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A class of integrable nonlinear evolution equations driven exclusively by extrinsic quadratic effects

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Abstract In this work, we investigate a class of integrable nonlinear evolution equations where extrinsic quadratic contributions are the only source of nonlinearity. Within the Inverse Scattering Transform (IST) framework and using the ∂-bar method based on the Riemann-Hilbert approach, we extend the analysis to systems governed purely by singular dispersion relations, and we derive integrable equations that lack intrinsic nonlinearities and are driven only by the interaction between the evolution equations and their spectral problems. By suitable choices of singular dispersion relations, we derive the "truncated" Nonlinear Schrödinger (NLS) and Korteweg-de Vries (KdV) Equations that exhibit localized coherent structures and singular asymptotic behaviors that arise independently of initial conditions. The integrability of these systems is confirmed through their associated Lax pairs. To demonstrate the physical relevance of these systems, we derive them in the context of laser-plasma interactions, where they model the formation of propagating localized structures and energy transfer dynamics.

1 Introduction

The Inverse Scattering Transform (IST) [1, 2] has won physicists' consideration by offering a method to solve universal nonlinear evolution equations such as the nonlinear Schrödinger (NLS) and Korteweg-de Vries (KdV) equations. In this method, also called the Nonlinear Fourier Transform, a nonlinear evolution $q_t(x,t) = \mathcal{L}[q(x,t)]$, where $\mathcal{L}$ is a nonlinear operator, is associated to a convenient spectral operator. As is well known, the spectral operators associated with the NLS and the KdV equations are the Zakharov-Shabat [3] and the Sturm-Liouville operators [4], respectively. In this method, $q(x,0)$ plays the role of a "potential" for the spectral operator inducing spectral data at time $t = 0$. A linear time evolution of these spectral data must be chosen obeying the so-called compatibility conditions [5]. Finally, $q(x,t)$ is reconstructed using the inverse scattering transform (IST).

An alternative approach, consists of writing the ∂-bar equation for the spectral operator [6]. In particular, the Zakharov-Shabat spectral operator (associated with the NLS equation) is written as follows:

$$\frac{\partial}{\partial \bar{k}}\psi(k) = \psi(k)R(k), \tag{1a}$$

$$\psi(k) = \mathbb{1} + O\left(\frac{1}{k}\right), \quad |k| \to \infty, \tag{1b}$$

where $\psi(k)$ and $R(k)$ are 2×2 matrices (for more details see Appendix A), $\partial/\partial\bar{k} = \partial/\partial k_R + \mathrm{i}\partial/\partial k_I$ is the $\bar{\partial}$-derivative with $k = k_R + \mathrm{i}\, k_I$, and $\mathbb{1}$ is the 2×2 unitary matrix. Note that a vanishing $\bar{\partial}$-derivative implies analyticity. $R(k)$ is the *jump operator* in the associated Riemann-Hilbert problem. The Sturm-Liouville spectral operator associated with the KdV equation is written as follows:

$$\frac{\partial}{\partial \bar{k}}\phi(k) = \phi(-k)r(k), \tag{2a}$$

$$\phi(k) = 1 + O\left(\frac{1}{k}\right), \quad |k| \to \infty, \tag{2b}$$

where $\phi(k)$ and $r(k)$ are given scalar distributions in $\mathbb{C}$ (for more details, see Appendix B), where $r(k)$ is the

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jump function in the associated Riemann-Hilbert problem. R and r can be called "reflection coefficients" since they are related to spectral data. At this stage, one should choose a "simple" x and t dependency for R and r which will also imply, through (1a)-(1b) and (2a)-(2b), a dependency of ψ and ϕ on x and t.

To obtain the NLS equation with *source* (5a)-(5b)-(5c) from (1a)-(1b), one has to choose (Appendix A)

$$R_t = [R\ ,\ \Omega]\,, \tag{3a}$$

$$R_x = [R\ ,\ \mathrm{i}k\sigma_3]\,, \tag{3b}$$

where Ω is a 2×2 matrix and is the sum of a diagonal matrix of k-polynomials and a diagonal matrix of non-analytic (singular) functions of k. Likewise, to obtain the KdV equation with *source* (23a)-(23b) from (2a)-(2b), one must choose (Appendix B):

$$r_x(k) = 2\mathrm{i}kr(k), \tag{4a}$$

$$r_t(k) = [\beta(k) - \beta(-k)]r(k)k, \tag{4b}$$

where $\beta(k)$ is a sum of a k-polynomial and a non-analytic (singular) function of k. The time-derivatives of reflection coefficients $R_t(k,t)$ and $r_t(k,t)$ are called *dispersion relations*. If the dispersion relations do not contain any singular (non analytic) term, namely $\partial\Omega/\partial\bar{k} = 0$ and $\partial\beta/\partial\bar{k} = 0$, then one recovers the usual NLS and KdV equations without "source". Suppose the dispersion relation contains a singular term and its polynomial part. In that case, the compatibility condition yields the coupling of $q_t(x,t) = \mathcal{L}[q(x,t)]$ and its spectral problem [5].

These systems are integrable using IST. The Lax pair for the NLS and KdV coupled to their spectral problem are given in their most general form in [7] and [8], respectively.

In a more general context, the link between the Riemann Hilbert problem, the ∂-bar method and Lax pairs is presented in [9]. In the present work, we point out that if the dispersion relation contains no polynomials but only singular terms, then the evolution equation does not contain any intrinsic nonlinear or dissipative term; see Eqs. (9a) and (27a). The integrability of these equations is discussed in the Appendices, and their Lax pairs are given. These set of "truncated" coupled equations possess solutions evolving toward localized coherent structures with a singular asymptotic behaviour (blow up). The formation of this localized structure, *independent from the initial condition* $q(x,0)$ is due to the "extrinsic" nonlinearity and the "blow up" to the continuous energy transfer from the external media described by the spectral problem into the system described by the evolution equation. This work shows how the two major prototypes of universal nonlinear evolutions, namely the NLS and KdV equations, exhibit these properties. The relevant physical examples demonstrate the importance of this type of equations in physics.

In Sec. (2), starting from the NLS equation with source, we show how the choice of a proper singular dispersion relation leads to the "truncated" NLS equation coupled to its spectral operator. The lax pair is given, confirming its integrability. A related physical example is derived and solved. In Sec. (3), starting from the forced KdV equation, we show how the choice of a proper singular dispersion relation yields the "truncated" KdV equation coupled to its spectral operator. The Lax pair for this system is also given and confirms its integrability. In Sec. (4), using the multi-scale analysis, a related physical example, namely, a system of resonant coupled waves is derived in the context of the laser-plasma interaction. In Sec. (5), we present the method of solution to the system obtained is Sec. (3), based on the ∂-bar approach and we solve analytically the system. Finally, in Sec. (6), major results are discussed.

2 The truncated NLS equation coupled to its spectral problem and related physical model

We consider the following NLS equation coupled to its spectral problem

$$-\mathrm{i}q_t(x,t) + \frac{1}{2}q_{xx}(x,t) - q(x,t)\,|q(x,t)|^2 = \mathrm{i}\alpha\int_{-\infty}^{+\infty} d\lambda\ r_1(\lambda,x,t)\overline{r_2}(\lambda,x,t)\ , \tag{5a}$$

$$r_{1,x}(k,x,t) + \mathrm{i}kr_1(k,x,t) = qr_2(k,x,t)\ , \tag{5b}$$

$$r_{2,x}(k,x,t) - \mathrm{i}kr_2(k,x,t) = \bar{q}r_1(k,x,t), \tag{5c}$$

where α is a real constant and the over-head bar, the complex conjugation (throughout the paper).

The system (5a)-(5b)-(5c) associated to following conditions:

$$q(x,0) \in L^1(\mathbb{R})\ \ ;\ \ \lim_{x\to+\infty} r_1(k,x,t) = A(k,t)\mathrm{e}^{-\mathrm{i}kx}\ \ ;\ \ \lim_{x\to-\infty} r_2(k,x,t) = 0. \tag{6}$$

In the ∂-bar approach [10], (5a)-(5b)-(5c) associated to (6) has a singular dispersion relation, namely

$$\Omega(k,t) = \mathrm{i}k^2\sigma_3 - \frac{1}{\pi}\iint_{\mathbb{C}} \frac{d\lambda_R\, d\lambda_I}{\lambda - k} f(\lambda,t)\sigma_3, \tag{7}$$

where σ_3 is the Pauli matrix and $f(\lambda)$ any given distribution in the complex plane for $\lambda = \lambda_R + \mathrm{i}\lambda_I$. A physical application of the system (5)-(6) can be found in [11] for $f(\lambda,t) = (\mathrm{i}\pi/6)\delta(\lambda_I)\delta(\lambda_R - k_0)|A(k_R,t)|$ with $a_1(k_0,x,0) = 0$. The proof of integrability based on the ∂-bar approach is exhibited in Appendix A, while the Lax pair of the system can be found in [7].

Now, if we reduce the dispersion relation into its singular part

$$\Omega_s = -\frac{2\mathrm{i}}{\pi} P \int_{-\infty}^{+\infty} \frac{d\lambda}{\lambda - k} f(\lambda,t)\sigma_3, \tag{8}$$

where P denotes the Cauchy principal value of the integral. By setting $a_j(k,x,t) = r_j(k,x,t)e^{-\mathrm{i}kx}$, $j = 1,2$, system (5)-(6) is reduced to the "truncated" coupled system

$$q_t(x,t) = \int_{-\infty}^{+\infty} d\lambda\; f(\lambda,t) a_1 \bar{a}_2, \tag{9a}$$

$$a_{1,x}(x,t) = q(x,t)a_2(x,t), \tag{9b}$$

$$a_{2,x}(x,t) - 2\mathrm{i}ka_2(x,t) = \bar{q}(x,t)a_1(x,t). \tag{9c}$$

Sys. (9), associated to initial/boundary conditions,

$$q(x,0) \in L^1(\mathbb{R})\ ;\ \lim_{x\to+\infty} a_1(k,x) = 1\ ;\qquad \lim_{x\to-\infty} a_2(k,x) = 0. \tag{10}$$

is also integrable. Its Lax pair is [7]

$$\mu_x + ik[\sigma_3,\mu] = Q\mu, \tag{11a}$$

$$\mu_t = (Cf)\mu\sigma_3 - \left((Cf)\,\mu\sigma_3\mu^{-1}\right)\mu, \tag{11b}$$

where $Cf(k)$ denotes the Cauchy integral of f:

$$Cf(k) = \frac{1}{2\mathrm{i}\pi}\int_{\mathbb{R}} \frac{f(l)}{l-k} dl, \qquad k \in \mathbb{C}, \tag{12}$$

and

$$Q = \begin{pmatrix} 0 & q \\ -\bar{q} & 0 \end{pmatrix}, \qquad \mu = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{13}$$

A physical application has been found [12] in the context of laser-plasma interaction, where a_1 and a_2 represent the small amplitude of laser beam components in the laboratory frame (T,z), namely

$$\begin{aligned}\mathcal{E} =& \varepsilon a_1(x,t)\exp\{[\mathrm{i}(\omega_1 T + k_1 z)]\} \\ &+ \varepsilon a_2(x,t)\exp\{[\mathrm{i}(\omega_2 T + k_2 z)]\} + O(\varepsilon^2),\end{aligned} \tag{14}$$

where $\varepsilon = (m_e/m_\mathrm{i})^2$ is the squared ratio of the electron and ion mass. q accounts for the fluctuations of electron density $n(z,T)$ around its average value n_0, through

$$\begin{aligned}&n(z,T)/(n_0) - 1 = \\ &\varepsilon(2k_1c^2)/(\mathrm{i}\omega^2)q(x,t)\exp\{[\mathrm{i}(\tilde{\omega}T + \tilde{k}z)]\} + O(\varepsilon^2),\end{aligned} \tag{15}$$

where s is the sound velocity, c is the speed of light, $\tilde{\omega}$ and $\tilde{k}$ are the frequency and the wave number mismatch between the two components a_1 and a_2 of the laser beam, respectively. $\tilde{\omega}$ and $\tilde{k}$ are expressed as follows:

$$\omega_1 = \omega_2 + \tilde{\omega}\ \ ;\ \ k_1 = k_2 + \tilde{k}. \tag{16}$$

In (9a), f is chosen to be

$$f = \frac{\omega_0^2}{\omega^2}\frac{Ze^2|A|^2}{2m_i m_e c_s^2 c}, \tag{17}$$

where $\omega_0 = 4\pi n_0 e^2/m_e$ is the plasma frequency, Ze is the plasma ion charge and A is the amplitude of the applied laser beam. We define the slow variables moving with the ion-acoustic waves as follows:

$$x = \epsilon(z + c_s T),\ \ t = \epsilon^2 T. \tag{18}$$

Within the framework defined by the variables (18), System (9) accounts for the Stimulated Brillouin Scattering effect and results from the low-frequency effect of the high-frequency electrostatic waves (ESW), $\mathcal{E}(x,t)$, by means of the ponderomotive force on the electrons, which acts as a source of ion-acoustic waves (ISW) $q(x,t)$. The time-asymptotic solution in the sharp line limit (i.e., as $A(k)$ tends to $\delta(k)$). Eqs. (9), completed by the initial-boundary conditions (10), read

$$a_1 = \frac{1+|\eta|^2}{1-|\eta|^2} \qquad a_2 = \frac{2\mathrm{i}\eta}{1-|\eta|^2}, \tag{19a}$$

$$q = -2\mathrm{i}\left(\frac{ft}{x}\right)^{1/2} \frac{\eta}{1-|\eta|^2}, \tag{19b}$$

where

$$\eta(x,t) = \frac{\mathrm{i}\rho\sqrt{\pi}}{2ft} u^{1/2}\left[1-\frac{27}{4u}+O(u^{-2})\right]e^{u/2}, \tag{20}$$

where

$$u(x,t) = 4\sqrt{fxt}, \qquad x>0, \tag{21}$$

and ρ is an arbitrary constant. One can see that $q(x,t)$ is a soliton-like solution for *any initial function* $q(x,0) \in L^1(\mathbb{R})$, representing the asymptotic behavior of the ion acoustic wave (IAW). Notice that $q(x,t)$ has singular points for $|\eta|^2 = 1$, that is, when $u(x,t)$ solves the equation

$$\left[\frac{2f}{\sqrt{\pi}\rho}\right]^2 t^2 e^{-u} = u\left[1-\frac{27}{4u}+O(u^{-2})\right]^{-2}. \tag{22}$$

3 The truncated KdV equation coupled to its spectral problem

We have proved [13] that the KdV equation coupled by its spectral problem

$$q_t + 6qq_x - q_{xxx} = \\ -2\frac{\partial}{\partial x}\int_{-\infty}^{+\infty} d\lambda\, \nu(\lambda)\,|u(\lambda,x,t)|^2, \tag{23a}$$

$$u_{xx} + k^2 u = qu, \tag{23b}$$

and associated with the following initial and boundary conditions

$$q(x,0) \in L^1(\mathbb{R}), \quad \lim_{x\to+\infty} u(k,x,t) = A(k,t)\mathrm{e}^{-\mathrm{i}kx}, \tag{24}$$

where $\nu(k)$ is a real function in $L^2(\mathbb{R})$, is integrable with the singular dispersion relation

$$\beta(k,t) = -4ik^3 - \mathrm{i}\int_{-\infty}^{+\infty} \frac{d\lambda}{\lambda-k}\nu(\lambda)\,|d(\lambda,t)|^2. \tag{25}$$

The proof of integrability of this System as well as of System (3) as a sub-case, based on the ∂-bar approach, is exhibited in Appendix B.

Now, if we reduce the dispersion relation to

$$\beta_r(k,t) = -\mathrm{i}\int_{-\infty}^{+\infty} \frac{d\lambda}{\lambda-k}\nu(\lambda)\,|d(\lambda,t)|^2, \tag{26}$$

System (23) is reduced to the "truncated" coupled system

$$q_t = -2\frac{\partial}{\partial x}\int_{-\infty}^{+\infty} d\lambda\, \nu(\lambda)\,|u(\lambda,x,t)|^2, \tag{27a}$$

$$u_{xx} + k^2 u = qu. \tag{27b}$$

The x-part of the Lax pair for System (27) is Eq. (27b) and its t-part is [8]:

$$u_t - \frac{1}{4k}\left[(H\nu)\left(u\bar{u} + \frac{1}{k^2}u_x\bar{u}_x + (u\hat{u})_x\right)\right]u_x \\ + \frac{1}{4\mathrm{i}}\left[(H\nu)\left(u\bar{u} - \frac{1}{k^2}u_x\bar{u}_x\right)\right]u = 0, \tag{28}$$

where

$$(H\nu)(k) = \frac{p}{\pi}\int_{\mathbb{R}} \frac{\nu(l)}{l-k}dl, \qquad k\in\mathbb{R}, \tag{29}$$

where H denotes the Hilbert transform and p denotes the principle value integral.

4 Derivation of a related physical example

In this section, we are going to use three set of variables: laboratory frame (ξ,τ), the dimensionless variables (X,T), defined in (32) and the slow variables (x,t), defined in (38). Let consider the propagation of a polarized electrostatic wave-packet (laser beam)

$$E(\xi,\tau) = \int_{-\infty}^{+\infty} d\omega \tilde{E}(\omega,\xi,T)\mathrm{e}^{-\mathrm{i}\omega\tau}, \tag{30}$$

in the laboratory frame (ξ,τ) with a frequency close to $\omega_0 = (4\pi e^2 n_0/m_e)^{1/2}$ in a uniform warm electron-cold ion plasma, where $\tilde{E}(\omega,\xi,T)$ is the slowly varying envelope of $E(\xi,\tau)$ and T the slow scaled time, defined below Eq. (32). The low frequency effect on the electrons due to the high frequency field E, results in the ponderomotive force

$$f_p = -\frac{e^2}{2m_e}\,\partial\xi \int_{-\infty}^{+\infty} d\omega\ \omega^{-2}\left|\tilde{E}(\omega,\xi,T)\right|^2, \tag{31}$$

which is the real part of $[\delta\xi(\partial E/\partial\xi)]$ around the average position $<\xi>$, where $\delta\xi$ is the solution of $m_e(\delta\xi)_{tt} = eE$. Using the small parameter of the system $\epsilon = (m_e m_i)^2$, where m_e is the mass of an electron and m_i the mass of an ion. The Debye wavelength is defined as $\lambda_D^2 = K_B T_e/4\pi n_0 e^2$, where T_e is the electron temperature and K_B is the Boltzmann constant. We set the following dimensionless set of variables

$$X = (\lambda_D)^{-1}\xi, \qquad T = \epsilon\omega_0\tau. \tag{32}$$

Using these variables and the dimensionless electrostatic field, the electrostatic potential and the velocity of ions, are defined as follows:

$$E' = E\frac{e}{\omega}\left(\frac{1}{2m_e K_B T_e}\right)^{-1/2} \quad ;$$

$$\phi' = \frac{e}{K_B T_e}\phi \ \ ; \ \ v_i' = v_i\left(\frac{m_e}{K_B T_e}\right)^{-1/2}, \tag{33}$$

where $-e$ is the charge of the electron. For $\omega^2 > {\omega_0}^2$, we can write the fluid-type equations [12, 13]

$$\frac{\partial\phi'}{\partial X} - \frac{1}{(1+q_e)}\frac{\partial q_e}{\partial X} - \frac{\partial}{\partial X}\int_{-\infty}^{+\infty} d\omega\ \nu(\omega)\left|E'(\omega,X,T)\right|^2 = 0, \tag{34a}$$

$$\frac{\partial v_i'}{\partial T} + v_i'\frac{\partial v_i'}{\partial X} = -\frac{\partial\phi'}{\partial X}, \tag{34b}$$

$$\frac{\partial q_i}{\partial T} + \frac{\partial}{\partial X}\left[(1+q_i)\ v_i'\right] = 0, \tag{34c}$$

$$\frac{\partial^2\phi'}{\partial X^2} = q_e - q_i. \tag{34d}$$

Next, we expand ϕ, q_e q_i and v_i', in powers of ε as

$$q_i = \varepsilon q_i^{(1)} + \varepsilon^2 q_i^{(2)} + O(\varepsilon^3), \tag{35}$$

and, we define $\tilde{E}'$ and $\mathcal{E}$ by applying a multi-scale expansion to the amplitude of the electric field

$$E' = \varepsilon\mathcal{E} + O(\varepsilon^2), \qquad \tilde{E}' = \varepsilon^{3/4}(E' + O(\varepsilon)) \ . \tag{36}$$

In the the laboratory frame (ξ,t), the set of Eqs. (34) gives at first order

$$\frac{\partial^2 q}{\partial t^2} - \frac{\partial^2 q}{\partial\xi^2} = \varepsilon^{1/2}\frac{\partial^2}{\partial\xi^2}\int_{-\infty}^{+\infty} d\omega\ \nu(\omega)|\mathcal{E}(\omega,\xi,t)|^2, \tag{37}$$

where $q = q_e^{(1)} = q_i^{(1)}$. Introducing the slow variables

$$x = \varepsilon^{1/2}(X - T), \qquad t = \epsilon T, \tag{38}$$

co-moving with ISW at speed $C_s = (K_B T_e/m_i)^{1/2} = \varepsilon\omega_0\lambda_D$, Eq. (37) provides the evolution equation (27a).

On the other hand, the Maxwell equation for ESW,

$$\left[\frac{\partial^2}{\partial\tau^2} - 3(V_{Te})^2\frac{\partial^2}{\partial\xi^2}\right]E(\xi,\tau) = -\omega_0^2(1+q_e)E(\xi,\tau) \tag{39}$$

is obtained on the basis of the dispersion relation [14] $\omega^2 = \omega_p^2 + 3V_{Te}^2 k^2$, where $V_{Te} = \lambda_D\omega_0$ is the thermal electron velocity, k is the wave number and ω_p is the plasma frequency $\omega_p^2 = \omega_0^2(1+q_e)$, and, q_e is the fractional change in the electron density of average value n_0, namely $n_e = n_0[1+q_e(\xi,t)]$. Using the scalings and the expansions (32), (35), (36) and (38), the Maxwell equation gives at first order, the spectral problem (27b).

5 Method of solution

Based on the physical context described in the previous section, system (27a)-(27b) accounts for the following physical quantities: $u(\omega,x,t)$ represents the incoming electric field (laser beam), ω the frequency of the Pump closed to the plasma frequency and $q(x,t)$, the fractional change on electron density. By letting $\mu(k)$ be the profile of the source, in the (x,t)-frame moving with IAW, the physical system can be written as

$$q_t = \frac{\partial}{\partial x}\int_{-\infty}^{+\infty} d\lambda\ \mu(\lambda)\,|u(\lambda,x,t)|^2, \tag{40a}$$

$$u_{xx} + \lambda^2 u = qu. \tag{40b}$$

Notice that for convenience, we have slightly changed the system (27a)-(27b) by letting $\mu = -2\nu$. We associate a very general initial condition to system (40) by imposing $q(x,0)$ to obey the Faddeev condition, namely,

$$\int_{-\infty}^{+\infty}(1+|x|)\,|q(x,0)|dx < \infty. \tag{41}$$

In other words, any background noise in the fractional change of ion density in plasma will satisfy this condition. We also associate to system (40a)-(40b), the following boundary conditions

$$u(k,x,t) \to T(k,t)e^{\mathrm{i}kx}, \qquad x \to +\infty, \tag{42a}$$

$$\to e^{\mathrm{i}kx} + R(k,t)e^{-\mathrm{i}kx}, \qquad x \to -\infty, \tag{42b}$$

where, $R(k,t)$ and $T(k,t)$, stand for the reflection and transmission coefficients, respectively. The boundary conditions (42) are quite generic. Indeed, conditions (42) correspond to a normalized incoming laser beam from left to right, which is partially transmitted from to the right and seen at the vicinity of $+\infty$ as $T(k,t)e^{ikx}$ and partially reflected to the left and seen at the vicinity of $-\infty$, as $R(k,t)e^{-ikx}$.

In this section, we will obtain the time-asymptotic solution of system (40a)-(40b) complemented by the generic, initial/boundary conditions (41)-(42). The integral equation could be either the Marchenko equation obtained by the usual IST method [15] or the Cauchy-Green equation obtained by the ∂-bar approach. Here, we adopt the ∂-bar approach, which allows for obtaining directly the singular dispersion relation from the evolution equation and for writing the time evolution of the reflection coefficient in its simplest form [5].

By extending the parameter k to the complex plane, let us define the complex-valued function $f(k,x,t)$ as follows:

$$f(k,x,t) = u(k,x,t)e^{-\mathrm{i}kx}, \tag{43}$$

and associate the following ∂-bar problem [16]

$$\frac{\partial}{\partial \bar{k}} f(k) = f(-k,t), \qquad k \in \mathbb{C}, \tag{44a}$$

$$f(k) = 1 + O(1/k), \qquad |k| \to \infty, \tag{44b}$$

to Eq.(40b). We consider the case where $q(x,0)$ has non-bound states, and we define $r(k,x,t)$ as follows:

$$r(k,x,t) = R(k,t)e^{2\mathrm{i}kx}\delta(k_I + 0), \tag{45}$$

where k_I stands for the imaginary part of k and δ for the Dirac distribution. We assume then, a simple time dependence for $r(k,x,t)$ in the form of

$$\frac{\partial}{\partial t} r(k) = [\beta(k) - \beta(-k)]r(k), \tag{46}$$

where β is a distribution obtained by the comparison of the evolution equation (40a) with

$$q_t = -\frac{1}{\pi}\iint_{\mathbb{C}} d\alpha \wedge d\bar{\alpha}\ u(\alpha)u(-\alpha)\frac{\partial}{\partial \bar{\alpha}}\beta(\alpha), \tag{47}$$

obtained from the compatibility condition $\partial_t(\partial_x^2 u) = \partial_x^2(\partial_t u)$. We remind that $d\alpha \wedge d\bar{\alpha} = -2\mathrm{i}\ d\alpha_R\ d\alpha_I$ where α_R and α_I are the real and imaginary parts of α, respectively. Notice that in Eq. (46), we have chosen odd coefficients because any even part can be scaled off through a gauge transformation of $f(\lambda)$. In our case, we choose:

$$\beta(k) = -\frac{1}{2}\mathrm{i}\int_{-\infty}^{+\infty}\frac{dl}{l-k}\,\frac{\mu(l)}{a\bar{a}(l)}, \tag{48}$$

where $a(k)$ is a scattering coefficient of the potential u in (40b). For clarity, let us recall briefly the definition of the scattering coefficients $a(k)$ and $b(k)$ and their relation to the reflection and transmission coefficients $R(k)$ and $T(k)$. Let us consider the Schrödinger equation

$$\psi_{xx}(k,x,t) + (k^2 - q)\psi(k,x,t) = 0, \quad k \in \mathbb{C}, \tag{49}$$

where ψ has the following asymptotic behavior

$$\psi(k,x,t) \to e^{\mathrm{i}kx}, \qquad x \to +\infty, \tag{50a}$$

$$\to a(k,t)e^{\mathrm{i}kx} + b(k,t)e^{-\mathrm{i}kx}, \quad x \to -\infty. \tag{50b}$$

For $k=\alpha>0$, we have [16, 17],

$$u(\alpha,x,t)=\frac{1}{a(k,t)}\psi(k,x,t)\big|_{k=\alpha}, \tag{51}$$

and

$$T(\alpha,t)=\frac{1}{a(k,t)}\big|_{k=\alpha}, \qquad R(\alpha,t)=\frac{b(k,t)}{a(k,t)}\big|_{k=\alpha}. \tag{52a}$$

Now, inserting Eq. (48) into Eq. (46), yields

$$\frac{\partial}{\partial t}R(k,t)=R(k,t)\int_{-\infty}^{+\infty}\frac{-\mathrm{i}k}{l^2-(k+\mathrm{i}\epsilon)^2}\frac{\mu(l)}{a\bar{a}(l)}dl. \tag{53}$$

The potential $q(x,t)$ is obtained by solving the basic integral equation

$$f(k,x,t)= 1+\frac{1}{2\mathrm{i}\pi}\iint_{\mathbb{C}}\frac{dl\wedge d\bar{l}}{l-k}\,f(-l,x,t)R(l,t)\,e^{2\mathrm{i}kx}\delta(k_I+\mathrm{i}\epsilon), \tag{54}$$

which solves the ∂-bar problem (44). The solution $q(x,t)$ of (40b) can be obtained from f as follows [5]:

$$q=-2\mathrm{i}\frac{\partial}{\partial x}f^{(1)}(x,t)\ , \tag{55}$$

where $f^{(1)}$ is the coefficient of $1/k$ in the Laurent series expansion of $f(k,x,t)$. We now have to solve the evolution given by Eq. (53), which can be written as

$$\frac{\partial}{\partial t}R(k,t)= R(k,t)\left(\frac{\pi\mu(k)}{a\bar{a}(k)}-\mathrm{i}k\ P\int_{-\infty}^{+\infty}\frac{dl}{l^2-k^2}\frac{\mu(l)}{a\bar{a}(l)}\right), \tag{56a}$$

$$\frac{\partial}{\partial t}\overline{R}(k,t)= \overline{R}(k,t)\left(\frac{\pi\mu(-k)}{a\bar{a}(k)}+\mathrm{i}k\ P\int_{-\infty}^{+\infty}\frac{dl}{l^2-k^2}\frac{\mu(l)}{a\bar{a}(l)}\right), \tag{56b}$$

where P is the principal value of the integral and $\overline{R}(k)=R(-k)$. Notice that the potential $q(x,t)$ is real. Therefore, $\overline{R}(k)$ must be equal to the complex conjugated of $R(k)$, and $\bar{a}(k)=a(-k)$. The assumption of a symmetric profile $\mu(k)$, namely $\mu(-k)=\mu(k)$ implies

$$\frac{\partial}{\partial t}(R\overline{R})=\frac{2\pi\mu}{a\bar{a}}R\overline{R}. \tag{57}$$

Now using the basic relation $a\bar{a}-b\bar{b}=1$ [15] and Eqs. (51)-(52), one can solve Eq. (57) by eliminating $a\bar{a}$, and obtain

$$|R(k,t)|^2= \frac{|R(k,0)|^2}{|R(k,0)|^2+[1-|R(k,0)|^2]\exp[-2\mu(k)\pi t]}. \tag{58}$$

Equations (56a)-(56b) can now be solved and yield

$$R(k,t)= \frac{R(k,0)e^{\mathrm{i}\theta(k,t)}}{\{|R(k,0)|^2+(1-|R(k,0)|^2)e^{-2\mu(k)\pi t}\}^{1/2}}, \tag{59a}$$

$$\theta(k,t)=\frac{k}{2\pi}P\int\frac{dl}{l^2-k^2}\times \ln\{|R(l,0)|^2+(1-|R(l,0)|^2)e^{-2\mu(l)\pi t}\}. \tag{59b}$$

Note that for $\mu>0$, when $t\to+\infty$, $|R(k,t)|\to1$. Here it worths to notice that we have the remarkable property that the phase of $R(k,t)$ approaches a constant as t yields to infinity. Indeed, using the fact that the function $|R(k,t)|^2=R(k)R(-k)$ is even in k, we have

$$\theta(k,t)\to\theta_\infty(k)=-\frac{k}{\pi}P\int\frac{dl}{l-k}\ln|R(l,0)|. \tag{60}$$

The solution f of the integral equation (54) approaches, asymptotically, the solution f_∞ of the integral equation

$$f_\infty(k,x)=1-\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{dle^{2\mathrm{i}lx}}{l-(k+\mathrm{i}0)}\times f_\infty(-l,x)\exp[\mathrm{i}\theta_\infty(l)+\mathrm{i}\arg R(l,0)]. \tag{61}$$

To evaluate the asymptotic behavior of IAW from Eqs. (61) and (55), we must choose a model for $R(k,0)$ [15]:

$$R(k,0)=-\exp\left(\frac{-wk}{k-\mathrm{i}\kappa}\right), \tag{62}$$

where the constant w measures the width in k space of the continuous spectrum, and κ determines how fast the spectum drops off in k space. For large k, $R(k,0) \to -e^{-w}$. Therefore, w must be chosen quite large. With this choice,

$$f(k,x,t) = 1 + \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{dl f(-l,x,t)}{l-(k+i\epsilon)}) \times \\ \exp\left[2ilx + \frac{\kappa l}{2(l+i\kappa)} - \frac{\kappa l}{l-i\kappa}\right]. \quad (63)$$

The behavior of Eq. (63) is evaluated by the steepest descent method. Let

$$g(l) = 2ilx + \frac{wl}{2(l+i\kappa)}, \qquad \tilde{g}(l) = 2ilx - \frac{wl}{2(l-i\kappa)}. \quad (64)$$

The stationary points are given by

$$x<0, \qquad \frac{\partial}{\partial l} g(l) = 0 \Rightarrow l^{\pm} = -i\kappa \pm \sqrt{\frac{\kappa w}{4|x|}}, \quad (65a)$$

$$x>0, \qquad \frac{\partial}{\partial l} \tilde{g}(l) = 0 \Rightarrow \tilde{l}^{\pm} = i\kappa \mp i\sqrt{\frac{\kappa w}{4x}}. \quad (65b)$$

For $x<0$, the path of integration can follow the path of the steepest descent, namely $\Im[g(l)] = \Im[g(l^{\pm})]$. In this case $(x<0)$,

$$f(k) \to 1 - \frac{1}{\pi} \sum_{\pm} \frac{R_0(l^{\pm})}{l^{\pm}-k} f(-l^{\pm}) \times \\ \sqrt{\frac{-2\pi}{g''(l^{\pm})}} \exp\left[g(l^{\pm})\right]. \quad (66)$$

Using (55), we obtain

$$q(x) = \frac{\partial}{\partial x} \sum_{\pm} \frac{-4il^{\pm} C^{\pm} \exp[g(l^{\pm})]}{l + C^{\pm} \exp[g(l^{\pm})]}, \quad (67)$$

where

$$C^{\pm} = \frac{R_0(l^{\pm})}{2\pi l^{\pm}} \sqrt{\frac{-2\pi}{g''(l^{\pm})}}. \quad (68)$$

The approximation in Eq. (66) is valid as long as the second term of the expansion is smaller than the first term, which occurs taken $\kappa w|x| \gg 1$.

For $x>0$, one can evaluate f by the method of stationary phase, following the path of integration $\Re[\tilde{g}(l)] = \Re[\tilde{g}(l^{-})]$, which yields

$$f(k) \to 1 + \frac{i}{\pi} \frac{f(-\tilde{l}^{-})}{\tilde{l}^{-}-k} \exp\left\{\left[\frac{w\tilde{l}^{-}}{2(\tilde{l}^{-}+i\kappa)}\right]\right\} \times \\ \sqrt{\frac{\pi e}{|\tilde{g}''(\tilde{l}^{-})|}} \exp\{[\tilde{g}(l^{-})]\}. \quad (69)$$

Again, using Eq. (55), we obtain

$$q(x) = \frac{\partial}{\partial x} \frac{-4i\tilde{l}^{-}\tilde{C} \exp\left[\tilde{g}(\tilde{l}^{-})\right]}{l + \tilde{C} \exp\{[\tilde{g}(\tilde{l}^{-})]\}}, \quad (70)$$

where

$$\tilde{C} = \frac{\sqrt{\kappa w e \pi}(4x)^{3/4}}{2\pi(w - \sqrt{\kappa w/4x})} \exp\left\{\left[-\left(\frac{w}{2}\right) \frac{2\kappa x - \sqrt{\kappa w x}}{4\kappa x - \sqrt{\kappa w x}}\right]\right\}. \quad (71)$$

In this case, the approximation is valid only if $\kappa w x \gg \kappa^2$. Otherwise, $\tilde{l}^{-}$ moves into the lower half plane and it would no more be possible to ignore the contribution of other terms in the Laurent expansion of f.

6 Results and conclusion

In this work, we aim to highlight two integrable systems of coupled waves that we call the "truncated NLS with source" (9) and the " forced truncated KdV equation" (27). The physical context we have chosen to discuss these systems is the laser-plasma interaction. However, these are systems of equations which can appear in many other physical contexts. Indeed, we deal with a low frequency initial "noise" as a generic initial condition $q(x,0)$ evolving in time toward a coherent structure due to a nonlinear interaction with high-frequency external fields. This allows us to refer to this phenomenon as a "universal behavior".

It is worth to note that the only reason we call Eq. (9a), "truncated NLS" even though (9a) does not contain the key terms $q|q|^2$ and q_{xx} of the NLS equation, is that in the context of the IST method, both (9a) and NLS equations are associated to the same spectral operator, namely to the Zakharov-Shabat operator. It is the same for the so-called "forced truncated KdV" equation (27a) which does not contain the key terms qq_{xx} and q_{xxx} of the KdV equation but again in the IST approach, both (27a) and the KdV equations have the Sturm-Liouville operator as the associated spectral problem in common.

In both cases described by Eqs. (9) and, (27), i.e. the "truncated NLS with source" and " forced truncated KdV equation" respectively, the only nonlinearities arise from the nonlinear quadratic contribution of an external field. In our physical context, this contribution results from the coupling between the IAW and the electrostatic waves. Using the integrability of these systems through the ∂-bar formulation of the Inverse Scattering Transform method (see Appendices A and B), we have demonstrated the formation of localized coherent structures due to this extrinsic nonlinearity, which is valid *for any generic initial condition.*

Appendix A

Here we prove the integrability of the system

$$Q_t - \frac{\mathrm{i}}{2}\sigma_3 Q_{xx} + \mathrm{i}\sigma_3 Q^3 = \mathrm{i}\left[\sigma_3, \frac{1}{2\pi\mathrm{i}}\iint_{\mathbb{C}} d\lambda\wedge d\bar{\lambda} g(\lambda)\mu(\lambda)\sigma_3\mu^{-1}(\lambda)\right], \quad \text{(A.1a)}$$

$$\psi_x(k) + \mathrm{i}[\sigma_3, \psi(k)] = Q\psi(k). \quad \text{(A.1b)}$$

A reduction of the above system (A.1) yields

$$\begin{cases} -\mathrm{i}q_t + \frac{1}{2}q_{xx} - q|q|^2 = \mathrm{i}\alpha\int_{-\infty}^{+\infty} d\lambda\ a_1\overline{a_2}\ , \\ a_{1,x} + \mathrm{i}ka_1 = qr_2\ , \\ a_{2,x} - \mathrm{i}ka_2 = \bar{q}r_1, \end{cases} \quad \text{(A.2)}$$

where α is a real constant, the over-head bar represents the complex conjugation, q is a function of (x,t) and a is a function of (k,x,t). We have

$$q(x,0) \in L^1(\mathbb{R}), \quad \lim_{x\to+\infty} a_1(k,x,t) = A(k,t)\mathrm{e}^{-\mathrm{i}kx}, \quad \lim_{x\to-\infty} a_2(k,x,t) = 0. \quad \text{(A.3)}$$

The integrability of the "truncated" system (9) can be seen as a sub-case of (A.2)-(A.3). The second and the third equations of System (A.2) are the vectorial form of the Zakharov-Shabat spectral problem [3] that can be written in the general matrix form [18]

$$\psi_x = U\ \psi, \qquad U = -k\ \sigma_3 + Q, \qquad Q = \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix}. \quad \text{(A.4)}$$

(A.2) is recovered by the reduction $r = \bar{q}$. Here, ψ is a 2×2 matrix, built with two independent column-vector solutions $(\psi_1\ ,\psi_2)$ and is completely determined by its asymptotic behavior. Then, the set of differential equations $\psi_x = U\ \psi$ can be equivalently written as a set of Volterra integral equations. For convenience, let us write these equations for the matrix $\mu(k,x)$, defined by

$$\mu = \psi\exp(\mathrm{i}k\sigma_3 x)\ \Rightarrow\ \mu_x = \mathrm{i}k[\mu\ ,\ \sigma_3] + Q\mu\ . \quad \text{(A.5)}$$

Two dependent solutions μ^+ and μ^- can be defined through

$$\begin{cases} \mu^+_{11} = 1 - \int_x^{+\infty} dx'\, q\,\mu^+_{21} \\ \mu^+_{21} = \int_{-\infty}^{x} dx'\, \bar{q}\,\mu^+_{11}\,\mathrm{e}^{2\mathrm{i}k(x-x')} \\ \mu^+_{12} = -\int_x^{+\infty} dx'\, q\,\mu^+_{22}\,\mathrm{e}^{-2\mathrm{i}k(x-x')} \\ \mu^+_{22} = 1 - \int_x^{+\infty} dx'\, \bar{q}\,\mu^+_{12} \end{cases} \quad \text{(A.6)}$$

$$\begin{cases} \mu^-_{11} = 1 - \int_x^{+\infty} dx'\, q\,\mu^-_{21} \\ \mu^-_{21} = \int_x^{+\infty} dx'\, \bar{q}\,\mu^-_{11}\,\mathrm{e}^{2\mathrm{i}k(x-x')} \\ \mu^-_{12} = -\int_{-\infty}^{x} dx'\, q\,\mu^-_{22}\,\mathrm{e}^{-2\mathrm{i}k(x-x')} \\ \mu^-_{22} = 1 - \int_x^{+\infty} dx'\, \bar{q}\,\mu^-_{12} \end{cases} \quad \text{(A.7)}$$

One can easily verify, by using the Leibniz formula, that μ satisfies Eq. (A.5). In the reduction $r = \bar{q}$, one has

$$Q = \sigma_1\ \bar{Q}\ \sigma_1, \qquad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad \text{(A.8)}$$

and it is easy to prove that

$$\mu^+(k,x) = \sigma_1\ \overline{\mu^-}(\bar{k},x)\ \sigma_1\ . \quad \text{(A.9)}$$

The set of Volterra equations (A.7) can be mapped into a Riemann-Hilbert problem as follows: μ^+, (respectively μ^-) is holomorphic in the complex upper half plane $\Im(k) >$

0 (respectively in the complex lower half plane $\Im(k)<0$). Let us write

$$\mu=\begin{cases}\mu^{+} & \text{in} \quad \Im(k)>0 \\ \\ \mu^{-} & \text{in} \quad \Im(k)<0\end{cases}. \tag{A.10}$$

The matrix μ is holomorphic every where in the complex plane, except on the real axis, where it is discontinuous. Writing the Riemann-Hilbert problem consists of expressing this discontinuity in terms of μ^+ (or μ^-). To do so, let us compute

$$D(k,x)= \\ \left[\mu_1^{+}(k+\mathrm{i}0,x)-\mu_1^{-}(k-\mathrm{i}0,x)\right]\mathrm{e}^{2\mathrm{i}kx}, \quad x\in\mathbb{R}. \tag{A.11}$$

from the integral equations (A.7) and obtain the following Volterra equations for D_1 and D_2, what constitutes the components of D,

$$\begin{cases} D_1=-\int_x^{+\infty} dx' \; q \; D_2 \; \mathrm{e}^{2\mathrm{i}k(x'-x)} \\ \\ D_2=\alpha^{+}(k)-\int_x^{+\infty} dx' \; \bar{q} \; D_1 \end{cases}. \tag{A.12}$$

Here,

$$\alpha^{+}(k)= \\ \int_{-\infty}^{+\infty} dx' \; \bar{q}(x') \; \mu_{11}^{+}(k,x') \; \mathrm{e}^{-2\mathrm{i}kx'}, \; \Im(k)=0^{+} \; . \tag{A.13}$$

An integral equation having the same Green function as Eq. (A.12) can be obtained readily from Eq. (A.7). Indeed, the vector

$$L=\mu_2^{+}(k+\mathrm{i}0,x)\alpha^{+}(k), \qquad k\in\mathbb{R}. \tag{A.14}$$

is also a solution of Eq. (A.12). General theorems about integral equations allow to prove that Eq. (A.12) possesses only the trivial solution $\alpha=0$. Then, we have $D=L$, which gives the following Riemann-Hilbert problem

$$\mu_1^{+}(k+\mathrm{i}0 \;,\; x)-\mu_1^{-}(k-\mathrm{i}0 \;,\; x)= \\ \alpha^{+}(k) \; \mathrm{e}^{2\mathrm{i}kx} \; \mu_2^{+}(k+\mathrm{i}0 \;,\; x) \; . \tag{A.15}$$

Hence,

$$D'(k,x)= \\ \left[\mu_2^{+}(k+\mathrm{i}0,x)-\mu_2^{-}(k-\mathrm{i}0,x)\right]\mathrm{e}^{-2\mathrm{i}kx}, \quad x\in\mathbb{R}. \tag{A.16}$$

This leads to the second Riemann-Hilbert problem:

$$\mu_2^{+}(k+\mathrm{i}0,x)-\mu_2^{-}(k-\mathrm{i}0,x)= \\ \alpha^{-}(k)\mathrm{e}^{-2\mathrm{i}kx}\mu_2^{-}(k-\mathrm{i}0,x). \tag{A.17}$$

Using Eq. (A.7), we obtain:

$$\alpha^{-}(k)=-\int_{-\infty}^{+\infty} q(x')\mu_{22}^{-}(k,x')\mathrm{e}^{2\mathrm{i}kx'}dx' \\ \equiv -\overline{\alpha^{+}}(k), \quad \Im(k)=0^{-}. \tag{A.18}$$

Eqs. (A.15) and (A.17) can be written as:

$$\mu^{+}-\mu^{-}=(\mu_1^{-},\mu_2^{+})S, \tag{A.19}$$

where

$$S(k,x)= \\ \mathrm{e}^{-\mathrm{i}k\sigma_3 s}\begin{pmatrix}0 & -\overline{\alpha^{+}}(k) \\ \alpha^{+}(k) & 0\end{pmatrix}\mathrm{e}^{\mathrm{i}k\sigma_3 x}, \quad k\in\mathbb{R}. \tag{A.20}$$

The matrix S satisfies the reduction $S(k,x)=-\sigma_1\bar{S}(k,x)\sigma_1$. The function $\alpha^{+}(k)$ is called the reflection coefficient. $\alpha^{+}(k)$ and $q(x)$ are equivalent in the sense that, given $q(x)$, one solves the Volterra equations (referencing Eq. (A.7)) and computes $\alpha^{+}(k)$ through:

$$\alpha^{+}(k)=\lim_{x\to+\infty}\mu_{21}^{+}(k,x)\mathrm{e}^{-2\mathrm{i}kx}. \tag{A.21}$$

This solves the direct problem.

The inverse problem consists of constructing $Q(x)$ from a given function $S(k,x)$ and solving the Riemann-Hilbert problem (A.19) to obtain $\mu(k,x)$. Once μ is obtained, $q(x)$ can be computed as follows: Write the Laurent expansion,

$$\begin{cases}\mu_{11}^{-}=1+\frac{1}{k}\mu_{11}^{-\,(1)}+\frac{1}{k^2}\mu_{11}^{-\,(2)}+\ldots. \\ \\ \mu_{21}^{-}=\frac{1}{k}\mu_{21}^{-\,(1)}+\frac{1}{k^2}\mu_{21}^{-\,(2)}+\ldots.\end{cases}, \tag{A.22}$$

obtained from Eq. (A.7). Then, insert Eq. (A.22) in Eq.(A.5) and use the Liouville theorem to obtain

$$q = 2\mathrm{i}\,\overline{\mu_{21}^{-}}^{(1)}, \tag{A.23}$$

which gives $q(x)$ from $\mu(k,x)$. To solve the Riemann-Hilbert problem, it is convenient to use the "∂-bar problem" formulation:

$$\frac{\partial\mu}{\partial\bar{k}} \doteq \frac{1}{2}\left[\mu(k)\,\delta^{+}(k_I) - \mu(k)\,\delta^{-}(k_I)\right], \tag{A.24}$$

where

$$\frac{\partial}{\partial\bar{k}} = \frac{\mathrm{i}}{2}\left(\frac{\partial}{\partial k_R} + \mathrm{i}\frac{\partial}{\partial k_I}\right), \tag{A.25}$$

with $k = k_R + \mathrm{i}\,k_I$, $\delta^{+} = \delta(k_I - \mathrm{i}0)$, $\delta^{-} = \delta(k_I + \mathrm{i}0)$, where δ is the Dirac distribution. Eq. (A.19) becomes

$$\frac{\partial\mu}{\partial\bar{k}} = \mu\,R, \qquad R = \frac{1}{2}\,S\begin{pmatrix}\delta^{+} & 0\\ 0 & \delta^{-}\end{pmatrix}, \tag{A.26}$$

and the generalized Cauchy formula reads

$$\mu(k,x) = \frac{1}{2\mathrm{i}\pi}\int_{\partial\mathcal{D}}\frac{d\lambda}{\lambda-k}\mu(\lambda,x) + \frac{1}{2\mathrm{i}\pi}\iint_{\mathcal{D}}\frac{d\lambda\wedge d\bar{\lambda}}{\lambda-k}\mu(\lambda,k)R(\lambda,k) \tag{A.27}$$

where $\mathcal{D}$ denotes the complex plane.

The Laurent expansion (A.22), allows for the computation of the first integral in Eq. (A.27) and, taking into account the analytic properties of μ, the second integral of (A.27) is reduced to an integration over the real line. Consequently, for the first column vector μ_1 with the reduction of (A.9), we have

$$\mu_1^{-}(k,x) = \begin{pmatrix}1\\0\end{pmatrix} + \frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{+\infty}\frac{d\lambda}{\lambda-k}\times \sigma_1\,\overline{\mu_1^{-}}(\bar{\lambda},x)\,\alpha^{+}(\lambda)\,\mathrm{e}^{2\mathrm{i}\lambda x}. \tag{A.28}$$

A similar equation holds for the second column vector.

Hence, the inverse problem (construction of $q(x)$ from $\alpha^{+}(k)$) is resolved by solving the Cauchy-Green integral equation and by calculating $q(x)$ from Eq. (A.23). Notice that the preceding formalism remains valid if $q(x)$ is assumed to depend also on a real external parameter t (time). Then, the eigenfunction $\mu(k,x)$ and the spectral transform $R(k,x)$ will also depend on t.

Next, we must construct integral evolutions from a simple choice of a given time dependence of the spectral transform $R(k,x,t)$. We start with the following generic ∂-bar problem

$$\begin{cases}\dfrac{\partial}{\partial\bar{k}}\,\mu(k) = \mu(k)\,R(k)\\[2ex] \mu(k) = 1 + O(\frac{1}{k})\ ,\ \ |k|\to\infty\end{cases}, \tag{A.29}$$

and ask for an (x,t)-dependence of R through the set of equations

$$\begin{cases}R_t = [R\ ,\ \Omega]\\ R_x = [R\ ,\ \Lambda]\end{cases}, \tag{A.30}$$

where Λ and Ω are given distributions of $k\in\mathbb{C}$, and are functions of x and t. It is a simple task to check the following relations

$$\begin{cases}\dfrac{\partial}{\partial\bar{k}}(\mu_x\mu^{-1} - \mu\Lambda\mu^{-1}) = -\mu\dfrac{\partial\Lambda}{\partial\bar{k}}\mu^{-1}\\[2ex] \dfrac{\partial}{\partial\bar{k}}(\mu_t\mu^{-1} - \mu\Omega\mu^{-1}) = -\mu\dfrac{\partial\Omega}{\partial\bar{k}}\mu^{-1}\end{cases}. \tag{A.31}$$

By integrating the above equations, one obtains:

$$\mu_x(k,x,t) = \left[U - \frac{1}{2\pi\mathrm{i}}\iint_{\mathbb{C}}\frac{d\lambda\wedge d\bar{\lambda}}{\lambda-k}\mu(\lambda,x,t)\times\frac{\partial\Lambda(\lambda,x,t)}{\partial\bar{\lambda}}\mu^{-1}(\lambda,x,t)\right]\mu(k,x,t) + \mu(k,x,t)\Lambda(k,x,t), \tag{A.32}$$

$$\mu_t(k,x,t) = \left[V - \frac{1}{2\pi\mathrm{i}}\iint_{\mathbb{C}}\frac{d\lambda\wedge d\bar{\lambda}}{\lambda-k}\mu(\lambda,x,t)\times\frac{\partial\Omega(\lambda,x,t)}{\partial\bar{\lambda}}\mu^{-1}(\lambda,x,t)\right]\mu(k,x,t) + \mu(k,x,t)\Omega(k,x,t). \tag{A.33}$$

The matrices U and V, referred to as the "constants of integration of the $\bar{\partial}$ operator," are given by:

$$\begin{cases} U(k,x,t) = \\ \qquad -\mathrm{Pol}_k\left[\mu(k,x,t)\Lambda(k,x,t)\mu^{-1}(k,x,t)\right], \\ V(k,x,t) = \\ \qquad -\mathrm{Pol}_k\left[\mu(k,x,t)\Omega(k,x,t)\mu^{-1}(k,x,t)\right]. \end{cases} \tag{A.34}$$

Here, $\mathrm{Pol}_k[\cdots]$ denotes the polynomial part in k of the given expression.

To obtain Eq. (A.5), we adopt the following choice:

$$\Lambda = \mathrm{i}k\sigma_3. \tag{A.35}$$

The compatibility condition $\mu_{xt} = \mu_{tx}$, when $\partial\Lambda/\partial\bar{k} = 0$, leads to:

$$U_t(k) - V_x(k) + [U(k), V(k)] = -\frac{1}{2\pi\mathrm{i}}\iint_{\mathbb{C}} \frac{d\lambda\wedge d\bar{\lambda}}{\lambda - k}\times \left[U(\lambda) - U(k), \mu(\lambda)\frac{\partial\Omega(\lambda)}{\partial\bar{\lambda}}\mu^{-1}(\lambda)\right]. \tag{A.36}$$

With the choice:

$$\Omega = \mathrm{i}k^2\sigma_3 + \frac{1}{2\mathrm{i}\pi}\iint_{\mathbb{C}} \frac{d\lambda\wedge d\bar{\lambda}}{\lambda - k} g(\lambda)\sigma_3, \tag{A.37}$$

the time evolution (A.36) becomes:

$$Q_t - \frac{\mathrm{i}}{2}\sigma_3 Q_{xx} + \mathrm{i}\sigma_3 Q^3 = \mathrm{i}\left[\sigma_3, \frac{1}{2\pi\mathrm{i}}\iint_{\mathbb{C}} d\lambda\wedge d\bar{\lambda} g(\lambda)\mu(\lambda)\sigma_3\mu^{-1}(\lambda)\right]. \tag{A.38}$$

This equation describes the evolution in (A.33). The explicit time evolution of the reflection coefficient $\alpha = \alpha^+(k)$ is then:

$$\alpha_t(k_R, t) = 2\mathrm{i}\left[k_R^2 - \frac{\mathrm{i}}{\pi}\iint_{\mathbb{C}} \frac{d\lambda_R d\lambda_I}{\lambda - (k_R - \mathrm{i}0)} g(\lambda)\right]\alpha(k_R, t). \tag{A.39}$$

In the specific case:

$$g(\lambda) = \frac{\mathrm{i}\pi}{6}\delta(\lambda_I)\delta(\lambda_R - k_0)|A(k_R)|, \tag{A.40}$$

we find:

$$\alpha(k,t) = \alpha(k,0)\exp\left[2\mathrm{i}\left(k^2 + \frac{1}{6}\int_{-\infty}^{+\infty}\frac{dk A(k)}{k_0 - k}\right)t\right]. \tag{A.41}$$

Clearly, the function $\alpha(k)$ has an essential singularity at $k = k_0$. Hence, $\alpha(k)$ is not defined for $k = k_0$ unless $a(k_0, 0) \equiv 0$, which is required by the system (A.2). For $q(x,0)$ vanishing at both ends of the x-axis, consistency requires that $a_1\bar{a}_2$ also vanishes. Consequently, using Eqs. (A.17) and (A.23), $\alpha(k)$ must vanish for $k = k_0$. This condition determines the parameter k_0 from $q(x,0)$, as the solution of:

$$\alpha(k_0, 0) = \int_{-\infty}^{+\infty} dx'\, \bar{q}(x')\overline{\mu_{22}^-(k_0, x', 0)}\mathrm{e}^{-\mathrm{i}k_0 x'} = 0. \tag{A.42}$$

This demonstrates that the initial value $q(x,0)$ determines the small correction to the wave number of the scattered ESW.

Appendix B

In this section, we prove the integrability of the system:

$$u_{xx} + \lambda^2 u = qu, \quad q = q(x,t), \quad u = u(\lambda, x, t), \tag{B1a}$$

$$q_t + 6qq_x - q_{xxx} = -\frac{\partial}{\partial x}\iint_{\mathbb{C}} d\lambda\wedge d\bar{\lambda}\, u(\lambda,x,t)u(-\lambda,x,t)\mu(\lambda,t), \tag{B1b}$$

where $\mu(\lambda,t)$ is an arbitrary distribution in $\mathbb{C}$, and $d\lambda\wedge d\bar{\lambda} = -2\mathrm{i}\, d\lambda_R$ for $\lambda = \lambda_R + \mathrm{i}\lambda_I$. These equations are associated with the initial and boundary conditions:

$$q(x,0) = q_0(x), \tag{B2a}$$

$$\lim_{x\to+\infty} u(\lambda,x,t) = d(\lambda,t)e^{-\mathrm{i}\lambda(x-\lambda^2 t)}. \tag{B2b}$$

The integrability of the "truncated system" (27) will be shown as a sub-case of (B1b)-(B2a)-(B2b).

We start by writing the $\bar{\partial}$-problem for the scalar ϕ:

$$\frac{\partial}{\partial\bar{\lambda}}\phi(\lambda) = \phi(-\lambda,t), \quad \lambda \in \mathbb{C}, \tag{B3a}$$

$$\phi(\lambda) = 1 + O(1/\lambda), \quad |\lambda| \to \infty, \tag{B3b}$$

where $r(\lambda)$ is a given distribution in $\mathbb{C}$. We restrict our study to the case where $\phi(\lambda)$ has only simple poles or discontinuities along lines in the λ-plane.

The solution of (B3a) satisfying (B3b) is given by the following integral equation:

$$\phi(\lambda) = 1 + \frac{1}{2i\pi}\iint_{\mathbb{C}} \frac{dl \wedge d\bar{l}}{l-\lambda}\phi(-l)r(l). \tag{B4}$$

This leads to the asymptotic series:

$$\phi(\lambda) = \sum_{j=0}^{n-1} \lambda^{-j}\phi^{(j)} + O(\lambda^{-n}), \quad \phi^{(0)} = 1. \tag{B5}$$

The (x,t)-dependence for ϕ is obtained by requiring the "simplest integrable" (x,t)-dependence for $r(\lambda)$:

$$\frac{\partial}{\partial x}r(\lambda) = [\alpha(\lambda) - \alpha(-\lambda)]r(\lambda), \tag{B6a}$$

$$\frac{\partial}{\partial t}r(\lambda) = [\beta(\lambda) - \beta(-\lambda)]r(\lambda). \tag{B6b}$$

The choice of $\alpha(\lambda) = \mathrm{i}\lambda$ fixes the principal spectral problem. For this choice, the function:

$$\psi(\lambda,x,t) = \phi(\lambda,x,t)e^{-\mathrm{i}\lambda x} \tag{B7}$$

solves the Schrödinger spectral problem for

$$q = -2\mathrm{i}\frac{\partial}{\partial x}\phi^{(1)}(x,t). \tag{B8}$$

The proof of the above statement is given in [19] and relies on a comparison of the asymptotic expansions of $\psi_{xx} + \lambda^2\psi$ and ψ. In our treatment, we assume $r(\lambda)$ is such that the integral equation (B4) has a unique solution.

ψ and u solve the same scalar second-order differential equation. The function u is defined by its asymptotic behavior, while ψ is determined by its behavior in the complex λ-plane. Therefore, to relate ψ and u, one must find the behavior of ψ as $x \to \infty$, which is feasible at least when $q(x,t)$ is piecewise continuous, bounded, and vanishes rapidly as $|x| \to \infty$.

The integral equation (B4) provides a solution (u,q) of the system (B1b) through (B8) and:

$$u(\lambda,x,t) = \psi(\lambda,x,t)d(\lambda,t)e^{\mathrm{i}\lambda^3 t}. \tag{B9}$$

We now relate the dispersion relation $\beta(\lambda)$ to integrable evolution equations in the (x,t)-space. The auxiliary spectral problem is constructed by analyzing the spectral Wronskian:

$$\widehat{W}[\psi(\lambda),F(\lambda)] = \frac{1}{2\mathrm{i}\lambda}[\psi(\lambda)F(-\lambda) - \psi(-\lambda)F(\lambda)] \stackrel{\text{def}}{=} b(\lambda). \tag{B10}$$

When $\beta(\lambda)$ is chosen as:

$$\beta(\lambda) = \mathrm{i}\lambda\sum_{j=0}^{n}\beta_{2j}\lambda^{2j} + \beta_s(\lambda), \tag{B11a}$$

$$\beta_s(\lambda) = O(1/\lambda), \quad |\lambda| \to \infty, \tag{B11b}$$

the well-known Korteweg-de Vries hierarchy of nonlinear evolution equations is recovered in the absence of $\beta_s(\lambda)$.

The nonlinear evolution equations are obtained from the compatibility condition:

$$\left[\frac{\partial^2}{\partial x^2} + \lambda^2 - q, \frac{\partial}{\partial t} - b\frac{\partial}{\partial x} + \frac{b_x}{2}\right]\psi = 0. \tag{B12}$$

This system is integrable by means of the spectral transform.

References

1. F. Calogero, A. Degasperis, *Spectral Transform and Solitons*, Elsevier (2011)
2. M. J. Ablowitz, H. Segur, *Solitons and the Inverse Scattering Transform*, SIAM (1981)
3. V. E. Zakharov, A. B. Shabat, Soviet Phys. JETP (1972)

4. C. S. Gardner, J. M. Greene, M. D. Kruskal, R. M. Miura, Phys. Rev. Lett. (1967)
5. J. Leon, Phys. Rev. A (1990)
6. P. D. Lax, Comm. Pure Appl. Math. (1968)
7. A. S. Fokas, A. Latifi, Open Comm. Nonlin. Math. Phys. **2**, (2022)
8. A. S. Fokas, A. Latifi, Zur. Mat. Fiz. Anal. Geo. (2023)
9. A. S. Fokas, *Recent Advances in Partial Differential Equations; Proc. Sympos. Appl. Math.* American Mathematical Society, (1998)
10. J. Leon, *J. of Math. Phys.* **29**, (1988)
11. A. Latifi, Nonlinearity, **29**, (2016)
12. J. Leon, Phys. Rev. Lett. (1991)
13. J. Leon, A. Latifi, J. Phys. A: Math. and General, (1990)
14. F.E. Chen, *Introduction to Plasma Physics and Controlled Fusion.* 4th ed., Springer, Cham, (2024)
15. M.J. Ablowitz, D.J. Kaup, A.C. Newell, H. Segur, Stud. Appl. Math. (1974)
16. R. Beals, R.R. Coifman, Comm. Pure App. Math. **37**, (1984)
17. D.J. Kaup, Phys. Rev. Lett. **59**, (1987)
18. D.J. Kaup, A.C. Newell, Ad. Math. **31** (1979)
19. M. Jaulent, M., Manna, L.M. Alonso, Inverse Prob. **4**, (1988)