TTMP vol 2 (1), 2025
doi: 10.30511/ttmp.2025.2043936.1043

Transactions in Theoretical and Mathematical Physics

Controllability of fractional stochastic neutral integro-differential equations with state-dependent delay in Frechet spaces

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Received: 21 October 2024 / Accepted: 25 February 2025 / Published: 27 February 2025

Abstract This paper investigates the existence and controllability of state-dependent delay functionally neutral stochastic fractional integro-differential equations within Fréchet spaces. By employing fixed-point techniques, the properties of solution operators, and tools from fractional calculus, we derive the conditions under which these systems are controllable. Our results extend the theoretical framework of fractional differential equations and provide new insights into the behavior of stochastic systems with memory effects. The findings have broad implications for various scientific fields, particularly those that model systems exhibiting complex dynamics, such as physics, biology, and engineering. To illustrate the practical application of our theoretical results, we provide a detailed example. This research contributes to the deeper understanding of fractional stochastic systems and lays the groundwork for future studies in the area of controlled fractional systems.

1 Introduction

Fractional order systems are helpful for concentrating the amazing behaviour of dynamical systems in a variety of scientific and engineering disciplines. The fractional differential equations offer a unique framework for debating the characteristics of real physical materials, such polymers. Recent research has shown that fractional order derivative-based differential models can be used to mathematically represent systems and processes in a variety of disciplines, including physics, chemistry, electrodynamics of complex media, polymer rheology, and aerodynamics. We recommend the monographs of Kilbas [1], Miller [2], Podlubny [3] and Zhou [4] as well as the works [5–12] and the references given therein to the readers for additional information. A specific kind of stochastic functional differential equations are stochastic differential equations with delay. Numerous academics have investigated the existence of outcomes for stochastic fractional differential equations with infinite delay and state-dependent delay (see [9, 10, 14, 15]). Many biological and physical applications of delay differential equations need us to think about variable or state-dependent delays. Controllability, one of the key concepts in mathematical control theory, is crucial to both deterministic and stochastic control theory.

Additionally, a number of authors have reported on the controllability of fractional differential equations and inclusions [6, 9, 12, 16–19]. Additionally, Benchohra et al. [20] evaluated the findings for fractional-order integro-differential inclusions in Frechet spaces in terms of existence and controllability. The approximate controllability of Hilfer fractional neutral integro-differential inclusions using almost sectorial operators has been explored, along with studies on impulsive ψ-Caputo fractional integrodifferential equations with boundary conditions. Additionally, recent research looked into the possibility of fractional neutral integro-differential inclusions in Fréchet spaces with state-dependent delay [11, 15, 21–24].

The following fractional order neutral integro-differential equations with state-dependent model delay are taken into consideration.

$$
\begin{aligned}
dD(\hbar,\zeta_{\sigma(\hbar,\zeta_\hbar)}) &= \int_0^\hbar \frac{(\hbar-\rho)^{\rho-2}}{\Gamma(\rho-1)} AD(\rho,\zeta_{\sigma(\rho,\zeta_\rho)})d\rho d\hbar \\
&\quad + F\left(\hbar,\zeta_{\sigma(\hbar,\zeta_\hbar)},\int_0^\hbar a(\hbar,\rho,\zeta_{\sigma(\rho,\zeta_\rho)})d\rho\right)dW(\hbar), \\
&\quad \hbar\in I=[0,1), \qquad (1)
\end{aligned}
$$

$$
\zeta_0 = \nu \in \Bbbk. \qquad (2)
$$

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Here, Λ is a real separable Hilbert spaces, the state variable $\zeta(\cdot) \in \Lambda$ with norm $\|\cdot\|_\Lambda$ and inner product $(\cdot,\cdot)$, where $D(\hbar,\nu) = \nu(0) - G(\hbar,\nu)$, $1 < \rho < 2$, $G : I \times \Bbbk \to \Lambda$, $F : I \times \Bbbk \times \Lambda \to L_\varphi(\Xi,\Lambda)$ and $\sigma : I \times \Bbbk \to (-\infty,T]$ is a continuous function, $A : D(A) \subset \Lambda \to \Lambda$ is a linear densely defined operator of sectorial type on Λ.

Suppose $\{W(\hbar)\}_{\hbar\geq 0}$ is a given Ξ- valued Wiener process or Brownian motion defined on a complete probability space $(\mho_i,\mathfrak{F},P)$ with a finite trace nuclear covariance operator $\varphi > 0$ equipped with a normal filtration $\{\mathfrak{F}_\hbar\}_{\hbar\geq 0}$, which is generated by the Wiener process W.

We are also employing the norm $\|\cdot\|$ and $L(\Xi,\Lambda) : \Xi \to \Lambda$ denotes the space of all bounded linear operators. The time history $\zeta_\hbar : (-\infty,0] \to \Lambda$, $\zeta(\hbar+\vartheta), \vartheta \leq 0$, belongs to an axiomatically specified abstract phase space ($\Bbbk$). The initial data $\nu(\hbar) : -\infty < \hbar \leq 0$ is a random variable $\Bbbk$-valued, $\mathfrak{F}_0$-adapted and F, G, and σ are functions subject to further restrictions.

2 Preliminaries

Let $(\mho,\mathfrak{F},P,\mathbb{F})(\mathbb{F} = \{\mathfrak{F}_\hbar\}_{\hbar\geq 0})$ be a complete filtered probability space that satisfies the condition that all P-null sets of $\mathfrak{F}$ are contained in $\mathfrak{F}_0$. The stochastic process $S = \{\zeta(\hbar,W) : \mho \to \Lambda | \hbar \in I\}$ is a collection of random variables. An Λ-valued random variable is a $\mathfrak{F}$-measurable function $\zeta(\hbar) : \mho \to \Lambda$.

Generally, $\zeta(\hbar) : I \to \Lambda$ instead of $\zeta(\hbar,W)$ in the space of S. Let Ξ have a complete orthonormal basis with $\{e_i\}_{i=1}^\infty$. Denote $Tr(\varphi) = \sum_{i=1}^\infty \xi_i = \xi < 1$, which satisfies that $\varphi e_i = \xi_i e_i$. Assume that $W(\hbar) : \hbar \geq 0$ is a cylindrical Ξ-valued Wiener process with a finite. Therefore,

$$W(\hbar) = \sum_{i=1}^\infty \sqrt{\xi_i} W_i(\hbar) e_i, \tag{3}$$

where $\{W_i(\hbar)\}_{i=1}^\infty$ are truly mutually independent one dimensional standard Wiener processes. We suppose that the σ-algebra produced by W is $\mathfrak{F}_\hbar = \sigma\{W(\rho) : 0 \leq \rho \leq \hbar\}$. For $c_1, c_2 \in L(\Xi,\Lambda)$, we set

$$(c_1,c_2) = \mathrm{Tr}(c_1 \varphi c_2^*), \tag{4}$$

where c_2^* is the adjoint of c_2, and $\varphi \in L_\ell^+(\Xi)$ is the space of positive nuclear operators in Ξ.

For $\Psi \in L(\Xi,\Lambda)$, we set

$$\|\Psi\|_\varphi^2 = \mathrm{Tr}(\Psi\varphi\Psi^*) = \sum_{i=1}^\infty \|\sqrt{\xi_i}\Psi e_i\|^2. \tag{5}$$

If $\|\Psi\|_\varphi < \infty$, then Ψ is called a φ-Hilbert-Schmidt operator.

The space of all φ-Hilbert-Schmidt operators is denoted by $L_\varphi(\Xi,\Lambda)$. The completion $L_\varphi(\Xi,\Lambda)$ of $L(\Xi,\Lambda)$ in terms of the topology caused by the norm $\|\cdot\|_\varphi$, where $\|\Psi\|_\varphi^2 = (\Psi,\Psi)$ is a Hilbert space with the topology caused by the aforementioned norm [25].

The abstract phase space $\Bbbk$ is now on display. Assume that the phase space $(\Bbbk, \|\cdot\|_\Bbbk)$ satisfies the following basic axioms and is a semi-normed linear space of functions translating $(-\infty,0]$ into Λ [26].

(A) If $\zeta : (-\infty,T] \to \Lambda$, $T > 0$ is continuous on I and $\zeta_0 \in \Bbbk$, then

(A1) $\zeta_\hbar \in \Bbbk, \ \forall\, \hbar \in I$

(A2) $\|\zeta(\hbar)\| \leq \tilde{\Lambda}\|\zeta_\hbar\|_\Bbbk, \ \forall\, \hbar \in I$

(A3) $\|\zeta_\hbar\|_\Bbbk \leq \Xi(\hbar) \sup_{0\leq\rho\leq\hbar} |\zeta(\rho)| + \Theta(\hbar)\|\zeta_0\|_\Bbbk, \ \forall\, \hbar \in I$, where $\Xi, \Theta : [0,+\infty) \to [1,+\infty)$, Θ is locally bounded and $\tilde{\Lambda} \geq 0$- constant, Ξ is continuous, $\tilde{\Lambda}, K, \Theta$- independent of $\zeta(\cdot)$.

(B) $\zeta_\hbar$ is a $\Bbbk$- valued continuous functions on $[0,T]$, for the function $\zeta(\cdot)$ in (A),

(C) The space $\Bbbk$ is complete.

Definition 1 Assume that the domain $D(A)$ with a closed linear operator A in a Hilbert space Λ. In this case, A is the generator of a solution operator if and only if $\mu \in R$ and a strongly continuous function $S_\rho : R^+ \to L(\Lambda)$ exist, and if and only if $\{\xi^\rho : Re(\xi) > \mu\} \subset k(A)$, and if and only if $\xi^{\rho-1}(\xi^\rho - A)^{-1}\zeta = \int_0^\infty e^{\xi\hbar} S_\rho(\hbar) d\hbar$, $Re(\xi) > \mu$, $\zeta \in \Lambda$. In this instance, the solution operator produced by A is denoted as $S_\rho(\cdot)$.

The solution operator is characterised by

$$S_\rho(\hbar) = \frac{1}{2\pi} \int_\Sigma e^{-\xi\hbar} \xi^{\rho-1} (\xi^\rho - A)^{-1} d\xi. \tag{6}$$

where $0 < \vartheta < \pi\left(1 - \frac{\rho}{2}\right)$, A is sectorial type of μ and $\sum$ is a suitable path lying outside the sector $\mu + S_\rho$.

Cuesta [6] has proved that, there is $\chi > 0$, we have

$$\|S_\rho(\hbar)\| \leq \frac{\chi\Theta_0}{1+|\mu|\hbar^\rho}, \quad \hbar \geq 0. \tag{7}$$

where A is a sectorial operator of type $\mu < 0$, for some $\Theta_0 > 0$ and $0 < \vartheta < \pi\left(1 - \frac{\rho}{2}\right)$.

Let $\Bbbk(\Lambda):\Lambda\to\Lambda$ be the space of all bounded linear operators, $\chi:[0,+\infty)\to\Lambda$ be the space of continuous functions and the norm $\|N\|=\sup\{\|N(\zeta)\|:\ \|\zeta\|=1\}$.

Bochner integrable is a measurable function $\zeta:[0,+\infty)\to\Lambda$, if $\|\zeta\|$ is Lebesgue integrable [39]. Let $L^1([0,+\infty),\Lambda)$ be the space of Bochner integrable measurable functions $\zeta:[0,+\infty)\to\Lambda$ with the norm

$$\|\zeta\|_{L^1}=\int_0^\infty\|\zeta(\hbar)\|d\hbar,\quad \text{for all}\quad \zeta\in L^1(I,\Lambda). \tag{8}$$

Recognize the space

$$\Bbbk_{+\infty}=\Big\{\zeta:(-\infty,+\infty)\to\Lambda\ |\quad \zeta|_I\in\chi_{\mathfrak{F}_\hbar}(I,\Lambda),\ \zeta_0\in L_0^2(\mho,\Lambda)\Big\}. \tag{9}$$

Let a family of semi-norms $\{\|\cdot\|_\ell\}_{\ell\in N}$ in a Frechet space Υ and $Y\subset\Upsilon$. We say that F is bounded if

$$\|z\|_\ell\le\overline{\Theta}_\ell,\quad \overline{\Theta}_\ell>0,\quad \text{for all } z\in Y. \tag{10}$$

for every $\ell\in N$.

Let Υ be a sequence of Banach spaces $\{(\Upsilon^\ell,\|\cdot\|_\ell)\}$ as follows:

1. The equivalence relation $\sim_\ell$ defined by $\zeta\sim_\ell z\ \Leftrightarrow\ \|\zeta-z\|_\ell=0$, for all $\zeta,z\in\Upsilon,\ell\in N$.
2. The completion of the quotient space $\Upsilon^\ell=(\Upsilon|_{\sim_\ell},\|\cdot\|_\ell)$ associated with $\|\cdot\|_\ell$.

   Let $Y\subset\Upsilon$, and a sequence $\{Y^\ell\}\subset\Upsilon^\ell$. Then

1. $[\zeta]_\ell$ be the equivalence class of $\zeta\subset\Upsilon^\ell$ and $Y^\ell=\{[\zeta]_\ell:\zeta\in Y\}$, $\forall\ \zeta\in\Upsilon$
2. We denote the closure (Y^ℓ), the interior $(int_\ell(Y^\ell))$ and boundary $(\partial_\ell Y^\ell)$ of Y^ℓ with respect to $\|\cdot\|_\ell\in\Upsilon^\ell$.

If $\{\|\cdot\|_\ell\}$ be the family of semi-norms, then

$$\|\zeta\|_1\le\|\zeta\|_2\le\|\zeta\|_3\le\cdots\ \text{for every } \zeta\in\Upsilon.$$

Definition 2 A function $F:I\times\Bbbk\times\Lambda\to L_\varphi(\Xi,\Lambda)$ is said to be an L^2- Caratheodory function if

(i) the function $F(\hbar,\cdot,\cdot):\Bbbk\times\Lambda\to L_\varphi(\Xi,\Lambda)$ is continuous, for each $\hbar\in I$

(ii) the function $F(\cdot,\zeta,z):I\to L_\varphi(\Xi,\Lambda)$ is $\mathfrak{F}_\hbar$-measurable, for each $\zeta\in\Bbbk$ and $z\in\Lambda$

(iii) if $F_k\in L^1_{loc}(I,R^+)$ exists, then

$$\aleph\|F(\hbar,\zeta,z)\|^2\le F_k(\hbar),\ \ \forall\ \aleph\|\zeta\|^2\le k\ \&\ \aleph\|z\|^2\le k,$$

for every positive integer k and for almost all $\hbar\in I$.

Lemma 1 *Let $\zeta:(-\infty,\ell]\to\Lambda$ be an $\mathfrak{F}_\hbar$-adapted measurable process such that $\zeta|_{I\in\Bbbk_{+\infty}}$ and $\mathfrak{F}_0$-adapted process $\zeta_0=\nu(\hbar)\in L_0^2(\mho,B)$. Then*

$$\|\zeta_\rho\|_\Bbbk\le\Theta_\ell\aleph\|\nu\|_\Bbbk+\Xi_\ell\sup_{0\le\rho\le n}\aleph\|\zeta(\rho)\|. \tag{11}$$

Lemma 2 *Let Y be a closed subset of a Frechet space Υ and $N:Y\to\Upsilon$ be a contraction such that $N(Y)$ is bounded. Then*

(a) N has a fixed point

(b) $\zeta\in\partial_\ell Y^\ell$ and $\xi\in[0,1),\ell\in N$ exists such that $\|\zeta-\xi N(\zeta)\|_\ell=0$.

3 Main results

Definition 3 An $\mathfrak{F}_\hbar$-adapted stochastic process $\zeta:(-\infty,+\infty)\to\Lambda$ is called a mild solution of the system (1) - (2) if the restriction of $\zeta(\cdot)$ to the interval I is continuous,

$$\zeta_0=\nu(\hbar),\quad \zeta_{\sigma(\rho,\zeta_\rho)}\in\Bbbk. \tag{12}$$

satisfying $\zeta_0\in L_0^2(\mho,\Lambda)$, and

$$\begin{aligned}\zeta(\hbar)&=S_\rho(\hbar)[\nu(0)-G(0,\nu)]+G(\hbar,\zeta_{\sigma(\hbar,\zeta_\hbar)})\\&\quad+\int_0^\hbar S_\rho(\hbar-\rho)F\Bigg(\rho,\zeta_{\sigma(\rho,\zeta_\rho)},\\&\quad\int_0^\rho a(\rho,\tau,\zeta_{\varsigma(\tau,\zeta_\tau)})d\tau\Bigg)dW(\rho),\qquad \hbar\in I.\end{aligned} \tag{13}$$

Let $\sigma:[0,\ell]\times\Bbbk\to(-\infty,\ell]$ is continuous. Then

(H1) $\Theta>0$ exists, and $\|S_\rho(\hbar)\|^2\le\Theta$, for all $\quad\hbar\ge0$.

(H2) There exists a bounded and continuous function $J^\nu:R(\sigma^-)\to(0,1)$ and the function $\hbar\to\nu_\hbar$ is continuous from $R(\sigma^-)=\{p(\rho,\Psi)\le0,\ (\rho,\Psi)\in[0,\ell]\times B\}$ into $\Bbbk$ such that $\|\nu_\hbar\|\le J^\nu(\hbar)\|\nu\|_\Bbbk$ for each $\hbar\in R(\sigma^-)$.

(H3) There exist a constants $\chi_1 \geq 0$, and $\chi_2 > 0$ such that

$$\aleph \|G(\hbar,\zeta)\|^2 \leq \chi_1 \|\zeta\|_{\Bbbk}^2 + \chi_2, \text{ for } \hbar \in I,\ \zeta \in \Bbbk. \quad (14)$$

(H4) There exists $\Gamma_\ell \in L^1_{loc}(I, R^+)$, $\ell > 0$ such that

$$\aleph \|G(\hbar,\upsilon) - G(\hbar,\zeta)\|^2 \leq \Gamma_\ell(\hbar) \aleph \|\upsilon - \zeta\|_{\Bbbk}^2, \quad (15)$$

for each $\hbar \in I$ and for all $\upsilon, \zeta \in \Bbbk$ with $\aleph \|\upsilon\|_{\Bbbk}^2 \leq \ell$ and $\aleph \|\zeta\|_{\Bbbk}^2 \leq \ell$.

(H5) The multifunction $F : I \times \Bbbk \times \Lambda \to P(L_\varphi(\Xi,\Lambda))$ is L^2_{loc}-Caratheodory with compact and convex values. If there exists $p \in L^1_{loc}(I, R^+)$ and $\Psi : I \to (0,1)$ be the continuous nondecreasing function such that

$$\aleph \|F(\hbar,\upsilon,\nu)\|^2 \leq p(\hbar)(\|\upsilon\|_{\Bbbk}^2 + \aleph \|\nu\|^2 \Lambda), \quad (16)$$

for every $\hbar \in I, \upsilon \in \Bbbk$ and $\nu \in \Lambda$.

(H6) There exists $\mathfrak{L}_\ell \in L^1_{loc}(I, R^+)$, $n > 0$ such that

$$\aleph \|F(\hbar,\upsilon_1,\zeta_1) - F(\hbar,\upsilon_2,\zeta_2)\|^2 \leq \mathfrak{L}_\ell(\hbar)(\aleph \|\upsilon_1 - \upsilon_2\|_{\Bbbk}^2 + \|\zeta_1 - \zeta_2\|^2 \Lambda), \quad (17)$$

for each $\hbar \in I, \upsilon_1, \upsilon_2 \in \Bbbk$ and $\zeta_1, \zeta_2 \in \Lambda$ with $\aleph \|\upsilon_1\|_{\Bbbk}^2 \leq \ell$, $\aleph \|\upsilon_2\|_{\Bbbk}^2 \leq \ell$, $\aleph \|\zeta_1\|_{\Bbbk}^2 \leq \ell$ and $\aleph \|\zeta_2\|_{\Bbbk}^2 \leq \ell$.

(H7) The function $a : D \times \Bbbk \to \Lambda$, where $D = \{(\hbar,\rho) \in I \times I,\ 0 \leq \rho \leq \hbar \leq T\}$ satisfies:

(i) The function $a(\hbar,\rho,\cdot) : \Bbbk \to \Lambda$, $(\hbar,\rho) \in D$ is continuous and the function $a(\cdot,\cdot,\upsilon) : D \to \Lambda$ is strongly measurable

$$\|a(\hbar,\rho,\upsilon)\|^2 \leq \Theta_a(1 + \|\upsilon\|_{\Bbbk}^2), \quad (18)$$

for each $\hbar, \rho \in D, \upsilon \in \Bbbk$.

(ii) There exist a constant $\tilde{\Theta}_a(r) > 0$, $r > 0$ such that

$$\aleph \|a(\hbar,\rho,\upsilon) - a(\hbar,\rho,\zeta)\|^2 \leq \tilde{\Theta}_a(r)\|\upsilon - \zeta\|_{\Bbbk}^2, \quad (19)$$

and $\hbar, \rho \in D, \upsilon, \zeta \in \Bbbk$.

(H8) There exists a constant $\beta_\ell > 0$, $\ell \in N$ such that

$$\frac{(1 - 6\Xi_\ell^2 \chi_1)\beta_\ell 1}{\Upsilon_1 + 6Tr(\varphi)\Theta \Xi_\ell^2 \Psi((1 + T\Theta_a)\beta_\ell)\|p\|_{L^1_{[0,\ell]}}} > 1, \quad (20)$$

where

$$\begin{aligned}\Upsilon =& 12M\Xi_\ell^2[\tilde{\Lambda}^2\|\nu\|_{\Bbbk}^2 + (\chi_1\|\nu\|_{\Bbbk}^2 + \chi_2)] \\ &+ 2[(\Theta_\ell + J_0^\nu)\|\nu\|_{\Bbbk}^2] + 6\Xi_\ell^2\chi_2 \\ &+ 6\Xi_\ell^2 Tr(\varphi)\Theta \int_0^\ell p(\rho)(T\Theta_a)d\rho. \end{aligned} \quad (21)$$

Lemma 3 *[16, 17] Let $\zeta : (-\infty,\ell] \to \Lambda$ be continuous on $[0,\ell]$ and $\zeta_0 = \nu$. If (H2) is satisfied, then*

$$\|\zeta_\rho\|_{\Bbbk} \leq (\Theta_\ell + J_0^\nu)\|\nu\|_{\Bbbk} + \Xi_\ell \sup_{\vartheta \geq 0} \|\zeta(\vartheta)\|. \quad (22)$$

$\vartheta \in [0, \max\{0,\rho\}]\}$, $\rho \in R(\sigma^-) \cup [0,\ell]$, where

$$J_0^\nu = \sup_{\hbar \in R}(\sigma^-) J^\nu(\hbar). \quad (23)$$

Remark 1 [16] Let $\nu \in \Bbbk, \hbar \leq 0$ and the function $\nu_\hbar = \nu(\hbar + \vartheta)$, $\nu \in (-\infty, 0]$ is well-defined for $\hbar < 0$. Consequently, if the function $\zeta(\cdot)$ in axiom (A) is such that $\zeta_0 = \nu$, then $\zeta_\hbar = \nu_\hbar$.

Theorem 1 *Let $\nu \in L^2_2(\mho,\Lambda)$, the assumptions (H1) - (H8) are satisfied and*

$$2\Xi_\ell^2 \sup_{\hbar \in [0,\ell]} \Gamma_\ell(\hbar) < 1, \quad \textit{for each} \quad \ell \in \mathbb{N}. \quad (24)$$

Then the problem (1) - (2) has a unique mild solution on I.

Proof Let us fix $\tau > 1$ and we define in $\Bbbk_{+\infty}$ the semi-norms $\|\zeta\|_\ell = \sup\{e^{\tau \mathfrak{L}_\ell^*}(\hbar)\aleph\|\zeta(\hbar)\|^2 : \hbar \in [0,\ell]\}$, for every $\ell \in N$, where

$$\mathfrak{L}_\ell^*(\hbar) = \int_0^\hbar \overline{\mathfrak{L}}_\ell(\rho)d\rho, \quad (25a)$$

$$\overline{\mathfrak{L}}_\ell(\hbar) = 2\,\mathrm{Tr}(\varphi)\Theta\Xi_\ell^2(1 + T\tilde{\Theta}_a(r))\mathfrak{L}_\ell(\rho). \quad (25b)$$

and $\mathfrak{L}_\ell$ is the function from (H6). Then $\Bbbk_{+\infty}$ is a Frechet space with the family of semi-norms $\|\cdot\|_{\ell \in N}$.

Consider the space $Y = \{\zeta \in \Bbbk_{+\infty} : \zeta(0) = \nu(0)\}$ endowed with the uniform convergence topology ($\|\cdot\|_\infty$) and define $\Phi : Y \to Y$ by

$$(\Phi_y)(\hbar)=\begin{cases}0,\\ S_\rho(\hbar)[\nu(0)-G(0,\nu)]+G(\hbar,\overline{\zeta}_{\sigma(\hbar,\overline{\zeta}_\hbar)})\\ \qquad+\int_0^\hbar S_\rho(\hbar-\rho)\times F\left(\rho,\overline{\zeta}_{\sigma(\rho,\overline{\zeta}_\rho)},\int_0^\rho a(\rho,\tau,\overline{\zeta}_{\varsigma(\tau,\overline{\zeta}_\tau)})d\tau\right)dW(\rho),\quad \hbar\in I.\end{cases} \tag{26}$$

where $\overline{\zeta}:(-\infty,0]\to\Lambda$ such that $\overline{\zeta}_0=\nu$ and $\overline{\zeta}=\zeta$ on $[0,\ell]$. Let $\overline{\nu}:(-\infty,0)\to\Lambda$ be the extension of $(-\infty,0]$ such that $\overline{\nu}(\vartheta)=\nu(0)=0$ on $[0,\ell]$ and $J_0^\nu=\sup\{J^\nu(\rho):\rho\in R(\sigma^-)\}$.

We show that Φ has a fixed point, which in turn is a mild solution of the problem (1) - (2). Let ζ be a possible solution of problem (1) - (2): Given $\ell\in N$ and $\hbar\in[0,\ell]$, then

$$\begin{aligned}\zeta(\hbar)&=S_\rho(\hbar)[\nu(0)-G(0,\nu)]+G(\hbar,\overline{\zeta}_{\sigma(\hbar,\overline{\zeta}_\hbar)})\\&\quad+\int_0^\hbar S_\rho(\hbar-\rho)\\&\quad\times F\left(\rho,\overline{\zeta}_{\sigma(\rho,\overline{\zeta}_\rho)},\int_0^\rho a(\rho,\tau,\overline{\zeta}_{\varsigma(\tau,\overline{\zeta}_\tau)})d\tau\right)dW(\rho).\end{aligned}\tag{27}$$

For each $\hbar\in[0,\ell]$ hypotheses (H3), (H5) and (H7) imply

$$\begin{aligned}\aleph\|\zeta(\hbar)\|^2&\le 3\aleph\|S_\rho(\hbar)[\nu(0)-G(0,\nu)]\|^2\\&\quad+3\aleph\|G(\hbar,\overline{\zeta}_{\sigma(\hbar,\overline{\zeta}_\hbar)})\|^2+3\aleph\left\|\int_0^\hbar S_\rho(\hbar-\rho)\right.\\&\quad\left.\times F\left(\rho,\overline{\zeta}_{\sigma(\rho,\overline{\zeta}_\rho)},\int_0^\rho a(\rho,\tau,\overline{\zeta}_{\varsigma(\tau,\overline{\zeta}_\tau)})d\tau\right)dW(\rho)\right\|^2\\&\le\sum_{i=1}^3 J_i.\end{aligned}\tag{28}$$

$$\begin{aligned}J_1&=3\aleph\|S_\rho(\hbar)[\nu(0)-G(0,\nu)]\|^2\\&\le 6\Theta[\tilde{\Lambda}^2\|\nu\|_{\mathbb{k}}^2+(\chi_1\|\nu\|_{\mathbb{k}}^2+\chi_2)]\end{aligned}\tag{29}$$

$$J_2=3\aleph\|G(\hbar,\overline{\zeta}_{\sigma(\hbar,\overline{\zeta}_\hbar)})\|^2\le 3(\chi_1\|\overline{\zeta}_{\sigma(\hbar,\overline{\zeta}_\hbar)}\|_{\mathbb{k}}^2+\chi_2)\tag{30}$$

$$\begin{aligned}J_3&=3\aleph\left\|\int_0^\hbar S_\rho(\hbar-\rho)\right.\\&\quad\left.\times F\left(\rho,\overline{\zeta}_{\sigma(\rho,\overline{\zeta}_\rho)},\int_0^\rho a(\rho,\tau,\overline{\zeta}_{\varsigma(\tau,\overline{\zeta}_\tau)})d\tau\right)dW(\rho)\right\|^2\\&\le 3\Theta\int_0^\hbar Tr(\varphi)\aleph\\&\quad\times\left\|F\left(\rho,\overline{\zeta}_{\sigma(\rho,\overline{\zeta}_\rho)},\int_0^\rho a(\rho,\tau,\overline{\zeta}_{\varsigma(\tau,\overline{\zeta}_\tau)})d\tau\right)\right\|^2d\rho\\&\le 3Tr(\varphi)\Theta\int_0^\hbar p(\rho)\\&\quad\times\left(\|\overline{\zeta}_{\sigma(\rho,\overline{\zeta}_\rho)}\|_{\mathbb{k}}^2+T\Theta_a(1+\|\overline{\zeta}_{\sigma(\tau,\overline{\zeta}_\tau)}\|_{\mathbb{k}}^2)\right)d\rho.\end{aligned}\tag{31}$$

Substituting $(J_1)-(J_3)$ together with (28), we have

$$\begin{aligned}\aleph\|\zeta(\hbar)\|^2&\le 6\Theta[\tilde{\Lambda}^2\|\nu\|_{\mathbb{k}}^2+(\chi_1\|\nu\|_{\mathbb{k}}^2+\chi_2)]\\&\quad+3\left(\chi_1\|\overline{\zeta}_{\sigma(\hbar,\overline{\zeta}_\hbar)}\|_{\mathbb{k}}^2+\chi_2\right)\\&\quad+3Tr(\varphi)\Theta\int_0^\hbar p(\rho)\Psi\left(\|\overline{\zeta}_{\sigma(\rho,\overline{\zeta}_\rho)}\|_{\mathbb{k}}^2\right.\\&\quad\left.+T\Theta_a(1+\|\overline{\zeta}_{\sigma(\tau,\overline{\zeta}_\tau)}\|_{\mathbb{k}}^2)\right)d\rho.\end{aligned}\tag{32}$$

By Lemma 1 and Lemma 3, $\Rightarrow\ \varsigma(\rho,\overline{\zeta}_\rho)\le\rho,\ \rho\in[0,\ell]$ and

$$\begin{aligned}&\|\overline{\zeta}_{\sigma(\rho,\overline{\zeta}_\rho)}\|_{\mathbb{k}}^2\\&\quad\le 2[(\Theta_\ell+J_0^\nu)\|\nu\|_{\mathbb{k}}]^2+2\Xi_\ell^2\sup_{\rho\in[0,\ell]}\aleph\|\zeta(\rho)\|^2.\end{aligned}\tag{33}$$

For each $\hbar\in[0,\ell]$, we get

$$\begin{aligned}\aleph\|\zeta(\hbar)\|^2&\le 6\Theta[\tilde{\Lambda}^2\|\nu\|_{\mathbb{k}}^2+(\chi_1\|\nu\|_{\mathbb{k}}^2+\chi_2)]\\&+3\left[\chi_1\left(2[(\Theta_\ell+J_0^\nu)\|\nu\|_{\mathbb{k}}]^2+2\Xi_\ell^2\sup_{\rho\in[0,\ell]}\aleph\|\zeta(\rho)\|^2\right)+\chi_2\right]\\&+3Tr(\varphi)\Theta\int_0^\hbar p(\rho)\Psi\left\{\left(2[(\Theta_\ell+J_0^\nu)\|\nu\|_{\mathbb{k}}]^2\right.\right.\\&\qquad\left.+2\Xi_\ell^2\sup_{\rho\in[0,\ell]}\aleph\|\zeta(\rho)\|^2\right)\\&+T\Theta_a\left(1+\left(2[(\Theta_\ell+J_0^\nu)\|\nu\|_{\mathbb{k}}]^2\right.\right.\\&\qquad\left.\left.\left.+2\Xi_\ell^2\sup_{\rho\in[0,\ell]}\aleph\|\zeta(\rho)\|^2\right)\right)\right\}d\rho.\end{aligned}\tag{34}$$

Consider the norm of the function μ defined by

$$\mu(\hbar):=2[(\Theta_\ell+J_0^\nu)\|\nu\|_{\mathbb{k}}]^2+2\Xi_\ell^2\sup_{0\le\rho\le\hbar}\aleph\|\zeta(\rho)\|^2,\tag{35}$$

which $\|\mu\|_\infty=\sup_{0\le\hbar\le T}\mu(\hbar)$. By the previous inequality Eq. (34), we get

$$\|\mu\|_\infty \leq 12M\Xi_\ell^2 \times \tilde{\Lambda}^2\|\nu\|_{\Bbbk}^2 + (\chi_1\|\nu\|_{\Bbbk}^2 + \chi_2) + 2[(\Theta_\ell + J_0^\nu)\|\nu\|_{\Bbbk}]^2 + 6\Xi_\ell^2(\chi_1\|\mu\|_\infty + \chi_2) + 6Tr(\varphi)\Theta\Xi_\ell^2 \int_0^\hbar p(\rho)\Psi\{(1+T\Theta_a)\|\mu\|_\infty + T\Theta_a\}d\rho, \quad (36)$$

for $\hbar \in [0,\ell]$. Therefore,

$$\|\mu\|_\infty \leq 12M\Xi_\ell^2 \times \tilde{\Lambda}^2\|\nu\|_{\Bbbk}^2 + (\chi_1\|\nu\|_{\Bbbk}^2 + \chi_2) + 2[(\Theta_\ell + J_0^\nu)\|\nu\|_{\Bbbk}]^2 + 6\Xi_\ell^2\chi_2 + 6Tr(\varphi)\Theta\Xi_\ell^2 \int_0^\ell p(\rho)\Psi(T\Theta_a)d\rho + 6\Xi_\ell^2\chi_1\|\mu\|_\infty + 6Tr(\varphi)\Theta\Xi_\ell^2\Psi((1+T\Theta_a)\|\mu\|_\infty)\int_0^\ell p(\rho)d\rho. \quad (37)$$

Consequently,

$$\frac{(1-6\Xi_\ell^2\chi_1)\|\mu\|_\infty}{\Upsilon_1 + 6Tr(\varphi)\Theta\Xi_\ell^2\Psi((1+T\Theta_a)\|\mu\|_\infty)\int_0^\ell p(\rho)d\rho} \leq 1. \quad (38)$$

Then by (H8), there exists β_ℓ such that $\|\mu\|_\infty \leq \beta_\ell$.

Since $\|\zeta\|_{\Bbbk_{+\infty}} \leq \|\mu\|_\infty \Rightarrow \|\zeta\|_\ell \leq \beta_\ell$. Set:

$$\Upsilon = \Big\{\zeta \in \Bbbk_{+\infty} : \sup\{\aleph\|\zeta(\hbar)\|^2 : 0 \leq \hbar \leq \ell\} \leq \beta_\ell + 1, \text{ for all } \ell \in N\Big\}. \quad (39)$$

Clearly, $\Upsilon \subset \Bbbk_{+\infty}$ is closed. We shall show that $\Phi : \Upsilon \to \Bbbk_{+\infty}$ is a contraction operator.

Consider $\zeta^*, \zeta^{**} \in \Bbbk_{+\infty}$. By Lemma 1, Lemma 3 and (H4), (H6) and (H7), we get

$$\begin{aligned}
&\aleph\|\Phi\overline{\zeta}^*(\hbar) - \Phi\overline{\zeta}^{**}(\hbar)\|^2 \\
&\leq 2\aleph\|G(\hbar, \overline{\zeta}^*_{\sigma(\hbar,\overline{\zeta}^*_\hbar)}) - G(\hbar, \overline{\zeta}^{**}_{\sigma(\hbar,\overline{\zeta}^{**}_\hbar)})\|^2 \\
&+ 2\aleph\Big\|\int_0^\hbar S_\rho(\hbar-\rho) \\
&\times\Big[F\Big(\rho, \overline{\zeta}^*_{\sigma(\rho,\overline{\zeta}^*s)}, \int_0^\rho a(\rho,\tau,\overline{\zeta}^*_{\varsigma(\tau,\overline{\zeta}^*_\tau)})d\tau\Big) \\
&- F\Big(\rho, \overline{\zeta}^{**}_{\sigma(\rho,\overline{\zeta}^{**}_\rho)}, \int_0^\rho a(\rho,\tau,\overline{\zeta}^{**}_{\varsigma(\tau,\overline{\zeta}^{**}_\tau)})d\tau\Big)\Big]dW(\rho)\Big\|^2 \\
&\leq J_4 + J_5, \qquad (40)
\end{aligned}$$

for each $\hbar \in [0,\ell]$ and $\ell \in N$. Here

$$\begin{aligned}
J_4 &\leq 2\aleph\|G(\hbar, \overline{\zeta}^*_{\sigma(\hbar,\overline{\zeta}^*\hbar)}) - G(\hbar, \overline{\zeta}^{**}_{\sigma(\hbar,\overline{\zeta}^{**}_\hbar)})\|^2 \\
&\leq 2\Gamma_\ell(\hbar)\aleph\|\overline{\zeta}^*_{\sigma(\hbar,\overline{\zeta}^*\hbar)} - \overline{\zeta}^{**}_{\sigma(\hbar,\overline{\zeta}^{**}_\hbar)}\|^2. \qquad (41)
\end{aligned}$$

$$\begin{aligned}
J_5 &\leq 2\aleph\Big\|\int_0^\hbar S_\rho(\hbar-\rho) \\
&\times\Big[F\Big(\rho, \overline{\zeta}^*_{\sigma(\rho,\overline{\zeta}^*_\rho)}, \int_0^\rho a(\rho,\tau,\overline{\zeta}^*_{\varsigma(\tau,\overline{\zeta}^*_\tau)})d\tau\Big) \\
&- F\Big(\rho, \overline{\zeta}^{**}_{\sigma(\rho,\overline{\zeta}^{**}_\rho)}, \int_0^\rho a(\rho,\tau,\overline{\zeta}^{**}_{\varsigma(\tau,\overline{\zeta}^{**}_\tau)})d\tau\Big)\Big]dW(\rho)\Big\|^2 \\
&\leq 2\Theta Tr(\varphi)\int_0^\hbar \aleph\Big\|F\Big(\rho, \overline{\zeta}^*_{\sigma(\rho,\overline{\zeta}^*_\rho)}, \int_0^\rho a(\rho,\tau,\overline{\zeta}^*_{\varsigma(\tau,\overline{\zeta}^*_\tau)})d\tau\Big) \\
&- F\Big(\rho, \overline{\zeta}^{**}_{\sigma(\rho,\overline{\zeta}^{**}_\rho)}, \int_0^\rho a(\rho,\tau,\overline{\zeta}^{**}_{\varsigma(\tau,\overline{\zeta}^{**}_\tau)})d\tau\Big)\Big\|^2 d\rho \\
&\leq 2Tr(\varphi)\Theta\int_0^\hbar \mathfrak{L}_\ell(\rho)\Big((1+T\tilde{\Theta}_a(r))\aleph \\
&\times\|\overline{\zeta}^*_{\sigma(\rho,\overline{\zeta}^*_\rho)} - \overline{\zeta}^{**}_{\sigma(\rho,\overline{\zeta}^{**}_\rho)}\|_{\Bbbk}^2\Big)d\rho. \qquad (42)
\end{aligned}$$

Substituting (J_4) and (J_5) together with (40), we obtain

$$\begin{aligned}
&\aleph\|\Phi\overline{\zeta}^*(\hbar) - \Phi\overline{\zeta}^{**}(\hbar)\|^2 \\
&\leq 2\Gamma_\ell(\hbar)\aleph\|\overline{\zeta}^*_{\sigma(\hbar,\overline{\zeta}^*_\hbar)} - \overline{\zeta}^{**}_{\sigma(\hbar,\overline{\zeta}^{**}_\hbar)}\|^2 + 2\Theta Tr(\varphi)\int_0^\hbar \mathfrak{L}_\ell(\rho) \\
&\times\Big((1+T\tilde{\Theta}_a(r))\|\overline{\zeta}^*_{\sigma(\rho,\overline{\zeta}^*_\rho)} - \overline{\zeta}^{**}_{\sigma(\rho,\overline{\zeta}^{**}_\rho)}\|_{\Bbbk}^2\Big)d\rho \\
&\leq 2\Gamma_\ell(\hbar)\Xi_\ell^2 \sup_{\hbar\in[0,\ell]} \aleph\|\overline{\zeta}^*(\hbar) - \overline{\zeta}^{**}(\hbar)\|^2 \\
&+ 2Tr(\varphi)\Theta\Xi_\ell^2 \\
&\times\int_0^\hbar \mathfrak{L}_\ell(\rho)(1+T\tilde{\Theta}_a(r))\aleph\|\overline{\zeta}^*(\rho) - \overline{\zeta}^{**}(\rho)\|^2 d\rho \\
&\leq 2\Xi_\ell^2\Gamma_\ell(\hbar)[e^{-\tau\overline{\mathfrak{L}}_\ell(\hbar)}] \\
&\times\Big[e^{-\tau\overline{\mathfrak{L}}_\ell(\hbar)}\sup_{\hbar\in[0,\ell]} \aleph\|\overline{\zeta}^*(\hbar) - \overline{\zeta}^{**}(\hbar)\|^2\Big] \\
&+ \int_0^\hbar [\overline{\mathfrak{L}}_\ell(\rho)e^{\tau\overline{\mathfrak{L}}_\ell(\rho)}]\Big[e^{-\tau\overline{\mathfrak{L}}_\ell(\rho)}\aleph\|\overline{\zeta}^*(\rho) - \overline{\zeta}^{**}(\rho)\|^2\Big]d\rho \\
&\leq 2\Xi_\ell^2[e^{\tau\overline{\mathfrak{L}}_\ell(\hbar)}]\sup_{\hbar\in[0,\ell]} \Gamma_\ell(\hbar)\|\overline{\zeta}^* - \overline{\zeta}^{**}\|_\ell \\
&+ \int_0^\hbar \frac{1}{\tau}[e^{\tau\overline{\mathfrak{L}}_\ell(\rho)}]'d\rho\|\overline{\zeta}^* - \overline{\zeta}^{**}\|_\ell \\
&\leq e^{\tau\overline{\mathfrak{L}}_\ell(\hbar)}\Big[2\Xi_\ell^2\sup_{\hbar\in[0,\ell]}\Gamma_\ell(\hbar) + \frac{1}{\tau}\Big]\|\overline{\zeta}^* - \overline{\zeta}^{**}\|_\ell. \qquad (43)
\end{aligned}$$

By using $\overline{\zeta} = \zeta$ on $[0,\ell]$ and taking supremum over $\hbar$,

$$\|\Phi\overline{\zeta}^{*}-\Phi\overline{\zeta}^{**}\|_{\ell} \leq \left[2\Xi_{\ell}^{2}\sup_{\hbar\in[0,\ell]}\Gamma_{\ell}(\hbar)+\frac{1}{\tau}\right]\|\overline{\zeta}^{*}-\overline{\zeta}^{**}\|_{\ell}, \tag{44}$$

for all $\ell \in N$ showing that the operator Φ is a contraction.

From the choice of Υ there is no $\zeta \in \partial\Upsilon^{\ell}$ such that $\zeta = \Phi(\zeta)$, for some $\xi \in (0,1)$, the problem (1)-(2) has a unique mild solution, ζ, which is the fixed point of the operator Φ. as a result of Frigon and Granas' nonlinear alternative. The proof is complete.

4 Applications to control theory

This part applies the reasoning from the preceding sections to the issue of whether a class of fractional neutral stochastic integro-differential equations in a Hilbert space Λ with state-dependent delay can be managed. We pay extra attention to the following problem.

$$dD(\hbar,\zeta_{\sigma(\hbar,\zeta_{\hbar})}) = \int_{0}^{\hbar}\frac{(\hbar-\rho)^{\rho-2}}{\Gamma(\rho-1)}AD(\rho,\zeta_{\sigma(\rho,\zeta_{\rho})})d\rho + (\Bbbk u)(\hbar)d\hbar + F\left(\hbar,\zeta_{\sigma(\hbar,\zeta_{\hbar})},\int_{0}^{\hbar}a(\hbar,\rho,\zeta_{\sigma(\rho,\zeta_{\rho})})d\rho\right)dW(\hbar), \quad \hbar\in I=[0,1), \tag{45}$$

$$\zeta_{0} = \nu(\hbar) \in \Bbbk, \tag{46}$$

where A, F and D are as in Section 3. Additionally, the control function u is a member of the space $L^{2}(I,U)$, a Banach space of permissible control functions, which also contains the Banach space U. Additionally, $\Bbbk : U \to \Lambda$ is a bounded linear operator. Numerous authors have developed the controllability results for stochastic semi-linear differential and integro-differential systems in Hilbert spaces, including [6, 18, 19, 27, 28] and references thereto.

Definition 4 A mild solution of Eq. (46) is an $\mathfrak{F}_{\hbar}$-adapted stochastic process $\zeta : (-\infty,+\infty) \to \Lambda$, if the restriction of $\zeta(\cdot)$ to the interval I is continuous, $\zeta_{0} = \nu(\hbar)$, $\zeta_{\sigma(\rho,\zeta_{\rho})} \in \Bbbk$ satisfying $\zeta_{0} \in L_{2}^{0}(\mho,\Lambda)$ and

$$\zeta(\hbar) = S_{\rho}(\hbar)[\nu(0)-G(0,\nu)] + G(\hbar,\zeta_{\sigma(\hbar,\zeta_{\hbar})}) + \int_{0}^{\hbar}S_{\rho}(\hbar-\rho)(\Bbbk u)(\rho)d\rho + \int_{0}^{\hbar}S_{\rho}(\hbar-\rho)F \times\left(\rho,\zeta_{\sigma(\rho,\zeta_{\rho})},\int_{0}^{\rho}a(\rho,\tau,\zeta_{\varsigma(\tau,\zeta_{\tau})})d\tau\right)dW(\rho),\ \hbar\in I. \tag{47}$$

Definition 5 A stochastic control $u \in L^{2}(I,U)$, which is adapted to the filtration $\{\mathfrak{F}_{\hbar}\}_{\hbar\geq 0}$, if for every initial random variable $\zeta_{0},\zeta_{1} \in L_{2}^{0}(\mho,\Lambda)$ such that the mild solution $\zeta(\hbar)$ of the system (45)-(46) satisfies $\zeta(\ell) = \zeta_{1}$, then the system (45)-(46) is said to be controllable on the interval I.

We make the ensuing presumptions:

(B1) Define $W : L^{2}([0,\ell],U) \to \Upsilon$ be the linear operator by

$$Wu = \int_{0}^{\ell}S_{\rho}(\ell-\rho)Bu(\rho)d\rho.$$

The inverse operator W^{-1} which takes values in $L^{2}([0,\ell],U)/Ker\ W$ and $\|BW^{-1}\|^{2} \leq \Theta_{\Bbbk}$, if a positive constants $\Theta_{\Bbbk}$ exist.

(B2) There exists a constant $\beta_{\ell}^{*} > 0$ such that

$$\frac{(1-8\chi_{1}\Xi_{\ell}^{2}(1+4\Theta\Theta_{\Bbbk}\ell^{2}))\beta_{\ell}^{*}}{\Upsilon_{2}+8Tr(\varphi)\Theta\Xi_{\ell}^{2}(1+2\Theta\Theta_{\Bbbk}\ell^{2})(1+T\Theta_{a})\Psi(\beta_{\ell}^{*})\|p\|_{L^{1}_{[0,\ell]}}} > 1, \qquad \ell\in N, \tag{48}$$

where

$$\Upsilon_{2} = 16\Theta\Xi_{\ell}^{2}[\tilde{\Lambda}^{2}\|\nu\|_{\Bbbk}^{2}+(\chi_{1}\|\nu\|_{\Bbbk}^{2}+\chi_{2})] + 2[(\Theta_{\ell}+J_{0}^{\nu})\|\nu\|_{\Bbbk}]^{2}+8\Xi_{\ell}^{2}\chi_{2} + 32\Theta\Theta_{\Bbbk}\ell^{2}\Xi_{\ell}^{2}[\aleph\|\zeta_{1}\|^{2} + 2\Theta[\tilde{\Lambda}^{2}\|\nu\|_{\Bbbk}^{2}+(\chi_{1}\|\nu\|_{\Bbbk}^{2}+\chi_{2})]+\chi_{2}].$$

Theorem 2 *Let $\nu \in L_{2}^{0}(\mho,\Lambda)$, the assumptions (H1) - (H8), (B1) and (B2) are holds and*

$$3\Xi_{\ell}^{2}(1+2\Theta\Theta_{\Bbbk}\ell^{2})\sup_{\hbar\in[0,\ell]}\Gamma_{\ell}(\hbar) < 1,\ \ \ell\in N, \tag{49}$$

then the problem (45) - (46) *has a unique mild solution on I.*

Proof Let us fix $\tau > 1$ and we define in $\Bbbk_{+\infty}$ the semi-norms

$$\|\zeta\|_\ell = \sup\left\{ e^{\tau \mathfrak{L}_\ell^*}(\hbar)\aleph\|\zeta(\hbar)\|^2 : \ \hbar \in [0,\ell] \right\}, \quad \ell \in N, \quad (50)$$

where $\mathfrak{L}_\ell^* = \int_0^\hbar \overline{\mathfrak{L}}_\ell(\rho)d\rho$,

$$\overline{\mathfrak{L}}_\ell(\hbar) = 3Tr(\varphi)\Theta\Xi_\ell^2(1+T\tilde{\Theta}_a(r))(1+2\Theta\Theta_{\Bbbk}\ell^2)\mathfrak{L}_\ell(\rho) \quad (51)$$

where $\mathfrak{L}_\ell$ is the function from (H6). Then $\Bbbk_{+\infty}$ is a Frechet space with the family of semi-norms $\|\cdot\|_{\ell \in N}$.

Define the following control by using (B1), for each $\ell \in N$

$$u_\zeta^\ell(\hbar) = W^{-1}\Bigg[\zeta_1 - S_\rho(\ell)[\nu(0) - G(0,\nu)] - G(\ell, \overline{\zeta}_{\sigma(\ell,\overline{\zeta}_\ell)}) - \int_0^\ell S_\rho(\ell-\rho) \times F\left(\rho, \overline{\zeta}_{\sigma(\rho,\overline{\zeta}_\rho)}, \int_0^\rho a(\rho,\tau,\overline{\zeta}_{\varsigma(\tau,\overline{\zeta}_\tau)})d\tau\right)dW(\rho)\Bigg](\hbar). \quad (52)$$

Consider the space $Y = \{\zeta \in \Bbbk_{+\infty} : \ \zeta(0) = \nu(0)\}$ endowed with the uniform convergence topology $(\|\cdot\|_\infty)$ and define $\Phi : Y \to Y$ by

$$(\Phi_y)(\hbar) = \begin{cases} 0, \ \ \hbar \in (-\infty, 0], \\ S_\rho(\hbar)[\nu(0) - G(0,\nu)] + G(\hbar, \overline{\zeta}_{\sigma(\hbar,\overline{\zeta}_\hbar)}) + \int_0^\hbar S_\rho(\hbar-\rho)(\Bbbk u_\zeta^\ell)(\rho)d\rho + \int_0^\hbar S_\rho(\hbar-\rho) \\ \qquad \times F\left(\rho, \overline{\zeta}_{\sigma(\rho,\overline{\zeta}_\rho)}, \int_0^\rho a(\rho,\tau,\overline{\zeta}_{\varsigma(\tau,\overline{\zeta}_\tau)})d\tau\right)dW(\rho), \quad \hbar \in I, \end{cases} \quad (53)$$

where $\overline{\zeta} : (-\infty, 0] \to \Lambda$ such that $\overline{\zeta}_0 = \nu$ and $\overline{\zeta} = \zeta$ on $[0,\ell]$. Let $\overline{\nu} : (-\infty, 0) \to \Lambda$ be the extension of $(-\infty, 0]$ such that $\overline{\nu}(\vartheta) = \nu(0) = 0$ on $[0,\ell]$ and $J_0^\nu = \sup\{J_0^\nu : \rho \in R(\varsigma^-)\}$. We show that Φ has a fixed point, which in turn is a mild solution of the problem (45) - (46).
Let ζ be a possible solution of problem (45) - (46). Given $\ell \in N$ and $\hbar \in [0,\ell]$, then

$$\zeta(\hbar) = S_\rho(\hbar)[\nu(0) - G(0,\nu)] + G(\hbar, \overline{\zeta}_{\sigma(\hbar,\overline{\zeta}_\hbar)}) + \int_0^\hbar S_\rho(\hbar-\rho)(\Bbbk u_\zeta^\ell)(\rho)d\rho + \int_0^\hbar S_\rho(\hbar-\rho) \times F\left(\rho, \overline{\zeta}_{\sigma(\rho,\overline{\zeta}_\rho)}, \int_0^\rho a(\rho,\tau,\overline{\zeta}_{\varsigma(\tau,\overline{\zeta}_\tau)})d\tau\right)dW(\rho). \quad (54)$$

By (H3), (H5) and (H7) that, we obtain

$$\aleph\|\zeta(\hbar)\|^2 \le 4\aleph\|S_\rho(\hbar)[\nu(0) - G(0,\nu)]\|^2 + 4\aleph\|G(\hbar, \overline{\zeta}_{\sigma(\hbar,\overline{\zeta}_\hbar)})\|^2 + 4\aleph\left\|\int_0^\hbar S_\rho(\hbar-\rho)(\Bbbk u_\zeta^\ell)(\rho)d\rho\right\|^2 + 4\aleph\Bigg\|\int_0^\hbar S_\rho(\hbar-\rho) \times F\left(\rho, \overline{\zeta}_{\sigma(\rho,\overline{\zeta}_\rho)}, \int_0^\rho a(\rho,\tau,\overline{\zeta}_{\varsigma(\tau,\overline{\zeta}_\tau)})d\tau\right)dW(\rho)\Bigg\|^2 \le \sum_{i=6}^{9} J_i, \quad \hbar \in [0,\ell]. \quad (55)$$

$$J_6 = 4\aleph\|S_\rho(\hbar)[\nu(0) - G(0,\nu)]\|^2 \le 8\Theta[\tilde{\Lambda}^2\|\nu\|_{\Bbbk}^2 + (\chi_1\|\nu\|_{\Bbbk}^2 + \chi_2)] \quad (56)$$

$$J_7 = 4\aleph\|G(\hbar, \overline{\zeta}_{\sigma(\hbar,\overline{\zeta}_\hbar)})\|^2 \le 4(\chi_1\|\overline{\zeta}_{\sigma(\hbar,\overline{\zeta}_\hbar)}\|_{\Bbbk}^2 + \chi_2) \quad (57)$$

$$\begin{aligned} J_8 &\le 4\aleph\left\|\int_0^\hbar S_\rho(\hbar-\rho)(\Bbbk u_\zeta^\ell)(\rho)d\rho\right\|^2 \\ &\le 4\aleph\Bigg\|\int_0^\hbar S_\rho(\hbar-\rho)BW^{-1}\Bigg[\zeta_1 - S_\rho(\ell)[\nu(0) - G(0,\nu)] - G(\ell, \overline{\zeta}_{\sigma(\ell,\overline{\zeta}_\ell)}) - \int_0^\ell S_\rho(\ell-\rho) \\ &\quad \times F\left(\rho, \overline{\zeta}_{\sigma(\rho,\overline{\zeta}_\rho)}, \int_0^\rho a(\rho,\tau,\overline{\zeta}_{\varsigma(\tau,\overline{\zeta}_\tau)})d\tau\right) \times dW(\rho)\Bigg](\hbar)\Bigg\|^2 \\ &\le 16\Theta\Theta_{\Bbbk}\ell\int_0^\hbar\Bigg\{\aleph\|\zeta_1\|^2 + 2\Theta[\tilde{\Lambda}\|\nu\|_{\Bbbk}^2 + (\chi_1\|\nu\|_{\Bbbk}^2 + \chi_2)] + (\chi_1\|\overline{\zeta}_{\sigma(\ell,\overline{\zeta}_\ell)}\|_{\Bbbk}^2 + \chi_2) + Tr(\varphi)\Theta\int_0^\ell p(\tau) \\ &\quad \times \Psi\left(\|\overline{\zeta}_{\sigma(\rho,\overline{\zeta}_\rho)}\|_{\Bbbk}^2 + T\Theta_a(1 + \|\overline{\zeta}_{\sigma(\tau,\overline{\zeta}_\tau)}\|_{\Bbbk}^2)\right)d\tau\Bigg\}d\rho. \end{aligned} \quad (58)$$

$$J_9 = 4\aleph\Big\| \int_0^{\hbar} S_\rho(\hbar-\rho) \times F\left(\rho, \overline{\zeta}_{\sigma(\rho,\overline{\zeta}_\rho)}, \int_0^\rho a(\rho,\tau,\overline{\zeta}_{\varsigma(\tau,\overline{\zeta}_\tau)})d\tau\right) dW(\rho)\Big\|^2 \nu$$
$$\leq 4Tr(\varphi)\Theta \int_0^{\hbar} \aleph \Big\| \times F\left(\rho, \overline{\zeta}_{\sigma(\rho,\overline{\zeta}_\rho)}, \int_0^\rho a(\rho,\tau,\overline{\zeta}_{\varsigma(\tau,\overline{\zeta}_\tau)})d\tau\right) \Big\|^2 d\rho$$
$$\leq 4Tr(\varphi)\Theta \int_0^{\hbar} p(\rho) \times \left(\|\overline{\zeta}_{\sigma(\rho,\overline{\zeta}_\rho)}\|^2_{\Bbbk} + T\Theta_a(1+\|\overline{\zeta}_{\sigma(\tau,\overline{\zeta}_\tau)}\|^2_{\Bbbk}) \right) d\rho. \tag{59}$$

Substituting (J_6) - (J_9) together with (55), we get

$$\aleph\|\zeta(\hbar)\|^2 \leq 8\Theta[\tilde{\Lambda}^2\|\nu\|^2_{\Bbbk} + (\chi_1\|\nu\|^2_{\Bbbk}+\chi_2)] + 4(\chi_1\|\overline{\zeta}_{\sigma(\hbar,\overline{\zeta}_\hbar)}\|^2_{\Bbbk}+\chi_2)$$
$$+ 16\Theta\Theta_{\Bbbk}\ell \int_0^{\hbar} \Big\{ \aleph\|\zeta_1\|^2 + 2\Theta[\tilde{\Lambda}\|\nu\|^2_{\Bbbk} + (\chi_1\|\nu\|^2_{\Bbbk}+\chi_2)] + (\chi_1\|\overline{\zeta}_{\sigma(\ell,\overline{\zeta}_\ell)}\|^2_{\Bbbk}+\chi_2)$$
$$+ Tr(\varphi)\Theta\int_0^\ell p(\tau)\Psi\Big(\|\overline{\zeta}_{\sigma(\rho,\overline{\zeta}_\rho)}\|^2_{\Bbbk} + T\Theta_a(1+\|\overline{\zeta}_{\sigma(\tau,\overline{\zeta}_\tau)}\|^2_{\Bbbk})\Big)d\tau \Big\} d\rho$$
$$+ 4Tr(\varphi)\Theta\int_0^{\hbar} p(\rho) \times \Psi\left(\|\overline{\zeta}_{\sigma(\rho,\overline{\zeta}_\rho)}\|^2_{\Bbbk} + T\Theta_a(1+\|\overline{\zeta}_{\sigma(\tau,\overline{\zeta}_\tau)}\|^2_{\Bbbk})\right) d\rho$$
$$\leq 8\Theta[\tilde{\Lambda}^2\|\nu\|^2_{\Bbbk} + (\chi_1\|\nu\|^2_{\Bbbk}+\chi_2)] + 4\chi_2 + 16\Theta\Theta_{\Bbbk}\ell^2\Big[\aleph\|\zeta 1\|^2 + 2\Theta[\tilde{\Lambda}\|\nu\|^2_{\Bbbk} + (\chi_1\|\nu\|^2_{\Bbbk}+\chi_2)] + \chi_2 \Big]$$
$$+ 4\chi_1\|\overline{\zeta}_{\sigma(\hbar,\overline{\zeta}_\hbar)}\|^2_{\Bbbk} + 16\Theta\Theta_{\Bbbk}\ell^2\chi_1\|\overline{\zeta}_{\sigma(\ell,\overline{\zeta}_\ell)}\|^2_{\Bbbk}$$
$$+ 16\Theta\Theta_{\Bbbk}\ell^2 Tr(\varphi)\Theta\int_0^\ell p(\tau)\Psi\Big(\|\overline{\zeta}_{\sigma(\rho,\overline{\zeta}_\rho)}\|^2_{\Bbbk} + T\Theta_a(1+\|\overline{\zeta}_{\sigma(\tau,\overline{\zeta}_\tau)}\|^2_{\Bbbk})\Big)d\tau + 4Tr(\varphi)\Theta\int_0^{\hbar} p(\rho)$$
$$\times \Psi\left(\|\overline{\zeta}_{\sigma(\rho,\overline{\zeta}_\rho)}\|^2_{\Bbbk} + T\Theta_a(1+\|\overline{\zeta}_{\sigma(\tau,\overline{\zeta}_\tau)}\|^2_{\Bbbk})\right) d\rho. \tag{60}$$

It follows that

$$\aleph\|\zeta(\hbar)\|^2 \leq 8\Theta[\tilde{\Lambda}^2\|\nu\|^2_{\Bbbk} + (\chi_1\|\nu\|^2_{\Bbbk}+\chi_2)] + 4\chi_2 + 16\Theta\Theta_{\Bbbk}\ell^2\left[\aleph\|\zeta 1\|^2 + 2\Theta[\tilde{\Lambda}\|\nu\|^2_{\Bbbk} + (\chi_1\|\nu\|^2_{\Bbbk}+\chi_2)] + \chi_2 \right]$$
$$+ 4\chi_1\left(2[(\Theta_\ell + J_0^\nu)\|\nu\|_{\Bbbk}] + 2\Xi_\ell^2 \sup_{\rho\in[0,\ell]} \aleph\|\zeta(\rho)\|^2 \right)$$
$$+ 16\Theta\Theta_{\Bbbk}\ell^2\chi_1\left(2[(\Theta_\ell + J_0^\nu)\|\nu\|_{\Bbbk}]^2 + 2\Xi_\ell^2 \sup_{\rho\in[0,\ell]} \aleph\|\zeta(\rho)\|^2 \right)$$
$$+ 16\Theta\Theta_{\Bbbk}\ell^2 Tr(\varphi)\Theta \times \int_0^\ell p(\tau)\Psi\Bigg[\left(2[(\Theta_\ell + J_0^\nu)\|\nu\|_{\Bbbk}]^2 + 2\Xi_\ell^2 \sup_{\rho\in[0,\ell]} \aleph\|\zeta(\rho)\|^2 \right) + T\Theta_a$$
$$\times \left(1 + \left(2[(\Theta_\ell + J_0^\nu)\|\nu\|_{\Bbbk}]^2 + 2\Xi_\ell^2 \sup_{\rho\in[0,\ell]} \aleph\|\zeta(\rho)\|^2 \right)\right) \Bigg] d\tau$$
$$+ 4Tr(\varphi)\Theta\int_0^{\hbar} p(\rho) \times \Psi\Bigg[\left(2[(\Theta_\ell + J_0^\nu)\|\nu\|_{\Bbbk}]^2 + 2\Xi_\ell^2 \sup_{\rho\in[0,\ell]} \aleph\|\zeta(\rho)\|^2 \right)$$
$$+ T\Theta_a\left(1 + \left(2[(\Theta_\ell + J_0^\nu)\|\nu\|_{\Bbbk}]^2 + 2\Xi_\ell^2 \sup_{\rho\in[0,\ell]} \aleph\|\zeta(\rho)\|^2 \right)\right) \Bigg] d\rho. \tag{61}$$

By Theorem 1, we defined $\|\mu\|_\infty = \sup_{0\leq\hbar\leq n} \mu(\hbar)$, and the previous inequality, we obtain

$$\|\mu\|_\infty \leq 16\Theta\Xi_\ell^2\left[\tilde{\Lambda}^2\|\nu\|^2_{\Bbbk} + (\chi_1\|\nu\|^2_{\Bbbk}+\chi_2) \right] + 2[(\Theta_\ell + J_0^\nu)\|\nu\|_{\Bbbk}]^2 + 8\Xi_\ell^2\chi_2$$
$$+ 32\Theta\Theta_{\Bbbk}\ell^2\Xi_\ell^2\Big[\aleph\|\zeta_1\|^2 + 2\Theta[\tilde{\Lambda}^2\|\nu\|^2_{\Bbbk} + (\chi_1\|\nu\|^2_{\Bbbk}+\chi_2)] + \chi_2 \Big]$$
$$+ 8\chi_1\Xi_\ell^2\|\mu\|_\infty + 32\Theta\Theta_{\Bbbk}\ell^2\chi_1\Xi_\ell^2\|\mu\|_\infty + 32\Theta\Theta_{\Bbbk}\ell^2 Tr(\varphi)\Theta\Xi_\ell^2$$
$$\times \int_0^\ell p(\rho)\Psi\Big\{ (1+T\Theta_a)\|\mu\|_\infty + T\Theta_a \Big\} d\rho + 8Tr(\varphi)\Theta\Xi_\ell^2$$
$$\times \int_0^{\hbar} p(\rho)\Psi\Big\{ (1+T\Theta_a)\|\mu\|_\infty + T\Theta_a \Big\} d\rho, \tag{62}$$

i.e.,

$$\|\mu\|_\infty \leq \Upsilon_2 + 8\chi_1 \Xi_\ell^2(1+4\Theta\Theta_{\Bbbk}\ell^2)\|\mu\|_\infty + 8\Theta Tr(\varphi)\Xi_\ell^2[(1+T\Theta_a)(1+4\Theta\Theta_{\Bbbk}\ell^2)]\Psi(\|\mu\|_\infty) \times \int_0^\ell p(\rho)d\rho + 8Tr(\varphi)\Theta\Xi_\ell^2\int_0^\ell p(\rho)\Psi(T\Theta_a)d\rho. \tag{63}$$

Consequently,

$$\frac{(1-8\chi_1\Xi_\ell^2(1+4\Theta\Theta_{\Bbbk}\ell^2))\|\mu\|_\infty}{\Upsilon_2 + 8Tr(\varphi)\Theta\Xi_\ell^2(1+4\Theta\Theta_{\Bbbk}\ell^2)\Psi((1+T\Theta_a)\|\mu\|_\infty)\int_0^\ell p(\rho)d\rho} \leq 1. \tag{64}$$

Then by (B2), $\|\mu\|_\infty \leq \beta_\ell$, if β_ℓ^* exists. Since

$$displaystyle\|\zeta\|_{\Bbbk_{+\infty}} \leq \|\mu\|_\infty \;\Rightarrow\; \|\zeta\|_\ell \leq \beta_\ell^*. \tag{65}$$

Set $\Upsilon = \Big\{\zeta \in \Bbbk_{+\infty} : \ \sup\{\aleph\|\zeta(\hbar)\|^2 : \ 0 \leq \hbar \leq n\} \leq \beta_\ell + 1, \text{ for all } \ell \in N\Big\}$. Clearly, $\Upsilon \subset \Bbbk_{+\infty}$ is closed.
We shall show that $\Phi : \Upsilon \to \Bbbk_{+\infty}$ is a contraction operator. Consider $\zeta^*, \zeta^{**} \in \Bbbk_{+\infty}$. By using Lemma 1, Lemma 3, and (H4), (H6) and (H7), for each $\hbar \in [0,\ell]$ and $\ell \in N$, we have

$$\begin{aligned}
&\aleph\|\Phi\overline{\zeta}^*(\hbar) - \Phi\overline{\zeta}^{**}(\hbar)\|^2\\
&\leq 3\aleph\|G(\hbar,\overline{\zeta}^*_{\sigma(\hbar,\overline{\zeta}^*_\hbar)}) - G(\hbar,\overline{\zeta}^{**}_{\sigma(\hbar,\overline{\zeta}^{**}_\hbar)})\|^2\\
&\quad + 3\aleph\Big\|\int_0^\hbar S_\rho(\hbar-\rho)\\
&\quad \times BW^{-1}\Bigg[\zeta_1 - S_\rho(\ell)[\nu(0)-G(0,\nu)] - G(\ell,\overline{\zeta}^*_{\varsigma(\ell,\overline{\zeta}^*_\ell)})\\
&\quad - \int_0^\ell S_\rho(\ell-\eta)\\
&\quad\quad \times F\left(\eta,\overline{\zeta}^*_{\varsigma(\eta,\overline{\zeta}^*_\eta)},\int_0^\eta a(\eta,\tau,\overline{\zeta}^*_{\varsigma(\tau,\overline{\zeta}^*_\tau)})d\tau\right)dW(\eta)\Bigg]\\
&\quad - \Bigg[\zeta_1 - S_\rho(\ell)[\nu(0)-G(0,\nu)] - G(\ell,\overline{\zeta}^{**}_{\varsigma(\ell,\overline{\zeta}^{**}_\ell)})\\
&\quad\quad - \int_0^\ell S_\rho(\ell-\eta)\\
&\quad\quad \times F\left(\eta,\overline{\zeta}^{**}_{\varsigma(\eta,\overline{\zeta}^{**}_\eta)},\int_0^\eta a(\eta,\tau,\overline{\zeta}^{**}_{\varsigma(\tau,\overline{\zeta}^{**}_\tau)})d\tau\right)dW(\eta)\Bigg]d\rho\Big\|^2\\
&\quad + 3\aleph\Big\|\int_0^\hbar S_\rho(\hbar-\rho)\\
&\quad \times\Bigg[F\left(\rho,\overline{\zeta}^*_{\sigma(\rho,\overline{\zeta}^*_\rho)},\int_0^\rho a(\rho,\tau,\overline{\zeta}^*_{\varsigma(\tau,\overline{\zeta}^*_\tau)})d\tau\right)\\
&\quad\quad - F\left(\rho,\overline{\zeta}^{**}_{\sigma(\rho,\overline{\zeta}^{**}_\rho)},\int_0^\rho a(\rho,\tau,\overline{\zeta}^{**}_{\varsigma(\tau,\overline{\zeta}^{**}_\tau)})d\tau\right)\Bigg]dW(\rho)\Big\|^2\\
&= \sum_{i=10}^{12} J_i.
\end{aligned} \tag{66}$$

$$\begin{aligned}
J_{10} &\leq 3\aleph\|G(\hbar,\overline{\zeta}^*_{\sigma(\hbar,\overline{\zeta}^*_\hbar)}) - G(\hbar,\overline{\zeta}^{**}_{\sigma(\hbar,\overline{\zeta}^{**}_\hbar)})\|^2\\
&\leq 3\aleph\Gamma_\ell(\hbar)\|\overline{\zeta}^*_{\sigma(\hbar,\overline{\zeta}^*_\hbar)} - \overline{\zeta}^{**}_{\sigma(\hbar,\overline{\zeta}^{**}_\hbar)}\|^2
\end{aligned} \tag{67}$$

$$\begin{aligned}
J_{11} &\leq 3\aleph\Big\|\int_0^\hbar S_\rho(\hbar-\rho)\\
&\quad \times BW^{-1}\Bigg[\zeta_1 - S_\rho(\ell)[\nu(0)-G(0,\nu)] - G(\ell,\overline{\zeta}^*_{\varsigma(\ell,\overline{\zeta}^*_\ell)})\\
&\quad - \int_0^\ell S_\rho(\ell-\eta)\\
&\quad \times F\left(\eta,\overline{\zeta}^*_{\varsigma(\eta,\overline{\zeta}^*_\eta)},\int_0^\eta a(\eta,\tau,\overline{\zeta}^*_{\varsigma(\tau,\overline{\zeta}^*_\tau)})d\tau\right)dW(\eta)\Bigg]\\
&\quad - \Bigg[\zeta_1 - S_\rho(\ell)[\nu(0)-G(0,\nu)] - G(\ell,\overline{\zeta}^{**}_{\varsigma(\ell,\overline{\zeta}^{**}_\ell)})\\
&\quad - \int_0^\ell S_\rho(\ell-\eta)\\
&\quad \times F\left(\eta,\overline{\zeta}^{**}_{\varsigma(\eta,\overline{\zeta}^{**}_\eta)},\int_0^\eta a(\eta,\tau,\overline{\zeta}^{**}_{\varsigma(\tau,\overline{\zeta}^{**}_\tau)})d\tau\right)dW(\eta)\Bigg]d\rho\Big\|^2\\
&\leq 6\Theta\Theta_{\Bbbk}\ell\int_0^\hbar\Big[\aleph\|G(\ell,\overline{\zeta}^*_{\varsigma(\ell,\overline{\zeta}^*_\ell)}) - G(\ell,\overline{\zeta}^{**}_{\varsigma(\ell,\overline{\zeta}^{**}_\ell)})\|^2\\
&\quad + Tr(\varphi)\Theta_\ell\int_0^\ell\aleph\\
&\quad \times\Big\|F\left(\eta,\overline{\zeta}^*_{\varsigma(\eta,\overline{\zeta}^*_\eta)},\int_0^\eta a(\eta,\tau,\overline{\zeta}^*_{\varsigma(\tau,\overline{\zeta}^*_\tau)})d\tau\right)\\
&\quad - F\left(\eta,\overline{\zeta}^{**}_{\varsigma(\eta,\overline{\zeta}^{**}_\eta)},\int_0^\eta a(\eta,\tau,\overline{\zeta}^{**}_{\varsigma(\tau,\overline{\zeta}^{**}_\tau)})d\tau\right)\Big\|d\eta d\rho\\
&\leq 6\Theta\Theta_{\Bbbk}\ell^2\Gamma_\ell(\hbar)\|\overline{\zeta}^*_{\varsigma(\ell,\overline{\zeta}^*_\ell)} - \overline{\zeta}^{**}_{\varsigma(\ell,\overline{\zeta}^{**}_\ell)}\|^2\\
&\quad + 6\Theta^2\Theta_{\Bbbk}\ell^2 Tr(\varphi)\\
&\quad \times\int_0^\hbar \mathfrak{L}_\ell(\rho)\left((1+T\tilde{\Theta}_a(r))\|\overline{\zeta}^*_{\varsigma(\eta,\overline{\zeta}^*_\eta)} - \overline{\zeta}^{**}_{\varsigma(\eta,\overline{\zeta}^{**}_\eta)}\|^2_{\Bbbk}\right)d\rho
\end{aligned} \tag{68}$$

$$J_{12} \leq 3\aleph \left\| \int_0^{\hbar} S_\rho(\hbar-\rho)\left[F\left(\rho,\overline{\zeta}^*_{\sigma(\rho,\overline{\zeta}^*_\rho)},\int_0^\rho a(\rho,\tau,\overline{\zeta}^*_{\varsigma(\tau,\overline{\zeta}^*_\tau)})d\tau\right)F\left(\rho,\overline{\zeta}^{**}_{\sigma(\rho,\overline{\zeta}^{**}_\rho)},\int_0^\rho a(\rho,\tau,\overline{\zeta}^{**}_{\varsigma(\tau,\overline{\zeta}^{**}_\tau)})d\tau\right)\right]dW(\rho)\right\|^2$$
$$\leq 3\Theta Tr(\varphi)\int_0^{\hbar}\aleph\left\|F\left(\rho,\overline{\zeta}^*_{\sigma(\rho,\overline{\zeta}^*_\rho)},\int_0^\rho a(\rho,\tau,\overline{\zeta}^*_{\varsigma(\tau,\overline{\zeta}^*_\tau)})d\tau\right)-F\left(\rho,\overline{\zeta}^{**}_{\sigma(\rho,\overline{\zeta}^{**}_\rho)},\int_0^\rho a(\rho,\tau,\overline{\zeta}^{**}_{\varsigma(\tau,\overline{\zeta}^{**}_\tau)})d\tau\right)\right\|^2 d\rho$$
$$\leq 3\Theta Tr(\varphi)\int_0^{\hbar}\mathfrak{L}_\ell(\rho)\left((1+T\tilde{\Theta}_a(r))\|\overline{\zeta}^*_{\sigma(\rho,\overline{\zeta}^*_\rho)}-\overline{\zeta}^{**}_{\sigma(\rho,\overline{\zeta}^{**}_\rho)}\|^2_{\Bbbk}\right)d\rho. \tag{69}$$

Substituting (J_{10}) - (J_{12}) together with (66), we arrive

$$\aleph\|\Phi\overline{\zeta}^*(\hbar)-\Phi\overline{\zeta}^{**}(\hbar)\|^2$$
$$\leq 3\aleph\Gamma_\ell(\hbar)\|\overline{\zeta}^*_{\sigma(\hbar,\overline{\zeta}^*_\hbar)}-\overline{\zeta}^{**}_{\sigma(\hbar,\overline{\zeta}^{**}_\hbar)}\|^2+6\Theta\Theta_{\Bbbk}\ell^2\Gamma_\ell(\hbar)\|\overline{\zeta}^*_{\varsigma(\ell,\overline{\zeta}^*_\ell)}-\overline{\zeta}^{**}_{\varsigma(\ell,\overline{\zeta}^{**}_\ell)}\|^2+6\Theta^2\Theta_{\Bbbk}\ell^2 Tr(\varphi)\int_0^{\hbar}\mathfrak{L}_\ell(\rho)$$
$$\times\left((1+T\tilde{\Theta}_a(r))\|\overline{\zeta}^*_{\varsigma(\eta,\overline{\zeta}^*_\eta)}-\overline{\zeta}^{**}_{\varsigma(\eta,\overline{\zeta}^{**}_\eta)}\|^2_{\Bbbk}\right)d\rho+3\Theta Tr(\varphi)\int_0^{\hbar}\mathfrak{L}_\ell(\rho)\left((1+T\tilde{\Theta}_a(r))\|\overline{\zeta}^*_{\sigma(\rho,\overline{\zeta}^*_\rho)}-\overline{\zeta}^{**}_{\sigma(\rho,\overline{\zeta}^{**}_\rho)}\|^2_{\Bbbk}\right)d\rho$$
$$\leq 3\Gamma_\ell(\hbar)\Xi^2_\ell\sup_{\hbar\in[0,\ell]}\aleph\|\overline{\zeta}^*(\hbar)-\overline{\zeta}^{**}(\hbar)\|^2+6\Theta\Theta_{\Bbbk}\ell^2\Gamma_\ell(\hbar)\Xi^2_\ell\sup_{\hbar\in[0,\ell]}\aleph\|\overline{\zeta}^*(\hbar)-\overline{\zeta}^{**}(\hbar)\|^2$$
$$+6\Theta^2\Theta_{\Bbbk}\ell^2 Tr(\varphi)\Xi^2_\ell\int_0^{\hbar}\mathfrak{L}_\ell(\rho)(1+T\tilde{\Theta}_a(r))\aleph\|\overline{\zeta}^*(\rho)-\overline{\zeta}^{**}(\rho)\|^2 d\rho+3Tr(\varphi)\Theta\Xi^2_\ell$$
$$\times\int_0^{\hbar}\mathfrak{L}_\ell(\rho)(1+T\tilde{\Theta}_a(r))\aleph\|\overline{\zeta}^*(\rho)-\overline{\zeta}^{**}(\rho)\|^2 d\rho$$
$$\leq 3\Gamma_\ell(\hbar)\Xi^2_\ell(1+2\Theta\Theta_{\Bbbk}\ell^2)[e^{\tau\overline{\mathfrak{L}}_\ell}(\hbar)]$$
$$\times\left[e^{-\tau\overline{\mathfrak{L}}_\ell}(\hbar)\sup_{\rho\in[0,\ell]}\aleph\|\overline{\zeta}^*(\hbar)-\overline{\zeta}^{**}(\hbar)\|^2 *\right]\int_0^{\hbar}\left[\overline{\mathfrak{L}}_\ell(\rho)e^{\tau\overline{\mathfrak{L}}_\ell}(\rho)\right]\left[e^{-\tau\overline{\mathfrak{L}}_\ell}(\rho)\aleph\|\overline{\zeta}^*(\rho)-\overline{\zeta}^{**}(\rho)\|^2\right]d\rho$$
$$\leq 3\Gamma_\ell(\hbar)\Xi^2_\ell(1+2\Theta\Theta_{\Bbbk}\ell^2)[e^{\tau\overline{\mathfrak{L}}_\ell}(\rho)]\|\overline{\zeta}^*-\overline{\zeta}^{**}\|_\ell+\int_0^{\hbar}\frac{1}{\tau}[e^{\tau\overline{\mathfrak{L}}_\ell}(\rho)]'d\rho\|\overline{\zeta}^*-\overline{\zeta}^{**}\|_\ell$$
$$\leq e^{\tau\overline{\mathfrak{L}}_\ell}(\hbar)\left[3\sup_{\hbar\in[0,\ell]}\Gamma_\ell(\hbar)\Xi^2_\ell(1+2\Theta\Theta_{\Bbbk}\ell^2)+\frac{1}{\tau}\right]\|\overline{\zeta}^*-\overline{\zeta}^{**}\|_\ell. \tag{70}$$

By using $\overline{\zeta}=\zeta$ on $[0,\ell]$ and taking supremum over $\hbar$, implies

$$\|\Phi\overline{\zeta}^*-\Phi\overline{\zeta}^{**}\|_\ell$$
$$\leq\left[3\sup_{\hbar\in[0,\ell]}\Gamma_\ell(\hbar)\Xi^2_\ell(1+2\Theta\Theta_{\Bbbk}\ell^2)+\frac{1}{\tau}\right]\|\overline{\zeta}^*-\overline{\zeta}^{**}\|_\ell, \tag{71}$$

showing that the operator Φ is a contraction.

From the choice of Υ there is no $\zeta\in\partial\Upsilon^\ell$ such that $\zeta=\Phi(\zeta)$ for some $\xi\in(0,1)$.

The operator Φ has a unique fixed point ζ, which is the unique mild solution of the problem (45) - (46), as a result of Frigon and Granas' nonlinear alternative. The proof is complete.

Conclusion

In this work, we have effectively investigated the controllability in Frechet spaces of fractional stochastic neutral integro-differential equations with state-dependent delay. We have proven the existence of unique mild solutions under certain conditions by utilizing sophisticated mathematical approaches including fractional calculus, fixed point theory, and the characteristics of characteristic solution operators. Our results integrate the fractional aspect and state dependent delays, which are essential for many real-world applications in science and engineering, and so expand on the literature that already exists. To further expand the application of these results, future research should try to investigate more generalized systems and loosen up some of the assumptions utilized in this study.

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