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Equivalent classes of square–integrable functions in the Hilbert space

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Abstract In this paper, we have discussed that the members of the Hilbert space are indeed not square-integrable functions themselves, but rather, the members are equivalent classes of these functions. Contemplating these functions compels us to distinguish between two operators. First, the ordinary derivative $\frac{d}{dx}$ is just the usual partial differentiation which acts on the scalar function and should not be considered as an operator which can act on the vectors in the Hilbert space. Second, the unbounded operator D_x, which is usually mixed up with the former.

1 Introduction

Quantum mechanics is one of the contexts in which the Hilbert space is used. A rigorous definition for a Hilbert space should be sought in mathematical analysis. However, this rigorous definition of the Hilbert space is not included in almost all practical applications of quantum mechanics. That is why, in most texts on quantum mechanics, especially those addressed by physicists, no attention is given to this. However, in the third chapter of the book [1], the author has discussed this subject in summary and footnotes. Ref. [2] is perhaps the only quantum mechanics book that has dealt with this issue more than others. If one were to trace the origins of the concept of a Hilbert space in mathematics and physics, it would be noticed that Hilbert space theory is part of functional analysis [3, 4].

In standard quantum mechanics, some complex Hilbert space describes a quantum mechanical system. For instance, the pure states of a single one-dimensional particle can be described by elements in the Hilbert space $L^2(\mathbb{R})$, as introduced in introductory courses in quantum mechanics. Generally, it is important to note that the L^2 space is the only space among the L^p spaces for which the norm comes from an inner product. The first natural attempt to define this space mathematically is the following:

$$L^2(\mathbb{R}) = \Big\{ f : \mathbb{R} \to \mathbb{C} \ : \ f_{[-n,n]} \text{ Riemann integrable, } \text{for } n \in \mathbb{N}, \int_{-\infty}^{\infty} |f(x)|^2 \, dx < \infty \Big\}. \tag{1}$$

In the definition (1), functions f are so-called square-integrable functions and Riemann integrable. In general, a Hilbert space is a set of vectors equipped with two operations: vector addition and scalar multiplication. It satisfies the axioms of a vector space over the field of complex numbers (or real numbers).

However, there are several problems with this approach. For example, there are plenty of square-integrable functions (according to definition (1)) which do not vanish at infinity [5]. Indeed, it is false that $f \in L^2(\mathbb{R})$ implies that $f \to 0$ as $x \to \infty$, as commonly believed. Additionally, the Hilbert space defined in (1) is not complete[1]. There exist some Cauchy sequences in $L^2(\mathbb{R})$ which do not converge in $L^2(\mathbb{R})$. For example, consider the sequence $\{\psi_j\}$, where ψ_j is a function such that $\psi_j(q_i) = 1$ for $i \leq j$, and $\psi_j(x) = 0$ otherwise. Here $\{q_i\}$ denotes an enumeration of the rational numbers[2].

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[1] There are plenty of examples which demonstrate that both the set of irrational and rational numbers are not complete. There are well-behaved sequences in each space that don't converge to an element of the space. These sequences are well-behaved in the sense that they do converge in $\mathbb{R}$.

[2] In fact, ψ_j is a sequence of functions such that $\psi_1, \psi_2, \psi_3, \ldots$. For instance, $\psi_3(q_1) = 1$, $\psi_3(q_2) = 1$, and $\psi_3(q_3) = 1$. So $\psi_3(x) = 1$ if x is one of the first three rationals; otherwise, $\psi_3(x) = 0$. Hence, each $\psi_j(x) = 0$ except for a finite number of points (to be more specific, except for three points)

Clearly, for each ψ_j, we have $\int_{-\infty}^{\infty} |\psi_j(x)|^2 \, dx = 0$, and this is true for $(\psi_i - \psi_j)$ hence for any i and j the amount $(\psi_i - \psi_j)$ is nonzero only in a finite number of points. So ψ is a Cauchy sequence. While the limit of this sequence is the function $\varphi(x)$ so that $\varphi(x) = \begin{cases} 1, & x \in \mathbb{Q}, \\ 0, & x \notin \mathbb{Q}, \end{cases}$ where $\mathbb{Q}$ is a set of rational numbers. Now, we can conclude that $|\varphi|^2$ is not Riemann integrable.

As a result, if the inner product is defined using Riemann integration, the Cauchy sequence $\{\psi_j\}$ does not converge in the space of square-integrable functions. However, if Lebesgue integration is used, the Cauchy sequence converges. Also, if the inner product is defined with Lebesgue integration, the Cauchy sequence ψ does converge.

In a later section, we will discuss the distinction between the operators $\frac{d}{dx}$ and Dx.

In some quantum mechanics literature, for example, in [1, 6], misleading notations are seen, which can confuse different operators. We must distinguish between the position operator X and the real number x. Also, $f(x)$ is a number[3]. The symbol $\frac{d}{dx}$ is not well-suited to denote an operator acting on elements of the Hilbert space. Instead, we denote by D_x an operator acting in Hilbert space such that

$(D_x f)(x) = f'(x)$, where $f'(x)$ is the usual derivative of $f(x)$, assuming f is differentiable.

2 Equivalent Classes of Square–Integrable Functions

According to definition (1), we can define the norm $\|f\| = \left(\int_{-\infty}^{\infty} |f(x)|^2 \, dx\right)^{1/2}$ as a norm on $L^2(\mathbb{R})$. However, there exist functions $f \in L^2(\mathbb{R})$, with $f \neq 0$, for which $\|f\| = 0$. As an example, consider the function $f(x) = \begin{cases} 1, & x = 0, \\ 0, & x \neq 0. \end{cases}$ This function is clearly not identically zero, but $\int_{-\infty}^{\infty} |f(x)|^2 \, dx = 0$. To resolve this, we introduce an equivalence relation: two functions $f, g \in L^2(\mathbb{R})$, where $\|f - g\|_2 = 0$. These functions are called *equivalent* if

$$\int_{-\infty}^{\infty} |f(x) - g(x)|^2 \, dx = 0. \tag{2}$$

According to Eq. (3), the functions f and g can be different on some poinbts. Therefore, when the operator D_x is applied to a function, we must account for this equivalence.

For example, suppose f is differentiable and g equals f except at a single point x_1. Then g is not differentiable at x_1, but $D_x g$ is still defined as $D_x f$, because f and g are equivalent.

As an example for this points, consider the square-integrable function $f(x) = \frac{1}{1+x^2}$, and define

$$g(x) = \begin{cases} f(x), & x \neq 0, \\ 0, & x = 0. \end{cases} \tag{3}$$

It is noted that the former function is differentiable everywhere and the latter is not differentiable and therefore not continuous at $x = 0$. In fact, the reason that we should consider the equivalence classes rather than functions themselves, is to make the inner product non singular. Here, the inner product of $(f - g)$ by itself is zero, by definition of Eq. (3). While the inner product of a vector by itself should be non-zero (positive) unless that vector is zero. Therefore, if Eq. (3)) is satisfied $(f - g)$ is taken to be equivalent to 0. It means that the action of any linear operator (acting on the Hilbert space) on $(f - g)$ should be zero. So, if $D_x f$ is defined, $D_x g$ or its generalization should be defined equal to $D_x f$. Another point that should be noted is that $D_x f$ is not necessarily square-integrable despite of f being a square-integrable function. So, the domain of D_x is not the whole Hilbert space, but it is an unbounded operator.

Also, it is worthwhile to point out, that if $(f - g)$ is non-zero only at a finite number of points, Eq. (3) will be true. More generally, if the set at which $(f - g)$ is non-zero, has zero measure, then the Eq. (3) is valid and f and g are called equivalent. In other words, if two functions f and g differ from each other on a set with zero measure, then they belong to the same equivalence class[4].

We are now ready to define the desired Hilbert space [7]. A Hilbert space in the context of the Lebesgue measure typically refers to a complete vector space with an inner product defined via the Lebesgue integral.

A classic example of a quantum system in an infinite-dimensional Hilbert space is a quantum particle constrained to move along the real line. In this case, the Hilbert space is defined as $\mathcal{H} := L^2(\mathbb{R}, dx)$,

where dx denotes the standard Lebesgue measure on $\mathbb{R}$. The members of this Hilbert space are equivalence classes of measurable functions $f : \mathbb{R} \to \mathbb{C}$ with unit norm, $\|f\|_2 = \int_{\mathbb{R}} |f(x)|^2 \, dx = 1$.

In this formal setting, momentum operators can be precisely defined using weak derivatives [8]. These definitions are elaborated in [7], where the discussion is extended to generalized momentum operators and domains [9].

[3]Note that f is a smooth and differentiable function and $f(x)$ is a pointwise evaluation at x, just like x itself. That is, $f(x)$ is the value of the function f at the point x.

[4]As an example, consider two functions f and g that differ only on the set of integers or a subset thereof. Since the measure of the integers is zero, $f \sim g$ in $L^2(\mathbb{R})$.

3 Difference between D_x and $\frac{d}{dx}$

This section discusses the differential operator, which acts on square-integrable functions in a Hilbert space. It is different from the usual differential operator defined in classical calculus.

One can define the operator D_x through its action on the Fourier transform of the function. For a function ψ, the Fourier transform satisfies the following identity:

$$F(D\psi)(k) = ik(F\psi)(k), \tag{4}$$

This gives a correspondence between $D_x\psi$ and multiplication by ik in the Fourier domain.

So, using above equation, we can deduce

$$(D_x\psi)(x) = \frac{1}{2\pi}\int\int e^{ikx-iky}\psi(y)\cdot(ik)\,dk\,dy. \tag{5}$$

If ψ is differentiable in the usual sense, then the right-hand side is equal to the derivative of this function. But the point is that, it can happen that the right-hand side exists, while ψ is not differentiable in the usual sense. Hence, Eq. (5) is some generalization of the definition of the derivative of ψ.

It should be npted that the operator D_x defined above is a dense in the Hilbert space. An operator is said to be densely defined in the Hilbert space if for any u and for any $\varepsilon > 0$, there exists a v such that $\|u - v\| < \varepsilon$ in the domain of the regarding operator.

Let $F(v)$ be the Fourier transform of v. It is shown that the Fourier transform of D_xv, denoted $F(D_xv)$, satisfies:

$$|F(D_xv)(k)|^2 = k^2|F(v)(k)|^2. \tag{6}$$

Now suppose $F(v)(k) = 0$ for $|k| > M$, where M is some constant. Then:

$$\int |F(D_xv)(k)|^2\,dk \leq M^2\int |F(v)(k)|^2\,dk. \tag{7}$$

For v in the Hilbert space, the right-hand side and hence the left-hand side will be finite. We immediately infer if v is in the Hilbert space and its Fourier transform vanishes for $|k|$ bigger than some constant, then v is in the domain of D_x.

This result implies that if a function $v \in L^2(\mathbb{R})$ has a compactly supported Fourier transform, i.e., vanishes for $|k| > M$, then $D_xv \in L^2(\mathbb{R})$, and v belongs to the domain of the operator D_x.

For any vector u in Hilbert space, the square-integrability is defined via its Fourier transform by:

$$\int |Fu(k)|^2\,dk < \infty. \tag{8}$$

This means that for any $\alpha > 0$, there exists an N such that:

$$\left|\int |(Fu)(k)|^2 - \int_{-N}^{N}|(Fu)(k)|^2dk\right| < 2\pi\alpha. \tag{9}$$

One can take ν to be a function with the following properties:

$$\mathcal{F}v(k) = \begin{cases} \mathcal{F}u(k), & |k| \leq N, \\ 0, & |k| > N. \end{cases}$$

Clearly, v in in the domain of D_x and $|u - v|^2 \leq \alpha$. So, for any $\varepsilon > 0$ ine take $\alpha = \frac{1}{2}\varepsilon^2$ and $|u - v| < \varepsilon$, while v is in the domain of D_x.

4 Conclusion

One of the key properties of the real numbers $\mathbb{R}$ is that all Cauchy sequences converge. This is proven in many real analysis texts, such as Chapter 1 of [10] or in [11].

However, we have shown that if the inner product is defined using Riemann integration, then $L^2(\mathbb{R})$ is *not* complete. This highlights the need for the Lebesgue integral. As discussed in this paper, if the inner product is defined using the Lebesgue integral, then $L^2(\mathbb{R})$ is a complete Hilbert space, meaning every Cauchy sequence converges within the space.

As a result, the members of the Hilbert space, contrary to common belief, are not square-integrable functions themselves but equivalence classes of such functions. To emphasise this difference, we may denote equivalence classes by a notation such as $[f]$.

As it is shown, this point is not only a curiosity or mathematical accuracy, but it leads to the distinction of two operators, which can remove some misconceptions. We should stress the difference between the two operators. The operator $\frac{d}{dx}$ acts on any object depending on x, be it scalar or vector. However the operator $D_x$is an unbounded operator which acts from right on vectors and convectors in the Hilbert space.

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References

1. D. J.Griffiths, D.F.Schroeter, *Introduction to quantum mechanics*, 3rd Cambridge University Press, (2018)
2. C. J.Isham, *Lectures on Quantum Theory: Mathematical and Structural Foundations*, ICP (1995).
3. K. S. Ranade, Fortschritte der Phy. **63**, (2015)
4. J. Dittrich, H.Kovarik, A.Laptev, J. Eur. Math. Soc. (2017)
5. F. Gieres, Rep. Prog. Phys. **63**, (2001)
6. S. Gasirowicz, *Quantum mechanics*, 3rd Edition, John Wiley, New York (2003)
7. V.Moretti, Int.J.Geom. Meth. Mod.Phys. **13**, (2016)
8. M. Jafari Matehkolaee, A. Hoseinian Abarghoei, Tran. Th. Math. Phys., **1**, (2024)
9. M. Jafari Matehkolaee and G. Rastegarzadeh, Pramana. J. Phys. **95**, (2021)
10. H.L.Royden, *Real Analysis*, Prentic Hall, (1988)
11. G.Folland, *Real Analysis: Modern Techniques and Their Applications*, Wiley, (1999)