnetic field where $H < H_{c1}$ and only form in a magnetic field lattice when $H > H_{c1}$. When analyzing the properties of an isolated vortex line, potential effects related to the order parameter should be disregarded. For temperatures close to the critical temperature T_c, only the Ginzburg-Landau (G-L) theory is applicable for calculating the properties of vortex lines. However, a microscopic theory is necessary for investigating more detailed aspects, such as the excitation spectra within vortex cores or the spatial dependence of the field, the current density, and order parameters at distances far from T_c. We will follow the approach used by Caroli et al. (1964) and later employed by Bardeen, Kummel, Jacobs, and Tewordt (1969), which uses the Bogoliubov method [1].

The Bogoliubov method is utilized in superconductors to describe the behavior of electrons in the presence of an effective potential arising from Cooper pairing and magnetic vortices. This approach aids in analyzing the energy spectrum and the distribution of electronic density of states in type-II superconductors. Moreover, it is employed to study the effects of magnetic fields and structural defects on electronic behavior within superconductors [2].

In the microscopic theory, the current density is related to the Bogoliubov wave functions, u_n, v_n, and the vector potential $\mathbf{A}$. We consider a specific gauge where the vector potential for the real gap function describes the magnetic field. The vortex line is expressed along the Z-axis in cylindrical coordinates. We compute the vortex line structure equations using the superconducting system's Hamiltonian and the wave functions in cylindrical coordinates. Two solutions are derived under boundary conditions: $r \to 0$ and $r \to \infty$. We then apply the WKB method's quasiclassical approximation to the Bogoliubov equations. We discuss the bound state of the vortex in the superconductor. Vortex-bound states play a significant role in the dynamic properties of superconductors, as they can influence local currents and energy transport. These bound states can act as energy channels within vortices, affecting the system's resistive behavior and electrical currents. Studying these bound states also provides valuable insights into quantum phase coherence and the effects of external fields on superconductors [3]. Subsequently, based on the energy of the bound state of the isolated vortex line, which depends on the magnetic quantum number, we calculate the bound state current density of the isolated vortex line as a function of temperature, and we compared some of the characteristics of vortices in superconductors with those in ultracold Fermi gases. Finally, we plot the current density as a function of temperature.

2 Vortex Line Structure Equations

We calculate the current density of the bound state of the vortex line using the WKB method and consider its temperature effects:

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$$\mathbf{J}(r) = \frac{e\hbar}{2mi}\sum_n \Big[f(E_n)u_n^*\big(\nabla - i\frac{e}{\hbar c}\mathbf{A}\big)u_n + \big(1-f(E_n)\big)\nu_n\big(\nabla - i\frac{e}{\hbar c}\mathbf{A}\big)\nu_n^* - c.c.\Big]. \quad (1)$$

In Eq. (1), $f(E_n)$ is Dirac's Fermi function, and the vector potential is obtained from Maxwell's equations. u_n and v_n are the Bogoliubov wave functions. We choose a gauge for the calculations. In the London model, as $r \to 0$, the current density is given by $j_\theta = \frac{ne}{2mr}\hbar$. The current density varies inversely with the radius r, increasing near the cylinder's axis. This characteristic is essential for studying Type-II superconductors' circular currents with vortex structures. In these systems, Cooper pairs generate angular momentum around the cylinder's axis, which increases as the radius decreases. In the microscopic theory, the current density is related to the Bogoliubov wave functions and the vector potential. The following equations describe the effect of the gauge function $\chi(r)$ on the vector potential, the Bogoliubov wave functions, and the gap function:

$$A(r) = \nabla\chi(r), \quad (2)$$

$$u_n(r) = u_n^{(0)}(r)e^{i\frac{e}{\hbar c}\chi(r)}, \quad (3)$$

$$v_n(r) = v_n^{(0)}(r)e^{-i\frac{e}{\hbar c}\chi(r)}, \quad (4)$$

$$\Delta(r) = \Delta^{(0)}(r)e^{i\frac{2e}{\hbar c}\chi(r)}. \quad (5)$$

where $u_n^{(0)}$, $v_n^{(0)}$, and $\Delta^{(0)}$ are the solutions with $\chi = 0$. $\Delta(r)$ represents the gap function of the system. In the London gauge, the form of the vector potential in the limit as $r \to 0$ is given by: $A_\theta = \frac{\hbar c}{2er}$,
and the vector potential in the vortex is defined as:

$$A_\theta(r) = \frac{\hbar c}{2er} + A'_\theta(r). \quad (6)$$

Here, $A'_\theta \to 0$ as $r \to 0$. When r is smaller than the penetration depth λ, and A'_θ is given by:

$$A'_\theta = -\frac{1}{2}rh_0, \quad (7)$$

where h_0 is the magnetic field in the $-z$ direction on the axis of the vortex line. The vector potential is crucial for describing magnetic fields and analyzing vortices in various physical systems. In superconductors, particularly type II, the potential is instrumental in defining magnetic fields and analyzing magnetic vortices. The Ginzburg-Landau model is utilized to investigate the behavior of vortices and the influence of external magnetic fields on superconducting currents. On the other hand, the vector potential is mainly used to study the mechanical and quantum properties of vortices in ultracold Fermi gases. In these systems, vortices are characterized as quantized structures within the superfluid, and the Schrödinger equation is applied to explore their dynamics and internal features. External magnetic fields have a less significant impact in these contexts. In both situations, the vector potential facilitates the description of magnetic fields and the analysis of vortices. The effect of external magnetic fields on vortex distribution is more evident in superconductors. In contrast, in ultracold Fermi gases, the emphasis is placed on the superfluid's internal dynamics and the vortices' quantized nature [4, 5]. We perform a gauge transformation $A_\theta \to A_\theta + \frac{1}{r}\frac{\partial\chi(\theta)}{\partial\theta}$, where $\chi(\theta) = -\frac{\hbar c}{2e}\theta$. In this measurement, the vector potential vanishes in the limit of small r. Nevertheless, the form of the Bogoliubov wave functions is explicitly determined as a function of θ:

$$u_n \to u_n e^{-i\frac{\theta}{2}}, \quad (8)$$

and,

$$v_n \to v_n e^{i\frac{\theta}{2}}. \quad (9)$$

In this gauge, the gap function depends on θ:

$$\Delta(r) \to \Delta(r)e^{-i\theta}. \quad (10)$$

On the basis of the G-L theory, we expect $|\Delta(r)| \propto r$ as $r \to 0$, and $|\Delta(r)| \to \Delta_\infty$ as $r \to \infty$, where Δ_∞ is the equilibrium gap. We consider a vortex line of unit strength along the z-axis of cylindrical coordinates (r, θ, z). The Bogoliubov wave functions are written as a two-component spinor:

$$\begin{pmatrix} U_n \\ V_n \end{pmatrix} = \hat{f}e^{ik_z z}e^{i\mu\theta}e^{-\frac{1}{2}i\tau_z\theta}, \quad (11)$$

where τ_i denote the Pauli matrices, 2μ is an odd integer, and $f(r)$ is given as:

$$f = \begin{pmatrix} f_+(r) \\ f_-(r) \end{pmatrix}. \quad (12)$$

Eq. (12) represents the radial part of the wave function in cylindrical coordinates, which is related to the Bogoliubov wave functions according to Eq. (11). Here, $f_+(r)$ is associated with U_n, and $f_-(r)$ is associated with V_n and k_z is the wave vector along the z-axis in the superconductor. Bogoliubov's equations to determine the structure of the vortex line are as follows [6, 7].

$$E_n u_n(r) = [H_0' + U(r)]u_n(r) + \Delta(r)v_n(r), \tag{13a}$$

$$E_n v_n(r) = -[H_0'^* + U(r)]v_n(r) + \Delta^*(r)u_n(r), \tag{13b}$$

where E_n represents the excitation energy of the system. Note that in a magnetic field the kinetic energy operator, $H' = -\frac{\hbar^2}{2m}(\nabla - \frac{ie}{\hbar c}A)^2$, contains an imaginary cross term that changes sign on taking the complex conjugate; hence $H_0' \neq H_0'^*$. The operator H_0' is given by:

$$H_0' = -\frac{\hbar^2}{2m}(\nabla - \frac{ie}{\hbar c}A)^2 - \mu. \tag{14}$$

The Hamiltonian system in a superconductor is defined as follows:

$$H' = \frac{1}{2m}\left[\mathbf{p} - \frac{e}{c}A\tau_z\right]^2 + U(r) - \mu, \tag{15}$$

where $U(r)$ is a mean (one-electron) potential and μ is the chemical potential [1]. The Bogoliubov-de Gennes (BdG) equations are employed in the physics of superconductors to describe quasiparticle excitations. In these equations, the effective potential experienced by electrons plays a crucial role in determining the behavior and energy distribution of quasiparticles. This potential can arise from external fields, localized impurities, or structural variations within the material. The presence of these potentials affects the energy structure and wave functions of the quasiparticles, leading to changes in their excitation behavior and superconducting properties. External magnetic or electric fields may also generate vortices, altering the quasiparticles' energy distribution.

Localized potentials, such as impurities or structural defects, also influence the behavior of electrons and Cooper pairs. These potentials can create localized or bound states, leading to modifications in the superconducting energy gap. Variations in the local potential may reduce the strength of Cooper pairs, destabilizing the superconducting state. Thus, the electronic potential affects the energy distribution and plays a critical role in shaping the overall properties of the superconducting system [3–5].

Eq.(13) can be rewritten as follows:

$$E\psi = [H'\tau_z + \Delta_1\tau_x + \Delta_2\tau_y]\psi. \tag{16}$$

The complex gap function is expressed as $\Delta(r) = \Delta_1(r) + i\Delta_2(r)$, which is useful for some applications.

The wave function ψ, according to Eq. (11), is a two-component spinor used in the calculations. The expression of the Laplacian in cylindrical coordinates, the substitution of the wave function from Eq. (11), and the explanation of the gap function $\Delta(r)$ help us derive the equations of the vortex line structure as stated in Eq. (16). The general result of the differential calculations in r is given by:

$$\begin{aligned}\tau_z\frac{\hbar^2}{2m}\Big[&-\frac{d^2\hat{f}}{dr^2} - \frac{1}{r}\frac{d\hat{f}}{dr} \\ &+ \left(\mu - \frac{er\tau_z}{\hbar c}A_\theta(r)\right)^2\frac{\hat{f}}{r^2} - k_\rho^2\hat{f}(r)\Big] \\ &+ \tau_x|\Delta(r)|\hat{f} = E\hat{f},\end{aligned} \tag{17}$$

where k_ρ is a separation constant given by $k_z^2 + k_\rho^2 = k_F^2$ and $A_\theta(r)$ is defined by Eq. (6). The vortex line structure equations are employed for the precise description of vortices and the distribution of magnetic fields and electric currents around them. Through these analyses, the microscopic characteristics of vortices can be understood. Additionally, these equations allow physicists to model the dynamic behavior of vortices under the influence of magnetic fields and external forces.

The exact solution of Eq. (17), when $A_\theta(r) \equiv 0$, $\Delta(r) \equiv \Delta_\infty$, and $r \to \infty$ are desired, is as follows:

$$\begin{aligned}f(r) =& \frac{\text{const}}{\sqrt{2}}\begin{pmatrix}\sqrt{1 \pm \frac{(E^2-\Delta_\infty^2)^2}{E}} \\ \sqrt{1 \mp \frac{(E^2-\Delta_\infty^2)^2}{E}}\end{pmatrix} \\ &\times H_\mu^{(1).(2)}\left(k_\rho^2 \pm \frac{2m}{\hbar^2}(E^2 - \Delta_\infty^2)^2\right)^{\frac{1}{2}} r,\end{aligned} \tag{18}$$

where $H_\mu^{(1).(2)}(k_\rho r)$ are the Hankel functions of the first and second kind. For another exact solution in the vortex core when $A_\theta'(r) = 0$ and $\Delta(r) = 0$, we have

$$f_\pm \approx A_\pm J_{\mu\mp\frac{1}{2}}[(k_\rho \pm q)r], \tag{19}$$

where $q = \frac{mE}{\hbar^2 k_\rho}$ and $J_{\mu\mp\frac{1}{2}}(kr)$ represents the Bessel functions of the first kind. We apply the quasi-classical approximation to the Bogoliubov equations and write the general form of the solution of Eq. (17) using Eqs. (18) and (19) as:

$$\hat{f}(r) = \hat{g}(r)H_{\mu}^{(1)}(k_\rho r) + \text{c.c.}, \tag{20}$$

where $H_{\mu}^{(1)}(k_\rho r)$ represents the Hankel function of the fast oscillation of the radial partof the wave function, while $\hat{g}(r)$ is calculated for slow changes in amplitude and phase, and is generated slowly by different functions $A'_\theta(r)$ and $|\varDelta(r)|$. We substitute Eq. (20) into Eq. (17) and ignore the terms $\frac{d^2 g}{dr^2}$ compared to $k_\rho \frac{dg}{dr}$. These terms are of the order $1/(k_\rho \xi)$ or $\varDelta/E_F$, where ξ is the coherence distance, E_F is the Fermi level energy. Additionally, we ignore terms involving A_θ^2.

$$-\frac{i\hbar^2}{m}\tau_z \beta(r)\frac{d\hat{g}}{dr} + \varDelta(r)\tau_x \hat{g} = \Big[(E + \frac{\mu e\hbar}{mcr})A_\theta(r)\Big]\hat{g}, \tag{21}$$

and where we have:

$$\beta(r) = \frac{k}{r}\left(r^2 - r_t^2\right)^{1/2}. \tag{22}$$

We write Eq. (21) in dimensionless form by changing the following variables:

$$x = \frac{2m\varDelta_\infty}{\hbar^2 k_\rho}\left(r^2 - r_t^2\right)^{1/2}, \tag{23}$$

$$\lambda = \frac{E}{\varDelta_\infty}, \tag{24}$$

$$F(x) = \frac{\mu e\hbar}{mcr\varDelta_\infty}A_\theta(r), \tag{25}$$

$$\delta(r) = \frac{\varDelta(r)}{\varDelta_\infty}. \tag{26}$$

In Eq. (23), $r_t = \mu/k_\rho$ represents the radial distance of the turning point. The equation for $\hat{g}(x)$ becomes:

$$-2i\tau_z\beta(r)\frac{d\hat{g}}{dx} + \delta(x)\tau_x\hat{g} = \Big[\lambda + F(x)\Big]\hat{g}. \tag{27}$$

To solve this equation, we express $\hat{g}$ in the following form:

$$\hat{g} = A\begin{pmatrix} e^{i\eta/2} \\ e^{-i\eta/2}\end{pmatrix} e^{i\xi}, \tag{28}$$

where A is a normalization factor, and η and ξ are in general complex functions of r. We substitute Eq. (28) into Eq. (27) and obtain two coupled equations for the coefficients of the Hankel function coefficients.

$$\frac{d\eta}{dx} + \delta(x)\cos(\eta) = \lambda + F(x), \tag{29}$$

$$2\frac{d\zeta}{dx} = i\delta(x)\sin(\eta). \tag{30}$$

These equations are to be solved subject to appropriate boundary conditions, which determine the eigenvalues λ. As x (or r) $\to \infty$, the following conditions hold: $F(x) \to 0$, $\delta(x) \to 1$, and $\eta(x) \to \eta_\infty$, where $\cos\eta_\infty = \lambda = E/\varDelta_\infty$. For the bound states in the core, $|E| < \varDelta_\infty$, η is real, and ξ is purely imaginary [8].

In the presence of a vortex, the superconducting pairing potential $\varDelta(r)$ vanishes locally at the vortex core and gradually increases away from the core. This spatial variation in the pairing potential creates localized quantum-bound states near the vortex core. These bound states, known as Bogoliubov-de Gennes states, are trapped within the vortex due to the confining potential caused by the spatial variations in the pairing potential.

These bound states are localized and have energies distinct from the continuum of quasiparticle states, forming discrete bound states within the superconducting gap. The energies of these bound states typically lie within the superconducting energy gap, meaning they exist at lower energies than the bulk superconducting gap predicted by BCS theory [9].

We use the asymptotic form of the Hankel function in the (WKB) approximation.

$$H_\mu^{(1).(2)} \sim \frac{\exp\left(\pm i\int_{r_t}^{r}\beta(r')\,dr'\right)}{(r^2 - r_t^2)^{1/4}}, \tag{31}$$

The quasi-classical approximation using the WKB method in superconductors, particularly in vortex lines, is an effective tool for calculating the energy and current of quantum-bound states in these systems. Bound states form near vortex lines due to spatial variations in the superconducting order parameter and pairing potential, and these states can be approximated using the WKB method. This method allows for approximating the wave functions and energies of these states and calculating the associated currents. In the next section, we will examine the current of a bound state in an isolated vortex line in a superconductor under the influence of temperature [10].

3 Current Density

To calculate the bound state density of the vortex line, we consider $A'_\theta(r) = 0$. From Eq. (1), with $A_\theta(r) = \hbar c/(2er)$, the equation for the current density is as follows:

$$j_\theta(r) = \frac{|e|\hbar}{m} \sum_n \Big\{ |v_n(r)|^2 \frac{\mu}{r} - \Big[(|u_n(r)|^2 + |v_n(r)|^2) \frac{\mu}{r} \Big] f(E_n) \Big\}, \tag{32}$$

where $|E| < \Delta_\infty$, and $n = (k_z, \mu)$.. We want to convert the summation into an integral.

$$j_\theta(r) = \frac{A^2|e|\hbar}{m} \int_{-kF}^{+k_F} \int_0^{rk_\rho} \Bigg\{ \frac{\mu r^{-1}}{\big[r^2 - (\mu/k_\rho)^2\big]^{1/2}} \mu r^{-1} - \frac{2\mu r^{-1}}{\big[r^2 - (\mu/k_\rho)^2\big]^{1/2}} \cdot \frac{1}{1 + e^{\frac{E}{K_B T}}} \Bigg\} d\mu \, dk_z. \tag{33}$$

The normalization constant A is given by:

$$A = \frac{1}{\sqrt{4\pi r_w}}, \tag{34}$$

where r_w represents the average extent of the wave function. In Eq. (32), we replace the bound state energy of the vortex line, which depends on μ.

The bound state energy in a superconductor's vortex core arises from quasiparticles trapped along the vortex line. These states, quantized around the vortex core, typically have energies below the superconducting energy gap and are calculated using the Bogoliubov-de Gennes equations.

These bound states play a critical role in the dynamics and magnetic behavior of superconductors dynamics and magnetic behavior with their energies being dependent on system properties such as the magnetic field and pairing interactions. In comparison, ultracold Fermi gases in superfluid states can also exhibit vortex lines and bound quasiparticle states. However, the intrinsic differences in particle types and interactions lead to variations in the energy of these states. In such systems, fermionic atoms pair up at extremely low temperatures, displaying behavior similar to superconductors. Yet, the energy of bound states is more sensitive to environmental parameters and experimental configurations [9–11]. Energy is calculated using Eq. (29) for a step pair potential in the vortex core, where $\delta(r) = 0$ for $r < r_c$ and $\delta(r) = 1$ for $r > r_c$, where, r_c is the vortex core radius, which is of the order of the coherence distance $\xi_0 = \hbar v_F/\pi\Delta_\infty$.

Finally, we obtain the following relations for energy:

$$E = -\Delta_\infty \sin\psi_0 = \Delta_\infty \Big[1 - \frac{1}{2}(x_\lambda^2/b^2)\Big], \tag{35}$$

$$\Psi_0 = -\frac{1}{2}\pi + \tan^{-1}(x_\lambda/b), \tag{36}$$

where λ is the penetration depth, and x_λ is the value of x corresponding to $r = \lambda$. Although this is a nonphysical approximation, it is qualitatively accurate when $r_c \ll \lambda$. b and b_c are defined as follows:

$$b = \frac{\mu}{\mu_c} b_c, \tag{37}$$

$$b_c = \frac{2mr_c\Delta_\infty r_c}{\hbar^2 k_\rho}, \tag{38}$$

$\mu_c = k_\rho r_c$ is the value of μ given at the turning point at $r = r_c$. The relationship between x and b_c is as follows:

$$x = b_c \sqrt{\left(\frac{r}{r_c}\right)^2 - \left(\frac{\mu}{\mu_c}\right)^2}, \tag{39}$$

where at $r = r_c$, we have:

$$x_c = x(r_c) = b_c \sqrt{1 - \left(\frac{\mu}{\mu_c}\right)^2}. \tag{40}$$

The vortex core radius in a superconductor corresponds to the distance from the vortex center where the superconducting order entirely collapses, allowing the magnetic flux to penetrate the material. This radius is approximately equal to the coherence length (ξ), which characterizes the spatial scale over which the superconducting order parameter decays. Within this core, the Cooper pairs are destroyed, and the supercurrent ceases, leading to the collapse of superconductivity and full magnetic penetration.

The core radius plays a significant role in determining the local current density around the vortex. The current density is significantly reduced near the vortex core because the superconducting order is weakened, and quasiparticles transition more easily to the normal state near the vortex core. However, as the distance from the core increases, the superconducting order is gradually restored, and the current density increases due to the reformation of Cooper pairs and the re-establishment of the supercurrent.

Suppose the core radius is smaller (associated with a shorter coherence length). In that case, the region of reduced current density is confined to a smaller area, resulting in a sharper increase in current density at a shorter distance from the core. Conversely, suppose the core radius is larger (associated with a longer coherence length). In that case, the reduced current density is spread over a larger region, causing the peak current density to occur farther from the core.

The vortex core radius directly influences superconductors' bound state current density distribution. Regions near the core exhibit reduced current density due to the collapse of the superconducting order. In contrast, farther from the core, the current density reaches higher values as superconductivity is restored [5–9].

By substituting Eqs. (34), (35), (37), and (38) into Eq. (33), we arrive at the bound state current equation for the isolated vortex line. The resulting integral is then solved numerically.

$$j_\theta(r) = \frac{1}{4\pi r_w}\frac{2e\hbar}{m}\int_{-k_F}^{k_F}\int_0^{rk_\rho}\left\{\frac{\mu}{r}\left[r^2-\left(\frac{\mu}{k_\rho}\right)^2\right]^{-\frac{1}{2}} - 2\left[r^2-\left(\frac{\mu}{k_\rho}\right)^2\right]^{-\frac{1}{2}}\left(\frac{\mu}{r}\right) \times \frac{1}{1+\exp\left(\frac{\Delta_\infty}{k_BT}\left[1-\frac{1}{2}\left(\frac{x_\lambda\mu_c\hbar^2k_\rho}{2\mu m\Delta_\infty r_c}\right)^2\right]\right)}\right\}d\mu\, dk_z. \quad (41)$$

To plot the current density (in arbitrary units) as a function of temperature, we assume $\hbar = 1$, $k_B = 1$, and consider $b/x_\lambda < 1$ and $x_\lambda > x_c$. Fig. 1 shows the current density as a function of $\frac{T}{E_F}$, and Fig. 2 shows the current density as a function of $\frac{E_F}{T}$, where E_F represents the Fermi energy.

As the temperature increases, the weakening of the Cooper pairs decreases the bound state current density of the isolated vortex line.

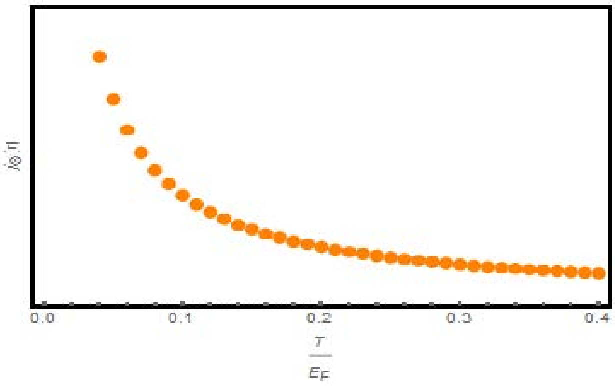

Fig. 1: The bound state current (in arbitrary units) of an isolated vortex line as a function of $\frac{T}{E_F}$.

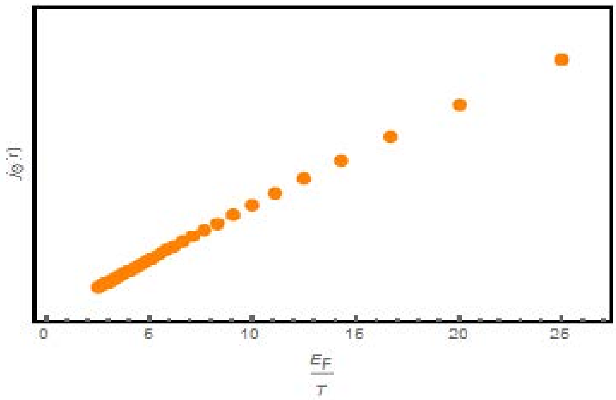

Fig. 2: The bound state current (in arbitrary units) of an isolated vortex line as a function of $\frac{E_F}{T}$.

4 Conclusion

We numerically investigated the isolated vortex line-bound state current with the WKB approximation and considered its temperature effects. According to the results, the bound state current density decreases as the temperature increases.

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