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A brief survey on the Selberg trace formula (the compact case)

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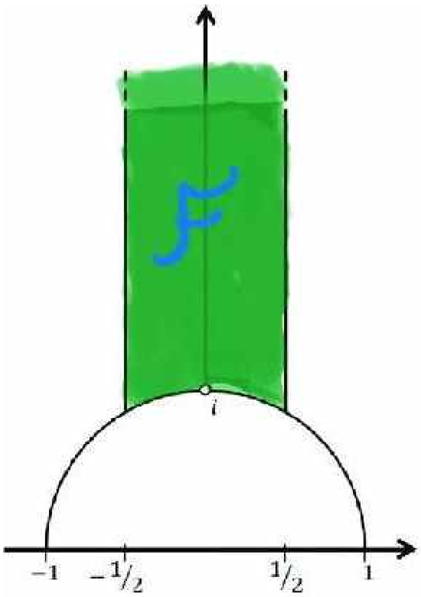
Abstract In this note, we aim to elucidate the fundamental mathematical concepts and mathematical ingredients underlying the Selberg trace formula. We explicitly present the formula for compact quotients and provide a brief overview of its interpretation in both mathematics and physics. The Selberg trace formula transcends the boundary of mathematics, establishing intriguing connections between classical mechanical entities such as volume, shape, and geodesics on a surface and quantum mechanical entities such as eigenvalues (frequencies), eigenfunctions, and resonances of the underlying geometry. A rudimentary understanding of complex analysis and hyperbolic geometry is assumed.

1 Introduction

I prepared this note for a conference on mathematical physics held in July 2024 at Qom university of technology in Qom city. Delivering a lecture on the Selberg trace formula sparked my curiosity to delve deeper into its intricate beauty and profound significance. I was fascinated by the beauty of the Selberg trace formula while preparing this report. This note serves as a (very) brief survey of the Selberg trace formula and some of its relatives which have far-reaching applications in various mathematical and physical domains, and I hope that this note will help to introduce it to others.

We begin by introducing a pivotal group known as the modular group, denoted by $\Gamma = SL(2,\mathbb{Z})$. This group acts naturally on the upper half plane $\mathscr{H}$ by linear-fractional transformations and the resulting quotient of this action is the modular surface denoted by $M = \mathscr{H}/SL(2,\mathbb{Z})$. For the interested reader, a brief introduction to hyperbolic geometry and complex analysis is essential. The modular surface $M = \mathscr{H}/SL(2,\mathbb{Z})$ is a rich mathematical entity, not only from a hyperbolic geometric perspective but also from a harmonic analysis standpoint. The following figure illustrates the renowned fundamental domain of the action of the group $SL(2,\mathbb{Z})$ on the upper half plane. This fundamental domain is also known as the Gauss fundamental domain, as Gauss had discovered it using quadratic forms (of negative discriminant) before the advent of hyperbolic geometry!



The resulting quotient surface, known as the modular surface, is a non-compact surface with finite volume (its volume is $\frac{\pi}{3}$). It is noteworthy that the geodesics on the modular surface are the projections of the geodesics in the upper half plane. Here is a picture of the modular surface and a closed geodesic on it.

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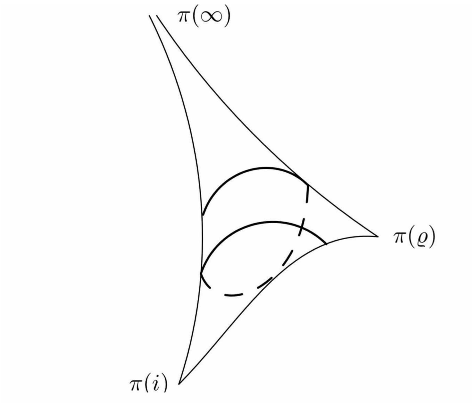


Furthermore, Studying quadratic forms of positive discriminant is closely intertwined with the study of the closed geodesics and their corresponding multiplicities on the aforementioned modular surface. A powerful tool for analyzing the behavior of closed geodesics on the surfaces (both compact or non-compact) is the Selberg trace formula. In this report, I aimed to understand this crucial formula and its significance in understanding closed geodesics on a surface. In fact, the Selberg trace formula establishes a connection between the geodesic geometry of a surface and its spectral geometry (eigenvalues of Laplacian). In this report I primarily focused on the compact case of the Selberg trace formula which is essential before delving into the more challenging case of the non-compact quotient.

The Selberg trace formula was originally discovered by Selberg in the 1950s (see [1]) influenced by the work of Maass, who employed techniques from harmonic analysis to study the spectral theory of hyperbolic surfaces. In essence, the Selberg trace formula is an identity that has a geometric side and deals with geometric entities such as closed geodesics of our underlying geometry (our surface) and a spectral side that involves eigenvalues of Laplacian of the underlying Riemann surface. These eigenvalues are crucial in quantum physics and play the role of modes (frequencies) of a domain or surface (for instance, the sound we hear from a drum). Before formulating the trace formula for hyperbolic surfaces, it is enlightening to consider the trace formula for the unit circle, which is essentially a reformulation of the Poisson summation formula. This formula for a suitable function h is as follows:

$$\sum_{n\in\mathbb{Z}} h(n) = \sum_{n\in\mathbb{Z}} \hat{h}(n), \tag{1}$$

where $\hat{h}(n) = \int_{\mathbb{R}} h(t)e^{2\pi int}dt$ is the Fourier transform of the test function h. Now, if we examine the above poisson summation formula more closely, we can reinterpret both sides of this formula as follows. In fact, the Poisson summation formula serves as the initial prototype of the trace formula. Consider $M = S^1$, a one-dimensional circle of radius 1. It is readily apparent that he eigenvalues of the Laplacian $\Delta = -\frac{d^2}{dx^2}$ on the circle S^1 are n^2 for $n = 0, \pm 1, \pm 2, \cdots$ with corresponding eigenvalues : $\phi_n(x) = \frac{1}{\sqrt{2\pi}}e^{inx}$. Define the linear operator T on the space of periodic functions as the following

$$[Tf](x) := \int_{[0,2\pi]} \rho(x,y)dy, \tag{2}$$

where the kernel $\rho(x,y) := \sum_{n\in\mathbb{Z}} h(n)\phi_n(x)\bar{\phi}_n(y)$. It is evident that $T\phi_n = h(n)\phi_n$. Consequently, the Poisson summation formula essentially entails computing the trace of the aforementioned linear operators in two different methods.

$$\mathrm{Tr}(T) = \sum_{n\in\mathbb{Z}} h(n) = \int_{\mathbb{R}} h(t)e^{2\pi int}dt. \tag{3}$$

Note that the left-hand side is a spectral quantity, while the right hand side is a geometric quantity, calculated as a sum over the closed geodesics on the manifold S^1 that have length $2\pi|n|$ for $n \in \mathbb{Z}$. In this way, we can see that trace formula for S^1 is actually a reformulation of the Poisson summation formula. We will also see that the strategy for higher-dimensional case of surfaces is quite similar to the above argument.

Furthermore, it is illuminating to examine the trace formula for the 2-dimensional sphere, as we possess explicit knowledge of the spectrum of the Laplacian and the structure of closed geodesics in this special case (similar to the case of circle). However, we omit this enlightening examples (the sphere) in this report and refer to [2, 3] for further details. In the context of hyperbolic surfaces, it is well-known that we have the following correspondence:
Geometric entities ⟷ spectral entities
(volume, shape, geodesics) ⟷ (eigenvalues, eigenfunctions, resonances)

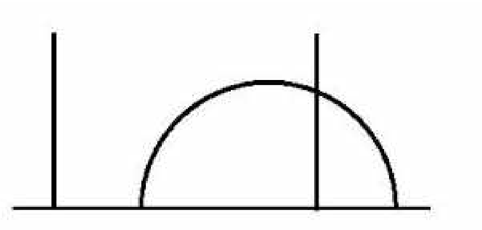
Additionally, we can provide motivation from a physics perspective. The classical-quantum correspondence principle in physics relates classical mechanical entities such as closed geodesics, to quantum mechanical entities like resonances and frequencies (modes).

2 Some background in hyperbolic geometry and hyperbolic surfaces

The upper half plane

$$\mathscr{H} = \{z \in \mathbb{C} | Im(z) > 0\}, \tag{4}$$

serves as a standard model for hyperbolic geometry. In this geometry, the straight lines are segments of semi-circles centered at a point on the real line (boundary at infinity) and lines perpendicular to the real line.



The group of isometries is the Mobius transformations (or linear-fractional transformations) defined as

$$\phi : z \mapsto \frac{az+b}{cz+d}, \tag{5}$$

where $a,b,c,d \in \mathbb{R}$ and $ad-bc > 0$. It is well-known that this group is equivalent to the Lie group $PSL(2,R)$ (the group of 2 by 2 matrices with real entries and positive determinants). Furthermore, it is straightforward to demonstrate that the hyperbolic metric:

$$ds^2 = \frac{dx^2+dy^2}{y^2}, \tag{6}$$

is invariant under every element ϕ in the isometry group $PSL(2,R)$, more precisely we have

$$\phi^*(ds^2) = ds^2. \tag{7}$$

The above metric is also known as the Poincare metric. It can be easily observed that the Gaussian curvature of this metric is constant and equal to -1. The area element of this metric is $dA(z) = \frac{dxdy}{y^2}$. One can see that the distance between two points $z,w \in \mathscr{H}$ can be obtained by the following formula:

$$d(z,w) = \log \frac{|z-\bar{w}|+|z-w|}{|z-\bar{w}|-|z-w|}. \tag{8}$$

We classify the transformations in the isometry group $PSL(2,R)$ based on the behavior of their fixed points. Suppose that $T(z) = \frac{az+b}{cz+d}$ is an element of $PSL(2,R)$, and z is a fixed point of this transformation. Then z must satisfy the following quadratic equation:

$$cz^2+(d-a)z-b = 0. \tag{9}$$

Generally this equation has 2 complex solutions.

(1) The transformation T is called elliptic if it has only one fixed point in the upper half plane (the other is in the lower half plane). In this case, geometrically, T is a rotation centered at the fixed point. One can see that the T is elliptic if and only if $|\mathrm{Tr}(T)| = |a+d| < 2$.

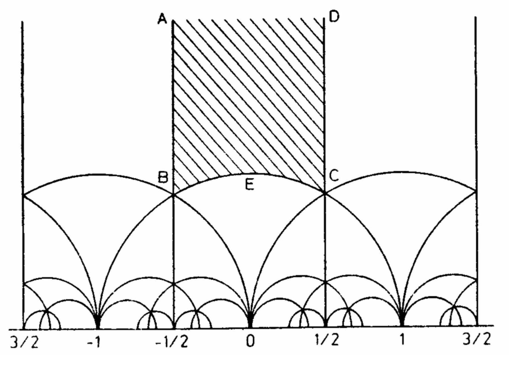
(2) The transformation T is referred to as parabolic if T has only one fixed point on the real line $\mathbb{R} = \partial\mathscr{H}$. In this case, T is conjugate to the translation $z \mapsto z+1$. One can see that the T is parabolic if and only if $|\mathrm{Tr}(T)| = |a+d| = 2$.

(3) The transformation T is called hyperbolic if it has two distinct fixed points on the boundary real line. In this case, T is conjugate to the transformation $z \mapsto e^a z$ for some real number a. One can see that the T is hyperbolic if and only if $|\mathrm{Tr}(T)| = |a+d| > 2$.

Table 1 Complete classification of hyperbolic isometries

Type	Fixed Points	Canonical Form	Trace
Elliptic	1 in $\mathscr{H}$	$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$	$2\cos\theta$
Parabolic	1 on $\partial\mathscr{H}$	$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	2
Hyperbolic	2 on $\partial\mathscr{H}$	$\begin{pmatrix} e^{l/2} & 0 \\ 0 & e^{-l/2} \end{pmatrix}$	$2\cosh(l/2)$

Now, we consider a Fuchsian group , which is a discrete subgroup of the isometry group $PSL(2,\mathbb{R})$. Our primary example in this context for a Fuchsian group is the discrete subgroup $SL(2,\mathbb{Z})$, which comprises 2 by 2 matrices with integer entries and determinant 1. This subgroup holds significant importance and is known as the modular group. It finds applications across various mathematical disciplines, ranging from number theory to physics. When a discrete group acts on the upper half plane $\mathscr{H}$ we can define a fundamental domain for this action. This domain is a polygonal region in the plane, ensuring that every orbit intersects the domains interior at most once (except for the boundary of this region). For instance, a fundamental domain for the modular group action is depicted in the following figure.



It's worth noting that if the fauchsian subgroup lacks parabolic and elliptic element, the quotient Riemann surface $S = \mathscr{H}/\Gamma$ is a compact Riemann surface with constant negative curvature -1. However, in the case $\Gamma = PSL(2,Z)$, the quotient surface, known as modular surface, is not compact. It possesses a cusp at infinity and also has finite volume. Geodesics on the modular surface S (or generally on any Riemann surface) correspond to the geodesics of the upper half plane under the action of the modular group. A function on the quotient $S = \mathscr{H}/\Gamma$ is a function on the upper half plane $\mathscr{H}$ which is invariant under the action of the Fuchsian group , meaning $f(gz) = f(z)$ for all $g \in \Gamma$. These functions are referred to as automorphic functions. Closed geodesics and their characterization on a Riemann surface (specially on the modular surface) holds immense importance in studying the surfaces geometry. They also serve as the foundation for comprehending the Selberg trace formula. When dealing with geodesics on the Riemann surface S, several key questions arise :

1) which geodesics are closed on the surface S.
2) Which closed geodesics are simple?
3) Are there dense geodesics on the surface, and how can they be characterized?

The answer to the first question is the following. Every hyperbolic matrix in $SL(2,\mathbb{Z})$ generates a closed geodesic and it is easy to see that any matrix in the modular group with trace greater than 2 is hyperbolic. Therefore, the closed geodesics correspond to conjugacy classes of hyperbolic elements in the Fuchsian group Γ.

The Laplacian operator on the upper half plane H with Poincare metric is defined as

$$\Delta := -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right). \tag{10}$$

Since the hyperbolic Laplacian is invariant under the action of the group $\Gamma = SL(2,Z)$, i.e. $go\Delta = \Delta og$ for $g \in SL(2,R)$, it descends on the quotients Riemann surface $S = \mathscr{H}/\Gamma$. The area measure $dA(z) = \frac{dxdy}{y^2}$ is also invariant under the action of the group and so it gives an area measure (volume form) over the quotient Riemann surface S. The natural Hilbert space that we consider is $L^2(\mathscr{H}/\Gamma, dA)$ with the inner product

$$< f,g >:= \int_F f(z)g\bar{(z)}dA(z), \tag{11}$$

for a fundamental domain F. An eigenvalue of Δ on $S = \mathscr{H}/\Gamma$ is a number λ which satisfies the equation $\Delta\phi = \lambda\phi$ where $\phi \in L^2(\mathscr{H}/\Gamma, dA)$ is a square integrable function and in the literature the corresponding eigenfunction is called a Maass form. It is a well-known fact that if the quotient Riemann surface is compact, its corresponding spectrum is discrete and tends to infinity. Additionally there exists an orthogonal basis of eigenfunctions $\{\phi_j\}$ for $L^2(\mathscr{H}/\Gamma, dA)$. If the quotient Riemann surface is not compact but of finite volume then the corresponding spectrum has two parts, one is the discrete part contained in the interval $[0,\infty)$ and the other is the continuous part and is in fact it is the interval $[\frac{1}{4},\infty)$. In the case of non-compact quotient we attempt to study the spectrum by introducing a new operator called the resolvent, denoted as $R(\lambda) := (\Delta - \lambda)^{-1}$. It is important to note that in the upper half plane, $\mathscr{H}$, we have $\Delta y^s = s(1-s)y^s$. Therefore we assume that our spectral parameter is $\lambda = s(1-s)$ and the resolvent is

$$R(s) := (\Delta - s(1-s))^{-1}. \tag{12}$$

It is well known that the resolvent admits a meromorphic continuation to the entire complex plane. The set $\lambda \in [0,\infty)$ corresponds to

$$s \in [\frac{1}{2},1] \cup \{s|Re(s) = \frac{1}{2}\}, \tag{13}$$

(see [4]). Recall that in the literature, the poles of the meromorphically extended resolvent are referred to as resonances. Another well-known result is that the continuous spectrum is parametrized by the Eisenstein series. Furthermore, the Eisenstein series can also be meromophically continued to the complex plane with respect to the variable s. Now, we endeavor to describe the Selberg trace formula in the case of compact quotients. Assume that the function $f \in C^\infty([0,\infty))$ decays sufficiently rapidly at infinity. In this case, we can define the integral operator by introducing the kernel.

$$k_f(z,w) := \sum_{g\in\Gamma} f(d(z,gw)), \tag{14}$$

and following the above kernel one can define the operator $K_f : L^2(S,dA) \to L^2(S,dA)$ as follows:

$$(K_f(g))(z) := \int_F k_f(z,w)g(w)dA(w). \tag{15}$$

The Selberg trace formula provides two distinct methods for computing the trace of the operator K_f. In the case of the compact quotient $S = \mathscr{H}/\Gamma$ as we mentioned, we have a discrete spectrum $\{\lambda_j\}$ and corresponding orthogonal eigenfunctions $\{\phi_j\}$. On the one hand, we have

$$\mathrm{Tr}(K_f) = \int_{\mathscr{H}/\Gamma} k_f(z,z)dA(z), \tag{16}$$

and on the other hand if we are given the eigenvalues $\{\kappa_j\}$ of the operator K_f, we can express the trace as

$$\mathrm{Tr}(K_f) = \sum_j \kappa_j. \tag{17}$$

3 The Selberg trace formula

Assume that the quotient Riemann surface $S = \mathscr{H}/\Gamma$ is compact and smooth, i. e., the discrete group Γ contains only hyperbolic elements. Recall that closed geodesics of $S = \mathscr{H}/\Gamma$ correspond to conjugacy classes of hyperbolic elements in the subgroup. Now, we can compute the trace of the operator K_f in terms of the length of the closed geodesics of the surface S. Assume that $L(\Gamma) := \{l(g) | g \in \Pi\}$ where Π is the set of hyperbolic conjugacy classes. Using the decomposition of the group in terms of the different types of conjugacy classes (hyperbolic, elliptic or parabolic) and after computation, the following formula for the trace of the operator K_f is obtained

$$\mathrm{Tr}(K_f) = f(0)\mathrm{Vol}(\mathscr{H}/\Gamma) + \sum_{l\in\Gamma}\sum_{n\in\mathbb{N}} \frac{1}{\sinh(\frac{kl}{2})}\int_{kl}^{\infty}\frac{f(\cosh t)}{\sqrt{2\cosh t - 2\cosh kl}}\sinh t dt. \tag{18}$$

Also from the spectral side we have

$$\mathrm{Tr}(K_f) = \sum_{j=0}^{\infty} h(\sqrt{\lambda_j - \frac{1}{4}}), \tag{19}$$

where h is an appropriate function and after simplification the Selberg trace formula for the compact quotient case takes the following form: group in terms of the different types of conjugacy classes (hyperbolic, elliptic or parabolic) and after computation, we obtain the following formula for the trace of the operator K_f:

$$\sum_{j=0}^{\infty} h(\sqrt{\lambda_j - \frac{1}{4}}) = \frac{\mathrm{Vol}(\mathscr{H}/\Gamma)}{4\pi}\int_{-\infty}^{\infty} rh(r)\tanh(\pi r)dr + \sum_{l\in L(\Gamma)}\sum_{k\in\mathbb{N}}\frac{1}{\sinh(\frac{kl}{2})}\hat{h}(kl), \tag{20}$$

where $\hat{h}$ is the Fourier transform of the function h. It is noteworthy that the first term on the right hand side constitutes a topological term. This trace formula can be considered as a natural generalization of the Poisson summation formula.

4 The formula for the non-compact case

In the non compact case, the Selberg trace formula becomes more intricate due to the fact that the operator K_f is no longer a trace class operator. Consequently, new terms and integrals corresponding to elliptic and parabolic elements emerge. Notably, on the spectral side, scattering terms related to cusp points appear, transforming this side of the trace formula as follows:

$$\sum_{j=0}^{\infty} h(\sqrt{\lambda_j - \frac{1}{4}}) - \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\phi'}{\phi}(\frac{1}{2}+ir)h(r)dr + \frac{1}{2}h(0)\mathrm{Tr}[\phi_{ij}(\frac{1}{2})], \tag{21}$$

where $\phi_{ij}(s)$ denotes the scattering matrix and $\phi(s) = \det[\phi_{ij}(s)]$. Similarly, on the length side, additional terms corresponding to elliptic and parabolic points are introduced, in addition to the previously mentioned term corresponding to hyperbolic elements as observed in the compact quotient case. For each elliptic element in the group, the following terms should be added

$$\sum_{j=1}^{n-1}\frac{1}{n\sin(\frac{\pi j}{n})}\int_{-\infty}^{\infty}\frac{e^{-\frac{2\pi j r}{n}}}{1-e^{-2\pi r}}h(r)dr, \tag{22}$$

where n is the order of the elliptic fixed point.

In the end, for a non-compact Riemann surface $\mathscr{H}/\Gamma$ like the group $\Gamma = SL(2,\mathbb{Z})$, with finite volume, the Selberg trace formula acquires additional terms and is as follows:

$$\begin{aligned}
&\sum_{j=0}^{\infty} h(r_j) - \frac{1}{4\pi}\int_{-\infty}^{\infty}\frac{\phi'}{\phi}\left(\frac{1}{2}+ir\right)h(r)\,dr \\
&\quad + \frac{1}{4}h(0)\mathrm{tr}\,\phi\left(\frac{1}{2}\right) \\
&= \frac{\mathrm{Vol}(\mathscr{F})}{4\pi}\int_{-\infty}^{\infty} rh(r)\tanh(\pi r)\,dr \\
&\quad + \sum_{\{\gamma\}}\sum_{k=1}^{\infty}\frac{l_\gamma}{2\sinh(kl_\gamma/2)}\hat{h}(kl_\gamma) \\
&\quad + \frac{1}{4}h(0) - \frac{\ln 2}{2\pi}\hat{h}(0) \\
&\quad + \sum_{\{R\}}\frac{1}{2m_R\sin(\pi/m_R)}\int_{-\infty}^{\infty}\frac{e^{-2\pi r/m_R}}{1+e^{-2\pi r}}h(r)\,dr.
\end{aligned} \tag{23}$$

Here, $\lambda_j = \frac{1}{4} + r_j^2$ are Laplacian eigenvalues, $\phi(s)$ is the scattering matrix determinant, $\{\gamma\}$ are primitive hyperbolic conjugacy classes (lengths l_γ), $\{R\}$ are elliptic conjugacy classes (orders m_R), and $\hat{h}$ is the Fourier transform of h.

5 Selberg Zeta function

One of the primary applications of the Selberg trace formula is the proof of the meromorphic continuation of a crucial zeta function known as the Selberg zeta function. The Selberg zeta function for Γ is defined as

$$Z_\Gamma(s) = \prod_{\{\gamma\}} \prod_{k=0}^{\infty} \left(1 - e^{-(s+k)l_\gamma}\right), \tag{24}$$

It satisfies the functional equation

$$Z_\Gamma(s) = Z_\Gamma(1-s) \times \exp\left(\frac{\mathrm{Vol}(\mathscr{H}/\Gamma)}{2\pi} \int_0^{s-1/2} r \tan(\pi r) dr\right). \tag{25}$$

The zeros occur at

- $s = \frac{1}{2} \pm i r_j$ (spectral zeros).
- $s = -k$, $k \in \mathbb{N}$ (trivial zeros).

One can also observe that the above product converges for $\mathrm{Re}(s) = 1$. It can be demonstrated using the trace formula that the Selberg zeta function can be analytically continued to the entire complex plane. The zeros of the Selberg zeta functions correspond to the eigenvalues of the hyperbolic Laplacian.

It would be intriguing to dissect the simple closed geodesics among all closed geodesics. For instance, if we modify the definition of the Selberg zeta function for a Riemann surface by multiplying over all simple closed geodesics, the resulting zeta function raises pertinent questions about which properties of the ordinary Selberg zeta function can induce this modified zeta function? For instance, can we extend the analytic continuation of this modified Selberg zeta function over the complex plane? Similarly, investigating the functional equation and the corresponding Riemann hypothesis holds potential. It may be worthwhile to devise a method and adapt the renowned Selberg trace formula for simple closed geodesics. This modified trace formula could then be employed to obtain the analytic continuation of the modified Selberg zeta function. Another interesting property of hyperbolic surfaces is the ergodicity of their geodesic flow. This implies that every function on the hyperbolic surface that is invariant under the geodesic flow must be a constant function. This type of ergodicity is called classical ergodicity. However, there exists a broader notion of ergodicity. Quantum ergodicity is another notion of ergodicity that holds significant importance in quantum chaos. Notably, early numerical simulations in chaotic billiards revealed intriguing phenomena involving highly excited quantum particles concentrating on periodic trajectories corresponding to classical particles. Quantum ergodicity is formally defined in terms of the Laplace eigenfunctions on a domain or, more generally, a surface or manifold. For each eigenfunction Φ_j we can associate a probability measure

$$\nu_j = |\Phi_j|^2 dA. \tag{26}$$

A central and intriguing conjecture in this research area, which now has been established as a theorem for a specific set of surfaces, points that the probability measures j converge to the Liouville measure on the unit tangent bundle of the corresponding Riemann surface. This phenomenon is referred to as quantum unique ergodicity (QUE), as discussed in [5]. For general compact manifolds with ergodic geodesic flow, it is known that for numerous choices of subsequences of eigenfunctions, the corresponding limits of the probability measures exist and are equal to the Liouville measure on the manifold (as detailed in [6]). However, this property does not hold for all sequences.

Another conjecture about the modular surface is that the simple closed geodesics on this special surface is simple, meaning that their multiplicity is 1 in the length spectrum. Through a well-known argument, it can be deduced that the corresponding quadratic forms possess class number one (according to the correspondence between closed geodesics and indefinite quadratic forms). We believe that the Selberg trace formula and Selberg zeta function are tools that will assist us in addressing the following problem. Let $K = \mathbb{Q}(\sqrt{d})$ be a real quadratic field and $\mathscr{O}_K$ be its ring of integers. The discrete group $\Gamma_K = PSL(2, \mathscr{O}_K)$ acts on the upper half plane, and we can consider the quotient surface $S = \mathscr{H}/\Gamma_K$. How is the geometry of this surface related to the arithmetic of the real quadratic number field K, such as class numbers and fundamental units, etc to the closed geodesics and spectrum of the surface S? We can also pose the same question for the complex surface (called Hilbert modular surface of dimension 2) $M = \mathscr{H} \times \mathscr{H}/\Gamma_K$. It is noteworthy that, according to a theorem by Siegel, the volume of the Hilbert modular surface is, in fact, the value of the Dedekind zeta function of the quadratic field K at -1. Another property is that the number of the cusps of the surface is equal to the class number of the number field K. Consequently, Gauss conjecture is equivalent to the following statement: There exist infinitely many modular surfaces M_K that possess only a single cusp under the action of the discrete group $\Gamma_K = PSL(2, \mathscr{O}_K)$. These inquiries are of interest and can be subject to investigation.

6 Conclusion

The Selberg trace formula presented above reveals that the left-hand side of the formula is solely dependent on the spectral parameters (eigenvalues of the Laplace operator), which corresponds to the modes or frequencies of the surface. These parameters are quantum mechanical entities. Conversely, the right hand side of the trace formula is dependent solely on the length spectrum, which are the classical mechanical parameters of the underlying universe (our surface). In essence, the left hand side of the trace formula resides within the realm of quantum physics, while the right hand side belongs to classical mechanics. In conclusion, this note also posed several intriguing questions relating the structure of the real quadratic fields to the geometry of the corresponding modular surfaces. Understanding these relationships may provide insights into the Gauss conjecture for class numbers of the real quadratic fields.

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References

1. A. Selberg, J. Indian Math. Soc. **20**, (1956)
2. P. Buser, *Geometry and spectra of compact Riemann surfaces*, Birkhauser Publication, (1992)
3. D. Hejhal, *The Selberg trace formula for* $PSL(2, \mathbb{R})$, Springer Verlag Publication, (1976)
4. S. Sarnak, Bull. Am. Math. Soc. **40**, (2003)
5. E. Lindenstrauss, Ann. Math. **163**, (2006)
6. S. Zelditch, *Quantum ergodicity and mixing of eigenfunctions,* Encyclopedia of mathematical physics, Oxford, Vol. 1-5, (2006)