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Hamiltonian Formulation of Scalar Density Field Theory

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Abstract In the ADM formalism, spacetime is foliated by spacelike hypersurfaces, dividing spacetime into space and time. In this formalism, the evolution of fields can be separated into evolution on these hypersurfaces and evolution from one hypersurface to the next. Teitelboim showed that by solving the algebra of the generators under certain assumptions, the Hamiltonian of scalar field, Maxwell's field, and general relativity can be obtained [1]. In this paper, we solve this algebra for scalar density field theory in arbitrary curved spacetime and demonstrate that scalar density fields with weights zero and one are permissible within the Hamiltonian formalism.

1 Introduction

The scalar density field is defined by its properties under a general coordinate transformation:

$$\phi(x') = \phi(x)\left|\frac{\partial x}{\partial x'}\right|^{\lambda}. \tag{1}$$

In this expression,$\left|\frac{\partial x}{\partial x'}\right|$ denotes the Jacobi determinant of the transformation, and x represents a point in "space". We regard the parameter λ as the weight of the scalar density field. By selecting $\lambda = 0$, the scalar field theory will be obtained. The effect of a weighted scalar field is not exclusively significant in curved spacetime. Experiments conducted in particle physics laboratories are all performed in an approximately flat spacetime. For instance, we know that the pseudoscalar pion field acquires a negative sign under a parity transformation, for which the Jacobi determinant is -1 [2]. If we assume that this field is described by equation (1), we find that the pion is a scalar density field with an odd weight. To determine the weight of pseudoscalar fields, one must resort to field theory in curved spacetime.

Spacetime is decomposed into space and time. This decomposition leads to the emergence of the Dirac algebra among the Hamiltonian generators. The diffeomorphism generators can be determined by identifying which class of tensors the fields of the theory belong to. Using the Dirac algebra, the time evolution generators can then be derived.

The only scalar density fields that can satisfy the entire algebra are those with weight one or zero. In Minkowski spacetime, their solutions are the same.

The Lagrangian formulation can be investigated by extending the three-dimensional diffeomorphism framework to the four-dimensional case. By expressing the corresponding terms in the Hamiltonian of scalar density field, it is shown that these terms do not yield the correct Hamiltonian form through a Legendre transformation, except in the case of a weightless scalar field.

2 ADM Decomposition

To describe this approach, we first decompose the four dimensional spacetime with metric $g_{\mu\nu}$ into space plus time. The spacetime is hyperbolic, and on every hyperbolic manifold, there exists a temporal function, such as τ, such that each surface of constant time is a Cauchy surface with metric h_{ij} where $i,j = \{1,2,3\}$ [3]. Therefore, the manifold can be foliated by these Cauchy surfaces. An important characteristic of a hyperbolic manifold is that every null or timelike curve intersects each Cauchy surface exactly once. Hence, τ can be considered as a representation of time, and the Cauchy surfaces can be regarded as representations of space [4]. The spatial part of the metric, h_{ij} can be obtained from the ADM metric [5, 6]:

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$$g_{\mu\nu} = \begin{pmatrix} -N^2 + N^i N_i & N_j \\ N_k & h_{jk} \end{pmatrix}. \tag{2}$$

N is called the lapse function, and N_i is known as the shift function. The upper index is defined using the relation $N^i = h^{ij} N_j$, where h^{ij} is the inverse of the metric with the condition $h_{ik} h^{kj} = \delta_i^j$.

The result of this decomposition is the emergence of Dirac algebra among the Hamiltonian generators of the theory. This algebra also arises from other ideas such as parametrized systems or the principle of path independence [7–9].

The generators are decomposed into one generator normal to the constant-time hypersurface $\mathscr{H}_\perp$ and three generators parallel to the hypersurface $\mathscr{H}_i$, which are also known as generators of infinitesimal diffeomorphisms [6].

We define the weight under spatial coordinate transformations like equation (1). For example, the determinant of the metric h is a scalar density with weight 2.

The Hamiltonian is obtained as a linear combination of these generators using the lapse and shift functions:

$$\mathscr{H} = N \mathscr{H}_\perp + N^i \mathscr{H}_i. \tag{3}$$

The algebra of these generators is as follows [9]:

$$\{\mathscr{H}_\perp(x), \mathscr{H}_\perp(x')\} = \delta_{,i}(x,x') \left(h^{ij}(x) \mathscr{H}_j(x) + h^{ij}(x') \mathscr{H}_j(x') \right). \tag{4}$$

$$\{\mathscr{H}_i(x), \mathscr{H}_\perp(x')\} = \delta_{,i}(x,x') \mathscr{H}_\perp(x). \tag{5}$$

$$\{\mathscr{H}_i(x), \mathscr{H}_j(x')\} = \delta_{,i}(x,x') \mathscr{H}_j(x) + \delta_{,j}(x,x') \mathscr{H}_i(x'). \tag{6}$$

The symbol δ denotes the Dirac delta distribution defined as [9]:

$$\int d^3x \, h^{\frac{1}{2}} \; \delta(x,x') \; F(x) = F(x'). \tag{7}$$

$\delta_{,j} := \frac{\partial \delta}{\partial x^j}$, and $\{,\}$ denotes the Poisson bracket given by

$$\{\phi(x), \pi(x')\} = \delta(x,x'), \tag{8}$$

implying that the weight of π, the conjugate momentum of the scalar density field, equals $1 - \lambda$.

Teitelboim, in his doctoral dissertation [1], has shown that if we decompose $\mathscr{H}_\perp$ into gravitational and matter parts, i.e., $\mathscr{H}_\perp = \mathscr{H}_\perp^{\text{matter}} + \mathscr{H}_\perp^{\text{gravitational}}$, the matter part is ultralocal with respect to the metric, meaning that it does not depend on the derivative of the metric field. This result follows from his first assumption that the matter part does not include the conjugate momentum of the metric. Another result of this assumption is that Dirac algebra (4) is closed for matter part (same as (6)):

$$\{\mathscr{H}_\perp^{\text{matter}}(x), \mathscr{H}_\perp^{\text{matter}}(x')\} = \delta_{,i}(x,x') \left(h^{ij}(x) \mathscr{H}_j^{\text{matter}}(x) + h^{ij}(x') \mathscr{H}_j^{\text{matter}}(x') \right), \tag{9}$$

$$\{\mathscr{H}_i^{\text{matter}}(x), \mathscr{H}_j^{\text{matter}}(x')\} = \delta_{,i}(x,x') \mathscr{H}_j^{\text{matter}}(x) + \delta_{,j}(x,x') \mathscr{H}_i^{\text{matter}}(x'). \tag{10}$$

His second and third assumptions are that $\mathscr{H}_\perp$ is ultralocal with respect to the conjugate momentum of the matter field and is of second order.

Teitelboim demonstrated that the algebra (4)-(6) can be solved as follows. Initially, by using the following rule to calculate the variation of a function of canonical variables such as F:

$$\delta F = \int d^3x \, h^{\frac{1}{2}} \{F, \; \mathscr{H}_\mu\} N^\mu. \tag{11}$$

If F belongs to a class of (weighted) tensor fields, the generators of infinitesimal diffeomorphism can be derived by comparing the result of (11) with its Lie derivative. Then from equations (5) and (6), we find that the weights of $\mathscr{H}_\perp$ and $\mathscr{H}_i$ are equal to 1. Then, from equation (4), $\mathscr{H}_\perp$ will be obtained.

In the next section, we will derive the properties of the Hamiltonian generators of matter for the scalar density field theory.

3 Hamiltonian Generators of Scalar Density Field

3.1 Diffeomorphism Generators

For an infinitesimal spatial diffeomorphism $x^i \to x'^i = x^i + N^i$ we have:

$$\delta\phi(y) = \phi'(y) - \phi(y) = \int d^3x \; \{\phi(y), \; \mathscr{H}_i(x)\} N^i(x). \tag{12}$$

From equation (1) we can find:

$$\delta\phi = \lambda\phi N^j_{,j} + N^j\phi_{,j}. \tag{13}$$

This implies that the generators of diffeomorphism can be expressed in the following form:

$$\mathscr{H}_i^{\text{matter}} = -\lambda\phi\pi_{,i} + (1-\lambda)\phi_{,i}\pi. \tag{14}$$

The transformation of the momentum conjugate to the scalar density field under a diffeomorphism can be investigated:

$$\delta\pi = \int d^3x\, N^i(x)\{\pi,\ \mathscr{H}_i(x)\} = (\pi N^i)_{,i} - \lambda\pi N^i{}_{,i}. \tag{15}$$

which is in a form that we expect from a scalar density with weight $1-\lambda$. Diffeomorphism generators satisfy Dirac algebra (6). The detailed calculation can be found in Appendix B.

3.2 Generator of Time Evolution

For the remainder of this paper, the superscript "matter" for generators will be dropped unless stated otherwise. For a field with zero weight, the generator must reduce to that of the Klein-Gordon field theory. Therefore, in accordance with Teitelboim's third assumption, we express the form of $\mathscr{H}_\perp$ as a second-order function of the matter momentum:

$$\mathscr{H}_\perp = \frac{1}{2}h^{\lambda-\frac{1}{2}}\pi^2 + V. \tag{16}$$

V includes a mass term, interaction terms, and local terms involving derivatives of the field. We assume that V is in the following form:

$$V = A[\phi] + B^i[\phi]\phi_{,i} + C^{ij}[\phi]\phi_{,ij} + D^{ij}[\phi]\phi_{,i}\phi_{,j}. \tag{17}$$

By substituting (16) into the Jacobi identity:

$$\{\{\phi(y),\ \mathscr{H}_\perp(x)\},\ \mathscr{H}_\perp(x')\} - \{\{\phi(y),\ \mathscr{H}_\perp(x')\},\ \mathscr{H}_\perp(x)\} = \{\phi(y),\ \{\mathscr{H}_\perp(x),\ \mathscr{H}_\perp(x')\}\}, \tag{18}$$

we find that our assumptions about the form of $\mathscr{H}_\perp$ could satisfy Jacobi identity. By a similar calculation to Appendix A one could prove that V includes only the first derivative, the second derivative, and the product of two first derivatives of the field. Terms involving other combinations of field derivatives must have zero coefficients.

3.2.1 Covariant Derivative of Scalar Density Field

Dirac algebra (5) forces the $\mathscr{H}_\perp$ to be a scalar density with weight one. Note that (5) is not closed for matter part and the gravitational part must be considered in diffeomorphism generators [1]:

$$\{\mathscr{H}_i^{\text{matter}}(x) + \mathscr{H}_i^{\text{gravitational}}(x), \mathscr{H}_\perp^{\text{matter}}(x')\} = \delta_{,i}(x,x')\mathscr{H}_\perp^{\text{matter}}(x), \tag{19}$$

The gravitational part of diffeomorphism generator is:

$$\mathscr{H}_i^{\text{gravitational}} = h_{jk,i}p^{jk} - 2(p^{jk}h_{ij})_{,k}. \tag{20}$$

p^{ij} is the momentum corresponding to the metric field with definition:

$$\{h_{ij}(x),\ p^{kl}(x')\} = \delta^k_i\delta^l_j\delta(x,\ x'). \tag{21}$$

The partial derivative of a scalar density field does not transform as a weighted tensor under coordinate transformations. Therefore, by defining connection γ_i, we must express the covariant derivative of the field in the following form, which transforms as a vector density with weight λ under coordinate transformations [10]:

$$\nabla_i\phi := \phi_{,i} + \lambda\gamma_i\phi. \tag{22}$$

Vector density is also defined by its properties under a general coordinate transformation:

$$V'_i(x') = V_l(x)\left|\frac{\partial x}{\partial x'}\right|^\lambda \frac{\partial x^l}{\partial x'^i}. \tag{23}$$

Although the connection is not unique, if we assume it depends only on the metric, the only form of the connection, using (5), would be as follows:

$$\gamma_i = -\frac{1}{2}h^{jk}h_{jk,i}. \tag{24}$$

This connection can be descriptively written as a canonical transformation of the scalar field ψ with zero weight:

$$g^{\lambda/2}\psi_{,i}=(\psi g^{\lambda/2})_{,i}+\lambda\left[-\frac{1}{2}g^{kj}g_{kj,i}\right](\psi g^{\lambda/2}). \tag{25}$$

In this case, the canonical transformation is defined as $\psi = \phi h^{\frac{-\lambda}{2}}$ and $\pi_\psi = \pi h^{\frac{\lambda}{2}}$. This choice of degrees of freedom preserves the Poisson bracket algebra. It is evident that applying this covariant derivative violates Teitelboim's first assumption, as $\mathscr{H}_\perp$ is not ultralocal with respect to the metric. Any other scalar density with weight λ such as ω (excluding ϕ), could be used to define a vector density of the form:

$$\nabla_i^\omega\phi = \phi_{,i} - (ln(\omega))_{,i}. \tag{26}$$

For example we could use determinant of momentum conjugate to metric p:

$$\nabla_i^p\phi = \phi_{,i} - p^{\frac{\lambda}{2}}(p^{\frac{-\lambda}{2}})_{,i}. \tag{27}$$

Which is also a canonical transformation from scalar field in the form $\psi = \phi p^{\frac{\lambda}{2}}$ and $\pi_\psi = \pi p^{-\frac{\lambda}{2}}$. But this connection is not consistent with first assumption of Teitelboim. we could also find an appropriate connection using momentum conjugate to scalar density field as:

$$\nabla_i^\pi\phi = \phi_{,i} - \frac{\lambda}{1-\lambda}(ln(\pi))_{,i}, \tag{28}$$

which is a canonical transformation in the form $\psi = \phi\pi^{\frac{-\lambda}{1-\lambda}}$ and $\pi_\psi = (1-\lambda)\pi^{\frac{1}{1-\lambda}}$. This connection violates Teitelboim's second assumption.

3.2.2 Calculating $\mathscr{H}_\perp$

If we write $\mathscr{H}_\perp$ using a connection of the form equation (24), by applying the Jacobi identity from equation (18) for a second-order Hamiltonian in terms of momenta similar to (16), we have:

$$\begin{aligned}\mathscr{H}_\perp = {} & \frac{1}{2}h^{\lambda-\frac{1}{2}}\pi^2 + \sum_n m_n h^{\frac{1}{2}-\lambda n}\phi^{2n} \\ & + B^i[\phi]\nabla_i\phi + C^{ij}[\phi]\nabla_i\nabla_j\phi + D^{ij}[\phi]\nabla_i\phi\nabla_j\phi.\end{aligned} \tag{29}$$

The parameters m_n are numerical constants. Since odd powers of the field would lead to the absence of a lower energy bound, we have discarded them. The coefficients B^i, C^{ij}, D^{ij} are ultralocal with respect to the matter field and are given by the following relations (detailed calculation can be found in Appendix A):

$$D^{ij}[\phi] = (\frac{1}{2}-\lambda)h^{ij}h^{\frac{1}{2}-\lambda}. \tag{30}$$

$$C^{ij}[\phi] = -\lambda\phi h^{ij}h^{\frac{1}{2}-\lambda}. \tag{31}$$

$$B^i[\phi] = (1-\lambda)\phi h^{\frac{1}{2}}h^{ij}(h^{-\lambda})_{,j}. \tag{32}$$

The solution (29), when combined with the properties (30), (31), and (32), as well as the Jacobi identity (18), also satisfies (4). It is necessary to check for weight of this solution again. Terms (30) and (31) satisfy equation (5) but (32) contains the partial derivative of the metric's determinant which is not a tensor. Therefore, we are forced to choose only $\lambda = \{0, 1\}$. It is evident that for a scalar field with $\lambda = 0$, the generator reduces to that of the Klein-Gordon field:

$$\mathscr{H}_\perp = \frac{1}{2}h^{-\frac{1}{2}}\pi^2 + \frac{1}{2}h^{ij}h^{\frac{1}{2}}\phi_{,i}\phi_{,j} + \sum_n m_n h^{\frac{1}{2}}\phi^{2n}. \tag{33}$$

And for $\lambda = 1$ we have:

$$\begin{aligned}\mathscr{H}_\perp = {} & \frac{1}{2}h^{\frac{1}{2}}\pi^2 - \frac{1}{2}h^{ij}h^{-\frac{1}{2}}\nabla_i\phi\nabla_j\phi \\ & + \sum_n m_n h^{\frac{1}{2}-n}\phi^{2n} - \phi h^{ij}h^{-\frac{1}{2}}\nabla_i\nabla_j\phi.\end{aligned} \tag{34}$$

4 Lagrangian Formulation

In the previous section, within the Hamiltonian formulation, the coordinate transformations and the definition of scalar density were established using equation (1) and in three-dimensional space. However, in studying the Lagrangian formulation, we must consider them within the framework of spacetime transformations [11]:

$$\phi(x') = \phi(x)\left|\frac{\partial x}{\partial x'}\right|^w. \tag{35}$$

Here, x denotes a spacetime coordinate, and w is the weight of scalar density field. In the Lagrangian formulation, the kinetic term is expressed as follows:

$$\mathscr{L} = \frac{1}{2} g^{\frac{1}{2}-w} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi. \tag{36}$$

Where the covariant derivative similar to (24) is:

$$\nabla_\mu \phi := \phi_{,\mu} - \frac{1}{2} w (ln(g))_{,\mu} \phi, \tag{37}$$

which is a vector density with weight w. g^w in equation (36) is added to ensure that the action is scalar. By performing a Legendre transformation, the Hamiltonian is obtained:

$$\begin{aligned}\mathscr{H} = -\frac{1}{2} N^{2w+1} h^{w-\frac{1}{2}} \pi^2 + \frac{w}{2} \phi \pi (ln(hN^2))_{,0} + \pi \nabla_\mu \phi N^i \\ - \frac{1}{2} N^{1-2w} h^{\frac{1}{2}-w} h^{ij} \nabla_i \phi \nabla_j \phi.\end{aligned} \tag{38}$$

Here we have used the metric (2), and the canonical momentum is expressed as:

$$\pi := \frac{\partial \mathscr{L}}{\partial \phi_{,0}} = g^{\mu 0} \nabla_\mu \phi g^{\frac{1}{2}-w}. \tag{39}$$

This Hamiltonian differs from (3) unless for $w = 0$, in which case it reduces exactly to the Klein-Gordon field.

5 Conclusions and Discussion

A weighted scalar field exhibits behavior distinct from the Klein-Gordon field, which can be used to describe the differences between scalar and pseudo-scalar fields under parity transformation. In this paper, by analyzing the solution of the Dirac algebra for the scalar density field within the Hamiltonian formulation, we investigated its properties, including its dependence on metric derivatives, which is inconsistent with Teitelboim's first assumption. We demonstrated that the Hamiltonian is constructed from combinations of field derivatives that differ from those of the Klein-Gordon theory. In the Hamiltonian formalism, scalar density fields with weight one and zero are permissible. Furthermore, this Hamiltonian cannot be derived via a Legendre transformation from the corresponding Lagrangian. However, for a scalar field with zero weight, all such inconsistencies vanish, and the generators of the Klein-Gordon field theory are recovered.

The approach presented in this paper is not the only method for studying the theory of scalar density fields. For example, appropriate connections were derived from the combination of the conjugate momentum of the metric and the conjugate momentum of the scalar density field. Using these connections would introduce new properties to the theory. However, they violate Teitelboim's first and second assumptions, respectively. Moreover, the use of h to adjust the weight of terms in the Hamiltonian and Lagrangian is not a critical choice. Employing other weighted tensors could potentially resolve some of the challenges in this theory.

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Appendix A: Calculation of Time Evolution Generators

In this section, we provide detailed calculation of equations (30), (31), (32). If we put equation (29) in Dirac algebra (9) the LHS is:

$$\begin{aligned}&\{\mathscr{H}_\perp(x), \mathscr{H}_\perp(x')\} = \\ &\int d^3y (\pi(x') h^{\lambda-\frac{1}{2}}(x') \delta(x',y) \big[B^i(x) \delta_{,i}(x,y) \\ &+ 2D^{ij}(x) \nabla_i \phi(x) \delta_{,i}(x,y) \\ &+ C^{ij}(x)(\delta_{,ij} + 2\gamma_j(x) \delta_{,i}(x,y) - \Gamma^k_{ij}(x) \delta_{,k}(x,y))\big] - x \leftrightarrow x') \\ &= \pi(x) h^{\lambda-\frac{1}{2}(x)} \big[B^i(x') \delta_{,i}(x,x') + 2D^{ij}(x') \nabla_j \phi(x') \delta_{,i}(x,x') \\ &- C^{ij}(x') \delta_{,ij}(x,x') + 2C^{ij}(x') \gamma_j(x') \delta_{,i}(x,x') \\ &\qquad - \Gamma^k_{ij}(x') \delta_{,k}(x,x') C^{ij}(x')\big] - x \leftrightarrow x'.\end{aligned} \tag{A.1}$$

Where Γ^i_{jk} is the Christoffel symbol corresponding to h^{ij}. Note that $\nabla_i \nabla_j \phi$ is calculated as:

$$\nabla_i \nabla_j \phi = (\nabla_j \phi)_{,i} + \lambda \gamma_i \nabla_j \phi - \Gamma^k_{ij} \nabla_k \phi. \tag{A.2}$$

Using arbitrary test functions $N(x)$, $N'(x')$ we can integrate over x, x':

$$\begin{aligned}&\int d^3x d^3x' \; N(x) N'(x') \{\mathscr{H}_\perp(x), \mathscr{H}_\perp(x')\} \\ &= \int d^3x \big[- (N \pi h^{\lambda-\frac{1}{2}})_{,i} \big[N' B^i + 2N' D^{ij} \nabla_j \phi + 2C^{ij} \gamma_j N' \\ &\qquad - N' \Gamma^i_{jk} C^{jk}\big] - (N \pi h^{\lambda-\frac{1}{2}})_{,ij} N' C^{ij} - N \leftrightarrow N' \big].\end{aligned} \tag{A.3}$$

After simplifying symmetric terms under interchange of x and x', we have:

$$
\begin{aligned}
&\int d^3x d^3x'\, N(x)N'(x')\{\mathscr{H}_\perp(x),\mathscr{H}_\perp(x')\} \\
&= \int d^3x(NN'_{,i} - N'N_{,i})\big[h^{\lambda-\frac{1}{2}}\pi B^i + 2\pi h^{\lambda-\frac{1}{2}} D^{ij}\nabla_j\phi \\
&+ 2C^{ij}\gamma_j\pi h^{\lambda-\frac{1}{2}} - \Gamma^i_{jk}\pi h^{\lambda-\frac{1}{2}}C^{jk} - 2\pi h^{\lambda-\frac{1}{2}}C^{ij}_{,j} \\
&\qquad + \big(C^{ij}\pi h^{\lambda-\frac{1}{2}}\big)_{,j}\big]. \qquad (A.4)
\end{aligned}
$$

Momentum and its derivatives are independent in phase space; therefore, we separate terms depending on momentum P^i and parts depending on derivative of momentum Q^{ij}.

$$
\begin{aligned}
&\int d^3x d^3x'\, N(x)N'(x')\{\mathscr{H}_\perp(x),\mathscr{H}_\perp(x')\} \\
&\qquad = \int d^3x(NN'_{,i} - N'N_{,i})[\pi P^i + \pi_{,i}Q^{ij}] \qquad (A.5)
\end{aligned}
$$

P^i and Q^{ij} are:

$$
Q^{ij} = C^{ij}h^{\lambda-\frac{1}{2}}. \qquad (A.6)
$$

$$
\begin{aligned}
P^i &= h^{\lambda-\frac{1}{2}}\pi B^i + 2\pi h^{\lambda-\frac{1}{2}}D^{ij}\nabla_j\phi + 2C^{ij}\gamma_j\pi h^{\lambda-\frac{1}{2}} \\
&\quad - \Gamma^i_{jk}\pi h^{\lambda-\frac{1}{2}}C^{jk} - 2\pi h^{\lambda-\frac{1}{2}}C^{ij}_{,j} + (C^{ij}h^{\lambda-\frac{1}{2}})_{,j}. \qquad (A.7)
\end{aligned}
$$

Similar calculation could be done on the RHS of (9):

$$
\begin{aligned}
&\int d^3x d^3x'\, N(x)N'(x')\delta_{,i}\big(\mathscr{H}_i(x)h^{ij}(x) + \mathscr{H}_i(x')h^{ij}(x')\big) \\
&= \int d^3x(NN'_{,i} - N'N_{,i})h^{ij}\big[-\lambda\phi\pi_{,i} + (1-\lambda)\pi\phi_{,i}\big]. \qquad (A.8)
\end{aligned}
$$

By equating (A.4) and (A.8) terms with derivative of momentum π must cancel each other so we have:

$$
C^{ij} = -\lambda\phi h^{ij}h^{\frac{1}{2}-\lambda}. \qquad (A.9)
$$

From the terms with derivative of field ϕ in (A.7) one can find:

$$
D^{ij} = (\frac{1}{2}-\lambda)h^{ij}h^{\frac{1}{2}-\lambda}, \qquad (A.10)
$$

and from the remaining terms:

$$
\begin{aligned}
0 &= B^i h^{\frac{1}{2}-\lambda} + 2(\frac{1}{2}-\lambda)h^{ij}\lambda\gamma_j\phi - 2\lambda h^i j\phi\gamma_j + \Gamma^i_{jk}\lambda\phi h^{kj} \\
&\quad + h^{-\frac{1}{2}+\lambda}\lambda\phi(h^{ij}h^{\frac{1}{2}-\lambda})_{,j} - \lambda\phi h^{ij}h^{\frac{1}{2}-\lambda}(h^{-\frac{1}{2}+\lambda})_{,j}. \qquad (A.11)
\end{aligned}
$$

After some simplification we obtain:

$$
B^i[\phi] = (1-\lambda)\phi h^{\frac{1}{2}}h^{ij}(h^{-\lambda})_{,j}. \qquad (A.12)
$$

Appendix B: Calculation of Diffeomorphism Generators

Diffeomorphism generators must satisfy Dirac algebra (5). If we put (14) in this algebra, we obtain:

$$
\begin{aligned}
&\{\mathscr{H}_i(x),\ \mathscr{H}_j(x')\} = \\
&\int d^3y\big[((1-\lambda)\pi(x)\delta_{,i}(x,y) - \lambda\delta(x,y)\pi(x)_{,i}) \\
&((1-\lambda)\delta(x',y)\phi_{,j}(x') \\
&-\lambda\phi(x')\frac{\partial\delta(x',y)}{\partial x'^j}) - ((1-\lambda)\delta(x,y)\phi_{,i}(x) - \lambda\phi(x)\delta_{,i}) \\
&\qquad ((1-\lambda)\pi(x')\frac{\partial\delta(x',y)}{\partial x'^j} - \lambda\delta(x',y)\pi_{,j}(x'))\big]. \qquad (B.13)
\end{aligned}
$$

By integrating using arbitrary test functions $N(x)$ and $N'(x')$, we have:

$$
\begin{aligned}
&\int dxdx'N(x)N'(x')\ \{\mathscr{H}_i(x),\ \mathscr{H}_j(x')\} = \\
&\int d^3x\big(-NN'(1-\lambda)^2\phi_{,j}\pi_{,i} - N'N_{,i}(1-\lambda)^2\pi\phi_{,j} \\
&-\lambda(1-\lambda)N'N\phi_{,j}\phi_{,i} + \lambda^2N'\phi[N_{,j}\pi_{,i} + N\pi_{,ij}] \\
&+\lambda(1-\lambda)N'\phi[\pi N]_{,ij} + NN'\lambda(1-\lambda)\phi_{,i}\pi_{,j} \\
&+\lambda^2N'\pi_{,j}(N\phi)_{,i} - (1-\lambda)^2N'\pi(N\phi_{,i})_{,j} \\
&\qquad -\lambda(1-\lambda)\pi N'[N\phi]_{,ij}\big). \qquad (B.14)
\end{aligned}
$$

After simplifying, we obtain:

$$
\begin{aligned}
&\int dxdx'N(x)N'(x')\ \{\mathscr{H}_i(x),\ \mathscr{H}_j(x')\} \\
&= \int dxdx'N(x)N'(x')\ \big(\mathscr{H}_i(x')\delta_{,j}(x,x') + \mathscr{H}_j(x)\delta_{,i}(x,x')\big). \\
&\qquad (B.15)
\end{aligned}
$$

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