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Asymptotic behaviors of solutions to a singular non-local fourth-order parabolic equation with gradient-type logarithmic nonlinearity

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Abstract In this paper, we consider the initial-boundary value problem to a singular non-local fourth-order parabolic equation with gradient-type logarithmic nonlinearity. We prove the global existence of weak solutions by utilizing the cut-off function technique and Galerkin's approximation method. We establish novel threshold criteria for the finite-time blow-up of solutions with initial value at arbitrary energy levels, and derive explicit upper bounds for the blow-up time under appropriate conditions. Furthermore, by employing energy estimates and specific ordinary differential inequalities, we characterize both non-extinction and extinction phenomena of solutions in finite time, and rigorously quantify their corresponding extinction rates.

1 Introduction

In this paper, we consider the initial-boundary value problem to a singular non-local fourth-order parabolic equation with gradient-type logarithmic nonlinearity:

$$\begin{cases} \frac{u_t}{|x|^s} + \Delta^2 u + \operatorname{div}\left(|\nabla u|^{p-2}\nabla u \log|\nabla u|\right) \\ \qquad = |u|^q - \frac{1}{|\Omega|}\int_\Omega |u|^q dx, & x\in\Omega,\quad t>0, \\ \frac{\partial u}{\partial \eta} = \frac{\partial \Delta u}{\partial \eta} = 0, & x\in\partial\Omega,\ t>0, \\ u(x,0) = u_0(x), & x\in\Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N\,(N>2)$ is a bounded domain with smooth boundary $\partial\Omega$, η denotes the outward unit normal vector on $\partial\Omega$, $x=(x_1,x_2,\cdots,x_N)$ with $|x|=\sqrt{x_1^2+x_2^2+\cdots+x_N^2}$, the initial data $u_0\in H^2(\Omega)$ satisfies $\int_\Omega u_0 dx = 0$, and the parameters satisfy $0\le s\le 2$, $p>0$, $q>0$. Note that when $s=0$, Eq. (1) reduces to the following fourth-order parabolic equation:

$$u_t + \Delta^2 u - \operatorname{div}(f(\nabla u)) = g(x,t). \tag{2}$$

This class of equations models epitaxial growth in nanoscale thin films [1–4], particularly focusing on surface roughening phenomena during thin film deposition. Here, u represents the height from the surface of the thin film, $\Delta^2 u$ denotes capillarity-driven surface diffusion, and $\operatorname{div}(f(\nabla u))$ describes upward atomic hopping. Recent studies on thin film equations have yielded significant advances in several key areas:

- For the case

$$\operatorname{div}(f(\nabla u)) = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) + \Delta u, \quad p>2, \tag{3}$$

King, Stein and Winkler [5] established not only the existence, uniqueness, and regularity of solutions in an appropriate function space but also characterized the structure of the ω-limit set by employing a semi-discrete approximation technique to study the asymptotic behavior of solutions. When

$$\operatorname{div}(f(\nabla u)) = -\mu\Delta u - \lambda\Delta|\nabla u|^2, \quad g(x,t) = f(x), \tag{4}$$

Winkler [6] proved the existence of global solutions in higher-dimensional settings. For the case where $g(x,t)=0$, $f(\nabla u)$ satisfies the growth condition

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$$\left|f'(\xi_1) - f'(\xi_2)\right| \le C\left(|\xi_1|^{\alpha-1} + |\xi_2|^{\alpha-1}\right)|\xi_1 - \xi_2|, \quad \forall \xi_1, \xi_2 \in \mathbb{R}^N, \quad \alpha > 1, \tag{5}$$

Sandjo, Moutari and Gningue [7] demonstrated the existence, uniqueness, and regularity of solutions in the function space $C^0([0,T]; L^p(\Omega))$, with $p = \frac{N\alpha}{2-\alpha}$ and $1 < \alpha < 2$, by employing $L_p - L_q$ estimates derived from Kato's method [8–10]. For further studies on thin film equations featuring the p-Laplacian nonlinearity $\operatorname{div}(f(\nabla u)) = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)$, we refer to [11–24, 26–29] and references therein.

• Under the influence of molecular and ionic effects, the conventional p-Laplacian nonlinearity $\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)$ is replaced by a gradient-type logarithmic nonlinearity of the form

$$\operatorname{div}\left(|\nabla u|^{p-2}\nabla u \log|\nabla u|\right). \tag{6}$$

In [30], Liu, Ma and Tang established rigorous lower bounds for both the blow-up time and the blow-up rate of solutions under $g(x,t) = 0$. For the case $g(x,t) = u^q \log u$, Liu, Li and Li [31] established a complete characterization of solution behavior (finite-time blow-up versus global existence) through a classification scheme based on both the initial energy and the Nehari energy. Additionally, the authors derived precise asymptotic estimates for the blow-up time in finite-time blow-up cases, as well as quantitative decay rates for global solutions. For the case $g(x,t) = u^q \log u$, Liu, Li and Li [31] established conditions for blow-up or global existence by classifying the initial energy and the Nehari energy. Furthermore, they showed asymptotic estimates for blow-up time and a large-time estimate of solutions, respectively. For the case $g(x,t) = -\alpha\Delta u - u^{q-1}u$, Lv and Fang [32] studied the asymptotic behavior of global weak solutions under the conditions $0 < q \le 1$ and $\alpha < \lambda_1$, where λ_1 denotes the first eigenvalue of $-\Delta u$ with null Dirichlet boundary conditions. For further studies on fourth-order pseudo-parabolic equations with gradient-dependent logarithmic nonlinearities of the form $\operatorname{div}\left(|\nabla u|^{p-2}\nabla u \log|\nabla u|\right)$, we refer to [33] and [34].

In a homogeneous, isotropic, rigid porous medium saturated with a compressible fluid, the balance of mass for the fluid phase is expressed as [35]:

$$\theta(x)\, u_t - \operatorname{div}(u\mathbf{V}) = f(u), \tag{7}$$

where $\theta(x)$ is the volumetric moisture content, $\mathbf{V}$ is the Darcy velocity, u is the fluid density, and $f(u)$ is the net volumetric source term. Several modifications were implemented in the parabolic model by incorporating terms of the form

$$\theta(x) = \frac{1}{|x|^s}, \quad \operatorname{div}(u\vec{V}) = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right), \tag{8}$$

have been discussed in [36–39]. Recently, Liu and Fang [40] studied a singular epitaxial thin-film growth equation with logarithmic nonlinearity

$$\frac{u_t}{|x|^s} + \Delta^2 u + c\Delta u = |u|^{p-2}u \log|u|, \quad x \in \Omega,\ t > 0, \tag{9}$$

where $2 < p < 2 + \frac{4}{N}, c < \lambda_1, \lambda_1$ is the first eigenvalue of $-\Delta$ with null Dirichlet boundary condition. Under the Navier boundary and initial conditions, the results of the local and global well-posedness, blow-up with arbitrary initial energy and the lifespan were derived. Furthermore, Liu and Fang [41] also investigated a singular parabolic p-biharmonic equation with logarithmic nonlinearity

$$\frac{u_t}{|x|^s} + \Delta\left(|\Delta u|^{p-2}\Delta u\right) = |u|^{q-2}u \log|u|, \quad x \in \Omega,\ t > 0, \tag{10}$$

where $\max\left\{1, \frac{2N}{N+4}\right\} < p \le q < p\left(1 + \frac{4}{N}\right)$. Subject to Navier boundary and initial conditions, the results of blow-up with arbitrary initial energy and extinction phenomena were presented. Lv and Fang [25] considered a singular parabolic p-biharmonic equation with gradient-type logarithmic nonlinearity

$$\frac{u_t}{|x|^s} + \Delta\left(|\Delta u|^{p-2}\Delta u\right) + \operatorname{div}\left(|\nabla u|^{q-2}\nabla u \log|\nabla u|\right) = 0, \quad x \in \Omega, t > 0, \tag{11}$$

where $2 < p \le q < p\left(1 + \frac{2}{N+2}\right)$. Subject to Navier boundary and initial conditions, the authors adopted the Hardy-Sobolev inequality to show the finite time blow-up result with arbitrary initial energy.

Inspired by the above-mentioned research work, in this paper, we will concentrate on the initial boundary value problem (1.1) for a singular non-local fourth-order parabolic equation with gradient-dependent logarithmic nonlinearity. And we will prove the global existence of weak solution, the finite time blow-up properties of weak solution under different initial energies and the finite time extinction and non-extinction phenomena of weak solution. The main difficulties stem from three key features: (i) the singular diffusion coefficient $\frac{1}{|x|^s}$; (ii) the logarithmic nonlinearity

$$\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\log|\nabla u|\right), \tag{12}$$

and (iii) the non-local source term

$$|u|^{q-2}u - \frac{1}{|\Omega|}\int_\Omega |u|^{q-2}u\,dx, \tag{13}$$

which collectively make the methods from [12, 16, 22, 23, 42–44] inapplicable for analyzing finite-time blow-up of solutions with arbitrary initial energy levels, extinction and non-extinction behavior of solutions in finite time. In this paper, to overcome these difficulties, we develop an approach that combines truncation function techniques with Levine's concavity method [45], along with logarithmic inequalities and Hardy-Sobolev inequalities. We prove the global existence of weak solutions by utilizing the cut-off function technique and Galerkin's approximation method. We establish novel threshold criteria for the finite-time blow-up of solutions with initial value at arbitrary energy levels, and derive explicit upper bounds for the blow-up time under appropriate conditions. Furthermore, by employing energy estimates and specific ordinary differential inequalities, we characterize both non-extinction and extinction phenomena of solutions in finite time, and rigorously quantify their corresponding extinction rates.

This paper is organized as follows. In Section 2, we introduce essential notations, definitions, and preliminary lemmas that are fundamental to our main results. Section 3 is devoted to proving the global existence of weak solutions by utilizing the cut-off function technique and Galerkin's approximation method. In Section 4, under suitable assumptions, we establish threshold criteria for finite-time blow-up of solutions with arbitrary initial energy levels and derive explicit upper bounds for the blow-up time. Finally, in Section 5, we analyze both non-extinction and extinction phenomena of solutions in finite time through careful energy estimates and analysis of key ordinary differential inequalities.

2 Perliminaries

In this section, we denote the norm of $L^p(\Omega)$ for $1 \le p \le +\infty$ by

$$\|\phi\|_p = \begin{cases} \left(\int_\Omega |\phi(x)|^p\,dx\right)^{\frac{1}{p}}, & \text{si } 1 \le p < +\infty, \\ \operatorname{ess\,sup}_{x\in\Omega}|\phi(x)|, & \text{si } p = +\infty. \end{cases} \tag{14}$$

and the norm of $H^2(\Omega)$ by

$$\|\phi\|_{H^2(\Omega)} = \sqrt{\|\phi\|_2^2 + \|\nabla\phi\|_2^2 + \|\Delta\phi\|_2^2}. \tag{15}$$

Considering the initial data $u_0 \in H^2(\Omega)$ with $\int_\Omega u_0 dx = 0$ and the Neumann boundary condition to Eq. (1). As in [5] and [12], the Hilbert space $H_N^2(\Omega)$ is defined by

$$H_N^2(\Omega) := \left\{\phi \in H^2(\Omega) \,\middle|\, \frac{\partial\phi}{\partial\eta}\bigg|_{\partial\Omega} = 0, \int_\Omega \phi\,dx = 0\right\}, \tag{16}$$

where $H_N^2(\Omega)$ is equipped with the inner product

$$(\phi,\psi) := \int_\Omega \Delta\phi\,\Delta\psi\,dx, \quad \forall\phi,\psi \in H_N^2(\Omega), \tag{17}$$

and the norm

$$\|\phi\|_{H_N^2(\Omega)} := \|\Delta\phi\|_2, \tag{18}$$

which is equivalent to the norm $\|\phi\|_{H^2(\Omega)}$.

We define the energy functional and Nehari functional associated to problem (1) as follows:

$$J(u) := \frac{1}{2}\|\Delta u\|_2^2 - \frac{1}{p}\int_\Omega |\nabla u|^p\log|\nabla u|\,dx + \frac{1}{p^2}\|\nabla u\|_p^p - \frac{1}{q+1}\|u\|_{q+1}^{q+1}, \tag{19}$$

$$I(u) := \|\Delta u\|_2^2 - \int_\Omega |\nabla u|^p\log|\nabla u|\,dx - \|u\|_{q+1}^{q+1}. \tag{20}$$

A direct calculation of (19) and (20) yields

$$J(u) = \frac{1}{p}I(u) + \frac{p-2}{2p}\|\Delta u\|_2^2 + \frac{1}{p^2}\|\nabla u\|_p^p + \frac{q+1-p}{p(q+1)}\|u\|_{q+1}^{q+1}. \tag{21}$$

Next, we provide a definition of a weak solution and discuss the existence of weak solutions.

Definition 1 (Weak solution) Let $T>0$. A function $u(x,t)$ is said to be a weak solution to problem (1) if $u(x,t)\in L^{\infty}\left(0,T;H_N^2(\Omega)\right)$ with $u_t\in L^2\left(0,T;L^2(\Omega)\right)$ satisfying the following conditions:
(i) for any $\varphi\in L^2\left(0,T;H^2(\Omega)\right)$ with $\left.\frac{\partial\varphi}{\partial\eta}\right|_{\partial\Omega}=0,\ t\in[0,T)$,

$$\int_0^t\int_\Omega\left[\frac{u_\tau}{|x|^s}\varphi+\Delta u\Delta\varphi-|\nabla u|^{p-2}\nabla u\log|\nabla u|\nabla\varphi\right.$$
$$\left.-\left(u^q-\frac{1}{|\Omega|}\int_\Omega|u|^q dx\right)\varphi\right]dxd\tau=0; \tag{22}$$

(ii) $\int_0^t\left\|\frac{u_\tau(\tau)}{|x|^{\frac{s}{2}}}\right\|_2^2 d\tau<+\infty$ and $u(x,0)=u_0\in H_N^2(\Omega)$.

Moreover, the weak solutions to problem (1) satisfy the following energy equality:

$$J(u_0)=J(u)+\int_0^t\left\|\frac{u_\tau(\tau)}{|x|^{\frac{s}{2}}}\right\|_2^2 d\tau,\ \forall t\in[0,T). \tag{23}$$

Remark 1 Let T be the maximal existence time of a weak solution $u(x,t)$ to Eq. (1), defined as follows:
(i) If $u(x,t)$ exists for all $0\le t<+\infty$, then $T=+\infty$, and the weak solution exists globally.
(ii) There exists a $t_0\in(0,+\infty)$ such that $u(x,t)$ exists for $0\le t<t_0$, but does not exist at $t=t_0$, then $T=t_0$, and the weak solution exists locally and blows up in finite time.

Next, we define a function

$$S(t):=\frac{1}{2}\left\|\frac{u(t)}{|x|^{s/2}}\right\|_2^2. \tag{24}$$

And we provide two definitions for the finite-time blow-up and extinction of weak solutions.

Definition 2 (Finite time blow-up) A weak solution $u(x,t)$ to Eq. (1) is called finite time blow-up if the maximal existence time $T<+\infty$ and

$$\lim_{t\to T^-}\sqrt{2S(t)}=\lim_{t\to T^-}\left\|\frac{u(t)}{|x|^{s/2}}\right\|_2=+\infty. \tag{25}$$

Definition 3 (Finite time extinction) A weak solution $u(x,t)$ to problem (1) is called finite time extinction if the maximal existence time $T<+\infty$ and

$$\lim_{t\to T^-}\sqrt{2S(t)}=\lim_{t\to T^-}\left\|\frac{u(t)}{|x|^{s/2}}\right\|_2=0. \tag{26}$$

Finally, we present four key inequalities that play crucial roles in the analysis presented in this paper.

Lemma 1 ([28]) *Let $\rho>0$. Then, the following elementary inequalities hold:*

$$\begin{cases}\xi^p\ln\xi\le(e\rho)^{-1}\xi^{p+\rho}, & \forall\xi\ge1,\\ |\xi^p\ln\xi|\le(ep)^{-1}, & \forall 0<\xi<1.\end{cases} \tag{27}$$

Lemma 2 (Bihari's inequality [46]) *Suppose that $T_0>0$ and $c_0>0$. Let $K:\mathbb{R}_+\to\mathbb{R}_+$ be a continuous nondecreasing function such that $K(t)>0$ for all $t>0$. Let $u_1(\cdot)$ be a Borel measurable bounded non-negative function on $[0,T]$, and let $v_1(\cdot)$ be a non-negative integrable function on $[0,T]$. If*

$$u_1(t)\le c_0+\int_0^t v_1(\tau)K\big(u_1(\tau)\big)d\tau,\quad\forall t\in[0,T]. \tag{28}$$

then,

$$u_1(t)\le G^{-1}\left[G(c_0)+\int_0^t v_1(\tau)d\tau\right]. \tag{29}$$

holds for all $t\in[0,T]$, and such that $G(c_0)+\int_0^t v_1(\tau)d\tau\in$ Dom$\left(G^{-1}\right)$, where

$$G(r_0)=\int_0^{r_0}\frac{d\tau}{K(\tau)},\quad r_0>0. \tag{30}$$

and G^{-1} is the inverse function of G.

Lemma 3 (Hardy-Sobolev inequality [47]) *Let $\mathbb{R}^N=\mathbb{R}^k\times\mathbb{R}^{N-k}$, where $2\le k\le N$, and $x=(y,z)\in\mathbb{R}^k\times\mathbb{R}^{N-k}$. For given λ and s satisfying $1<\lambda<N$, $0\le s\le\lambda$, $s<k$, and $\delta(s,N,\lambda)=\frac{\lambda(N-s)}{N-\lambda}$, there exists a constant $H(s,N,\lambda,k)>0$ such that*

$$\int_{\mathbb{R}^N}\frac{|u(x)|^\delta}{|y|^s}dx\le H\left(\int_{\mathbb{R}^N}|\nabla u(x)|^\lambda dx\right)^{\frac{N-s}{N-\lambda}},$$
$$\forall u\in W^{1,\lambda}\left(\mathbb{R}^N\right). \tag{31}$$

Remark 2 In particular, we set $k=N$ in Eq. (23), which implies $x=y\in\mathbb{R}^N$. We define $u(x)=0$ for $x\in\mathbb{R}^N\backslash\Omega$, thus,

$$\int_\Omega\frac{|u(x)|^\delta}{|x|^s}dx\le H\left(\int_\Omega|\nabla u(x)|^\lambda dx\right)^{\frac{N-s}{N-\lambda}},$$
$$\forall u\in W^{1,\lambda}(\Omega). \tag{32}$$

If $\delta = \frac{\lambda(N-s)}{N-\lambda} = 2$ in Eq. (32), then, by the $0 \le s \le 2$, $N > 2$, and the Rellich-Kondrachov theorem, we have

$$\int_{\Omega} \frac{|u(x)|^2}{|x|^s} dx \le H \left(\int_{\Omega} |\nabla u(x)|^{\frac{2N}{N-s+2}} dx \right)^{\frac{N-s+2}{N}} \le C_H \|\Delta u\|_2^2, \quad (33)$$

where $C_H = HB_0$, B_0 is the optimal embedding constant of the embedding $H_0^2(\Omega) \hookrightarrow W_0^{1,\frac{2N}{N-s+2}}(\Omega)$. Due to the boundedness of Ω, there exists a sufficiently large $L > 0$ such that $\Omega \subset B_L(0)$, and $|x| \le L$ for all $x \in \Omega$. Hence, we introduce the cut-off function

$$\rho_n := \min\left\{|x|^{-s}, n\right\}, \quad n \in \mathbb{N}^+. \quad (34)$$

to handle the singular diffusion coefficients $\frac{1}{|x|^s}$ in Eq. (1). Then, it follows from Eq. (33) that

$$\min\left\{L^{-s}, n\right\} \|u\|_2^2 \le \int_{\Omega} \rho_n |u|^2 dx \le C_H \|\Delta u\|_2^2, \quad \forall n \in \mathbb{N}^+. \quad (35)$$

Lemma 4 **([20])** *Suppose that $\theta > 0$, $\alpha > 0$ and $\beta > 0$. Let $\mu(t)$ be a nonnegative and absolutely continuous function satisfying*

$$\mu'(t) + \alpha \mu^{\theta}(t) \ge \beta. \quad (36)$$

Then, for $0 < t < +\infty$, it holds that

$$\mu(t) \ge \min\left\{\mu(0), \left(\frac{\beta}{\alpha}\right)^{\frac{1}{\theta}}\right\}. \quad (37)$$

3 Global existence

In this section, we establish the global existence of weak solutions utilizing the cut-off function technique and Galerkin's approximation method

Theorem 1 *Let $u \in H_N^2(\Omega)$. If*

$$\max\left\{2, \frac{2N}{N+2}\right\} < p < \min\left\{\frac{2N}{N-2}, \frac{2N+8}{N+2}\right\}. \quad (38)$$

and

$$1 < q < \min\left\{\frac{N+4}{N-4}, 1 + \frac{8}{N}\right\}. \quad (39)$$

where $N \ge 4$. Then Eq. (1) admits a global weak solution $u = u(x,t) \in L^{\infty}\left(0, +\infty; H_N^2(\Omega)\right)$ with $\frac{u_t}{|x|^s} \in L^2\left(0, +\infty; L^2(\Omega)\right)$.

Proof . We intend to utilize the cut-off function technique and Galerkin's approximation method to demonstrate this theorem in four steps.

Step 1: Approximate problem We denote the solutions corresponding to ρ_n of Eq. (1) as u_n for any $n \in \mathbb{N}_+$. Let $\{\vartheta_i\}_{i=1}^{+\infty}$ be a completed orthogonal basis of $H_N^2(\Omega)$. Suppose that the finite dimensional space $U_l := \operatorname{span}\{\vartheta_1, \vartheta_2, \cdots, \vartheta_l\}, l \in N_+$. Then the approximate solution of problem (1) can be constructed as

$$u_n^l(x,t) := \sum_{i=1}^{l} c_{ni}^l(t)\, \vartheta_i(x), \quad (40)$$

which satisfies

$$\begin{cases} \left(\rho_n u_{nt}^l, \hat{\vartheta}\right) + \left(\Delta u_n^l, \Delta \hat{\vartheta}\right) - \left(\left|\nabla u_n^l\right|^{p-2} \nabla u_n^l \log\left|\nabla u_n^l\right|, \nabla \hat{\vartheta}\right) \\ \qquad = \left(\left|u_n^l\right|^q - \frac{1}{|\Omega|} \int_{\Omega} \left|u_n^l\right|^q dx, \hat{\vartheta}\right), \\ u_n^l(x,0) = u_{n0}^l, \end{cases} \quad (41)$$

for any $\hat{\vartheta} \in H_N^2(\Omega)$. For a differential equation involving $c_{ni}^l(t)$, the following Cauchy problem can be obtained by taking $\hat{\vartheta} = \vartheta_i$ for $i = 1, 2, \cdots, l$:

$$\begin{cases} \left(\rho_n u_{nt}^l, \vartheta_i\right) + \left(\Delta u_n^l, \Delta \vartheta_i\right) \\ \qquad - \left(\left|\nabla u_n^l\right|^{p-2} \nabla u_n^l \log\left|\nabla u_n^l\right|, \nabla \vartheta_i\right) \\ \qquad = \left(\left|u_n^l\right|^q - \frac{1}{|\Omega|} \int_{\Omega} \left|u_n^l\right|^q dx, \vartheta_i\right), \\ \left(u_{n0}^l, \vartheta_i\right) = b_{ni}^l, \end{cases} \quad (42)$$

where b_{ni}^l are constants satisfying

$$u_n^l(x,0)=\sum_{i=1}^{l} b_{ni}^l \vartheta_i(x) \longrightarrow u_0(x) \tag{43}$$

in $H_N^2(\Omega)$ as $n\to+\infty$ and $l\to+\infty$. By the Picard iteration theorem of ODEs, there exists a $T>0$ such that $c_{ni}^l(t)\in C^1[0,T]$. And thus, it follows from Eq. (40) that

$$u_n^l \in C^1\left([0,T], H_N^2(\Omega)\right). \tag{44}$$

From this, we obtain the local existence of solution to Eq. (1).

Next, we prove the global existence of solution by estimating on the boundness of u_n^l in any finite time.

Step 2: Priori estimates Multiplying the first of the equations of Eq. (42) by $c_{ni}^l(t)$ and summing over $i=1,2,\cdots,l$, we obtain

$$\begin{aligned}&\left(\rho_n u_{nt}^l, u_n^l\right)+\left(\Delta u_n^l, \Delta u_n^l\right)\\&\quad-\left(\left|\nabla u_n^l\right|^{p-2}\nabla u_n^l \log\left|\nabla u_n^l\right|, \nabla u_n^l\right)\\&\quad=\left(\left|u_n^l\right|^q-\frac{1}{|\Omega|}\int_\Omega\left|u_n^l\right|^q dx, u_n^l\right).\end{aligned} \tag{45}$$

Integrating Eq. (45) over $(0,t)$, we obtain

$$\begin{aligned}&\frac{1}{2}\int_\Omega \rho_n\left|u_n^l\right|^2 dx-\frac{1}{2}\int_\Omega \rho_n\left|u_{n0}^l\right|^2 dx+\int_0^t\left\|\Delta u_n^l\right\|_2^2 d\tau\\&\quad=\int_0^t\int_\Omega\left|\nabla u_n^l\right|^p\log\left|\nabla u_n^l\right| dx d\tau\\&\quad+\int_0^t\int_\Omega\left(\left|u_n^l\right|^q-\frac{1}{|\Omega|}\int_\Omega\left|u_n^l\right|^q dx\right)u_n^l dx d\tau.\end{aligned} \tag{46}$$

Setting $F(t)=\frac{1}{2}\int_\Omega \rho_n\left|u_n^l\right|^2 dx+\int_0^t\left\|\Delta u_n^l\right\|_2^2 d\tau$, then $F(0)=\frac{1}{2}\int_\Omega \rho_n\left|u_{n0}^l\right|^2 dx$. Therefore, we rewrite Eq. (46) as

$$\begin{aligned}F(t)-F(0)&=\int_0^t\int_\Omega\left|\nabla u_n^l\right|^p\log\left|\nabla u_n^l\right| dx d\tau\\&\quad+\int_0^t\int_\Omega\left(\left|u_n^l\right|^q-\frac{1}{|\Omega|}\int_\Omega\left|u_n^l\right|^q dx\right)u_n^l dx d\tau.\end{aligned} \tag{47}$$

By Lemma 1, Nirenberg's inequality and Young's inequality with ε, there exists a constant C_1 and a sufficiently small $\rho_1>0$ such that

$$\begin{aligned}&\int_\Omega\left|\nabla u_n^l\right|^p\log\left|\nabla u_n^l\right| dx\\&\le(e\rho_1)^{-1}\left\|\nabla u_n^l\right\|_{p+\rho_1}^{p+\rho_1}\\&\le(e\rho_1)^{-1}C_1^{p+\rho_1}\left\|\Delta u_n^l\right\|_2^{r_1(p+\rho_1)}\left\|u_n^l\right\|_2^{(1-r_1)(p+\rho_1)}\\&\le(e\rho_1)^{-1}C_1^{p+\rho_1}\varepsilon\left\|\Delta u_n^l\right\|_2^2\\&+(e\rho_1)^{-1}C_1^{p+\rho_1}\frac{2-r_1(p+\rho_1)}{2}\\&\times\left(\frac{2\varepsilon}{r(p+\rho_1)}\right)^{-\frac{r_1(p+\rho_1)}{2-r_1(p+\rho_1)}}\left\|u_n^l\right\|_2^{2m_1}\\&=C_1(\varepsilon)\left\|\Delta u_n^l\right\|_2^2+\bar{C}_1(\varepsilon)\left\|u_n^l\right\|_2^{2m_1},\end{aligned} \tag{48}$$

where

$$r_1=\frac{(N+2)(p+\rho_1)-2N}{4(p+\rho_1)}\in(0,1), \tag{49}$$

$$m_1=\frac{(1-r_1)(p+\rho_1)}{2-r_1(p+\rho_1)}>1. \tag{50}$$

Additionally, following a similar approach to Eq. (48), we know that there exists a constant C_2 such that

$$\begin{aligned}&\int_\Omega\left(\left|u_n^l\right|^q-\frac{1}{|\Omega|}\int_\Omega\left|u_n^l\right|^q dx\right)u_n^l dx\\&\le\int_\Omega\left|u_n^l\right|^{q+1}dx-\frac{1}{|\Omega|}\int_\Omega\left|u_n^l\right|^q dx\int_\Omega u_n^l dx\\&=\int_\Omega\left|u_n^l\right|^{q+1}dx\\&\le C_2^{q+1}\left\|\Delta u_n^l\right\|_2^{r_2(q+1)}\left\|u_n^l\right\|_2^{(1-r_2)(q+1)}\\&\le C_2^{q+1}\varepsilon\left\|\Delta u_n^l\right\|_2^2\\&+C_2^{q+1}\frac{2-r_2(q+1)}{2}\left(\frac{2\varepsilon}{r_2(q+1)}\right)^{-\frac{r_2(q+1)}{2-r_2(q+1)}}\left\|u_n^l\right\|_2^{2m_2}\\&=C_2(\varepsilon)\left\|\Delta u_n^l\right\|_2^2+\bar{C}_2(\varepsilon)\left\|u_n^l\right\|_2^{2m_2},\end{aligned} \tag{51}$$

where

$$r_2=\frac{N(q-1)}{4(q+1)}\in(0,1),\quad m_2=\frac{(1-r_2)(q+1)}{2-r_2(q+1)}>1. \tag{52}$$

Thus, it can be obtained from Eq. (35) and Eqs (47)-(51) that

$$F(t)-F(0)\leq\int_0^t\left(C_1(\varepsilon)\left\|\Delta u_n^l\right\|_2^2+\bar{C}_1(\varepsilon)\left\|u_n^l\right\|_2^{2m_1}\right)d\tau+\int_0^t\left(C_2(\varepsilon)\left\|\Delta u_n^l\right\|_2^2+\bar{C}_2(\varepsilon)\left\|u_n^l\right\|_2^{2m_2}\right)d\tau$$

$$\leq C_1(\varepsilon)\left(\frac{1}{2}\int_\Omega\rho_n\left|u_n^l\right|^2dx+\int_0^t\left\|\Delta u_n^l\right\|_2^2d\tau\right)+C_2(\varepsilon)\left(\frac{1}{2}\int_\Omega\rho_n\left|u_n^l\right|^2dx+\int_0^t\left\|\Delta u_n^l\right\|_2^2d\tau\right)$$
$$+\bar{C}_1(\varepsilon)\int_0^t\frac{1}{\min\{L^{-s},n\}^{m_1}}\cdot\min\{L^{-s},n\}^{m_1}\left\|u_n^l\right\|_2^{2m_1}d\tau+\bar{C}_2(\varepsilon)\int_0^t\frac{1}{\min\{L^{-s},n\}^{m_2}}\cdot\min\{L^{-s},n\}^{m_2}\left\|u_n^l\right\|_2^{2m_2}d\tau$$

$$\leq C_3(\varepsilon)F(t)+\frac{\bar{C}_1(\varepsilon)}{\min\{L^{-s},n\}^{m_1}}\int_0^t\left(\int_\Omega\rho_n\left|u_n^l\right|^2dx\right)^{m_1}d\tau+\frac{\bar{C}_2(\varepsilon)}{\min\{L^{-s},n\}^{m_2}}\int_0^t\left(\int_\Omega\rho_n\left|u_n^l\right|^2dx\right)^{m_2}d\tau$$

$$\leq C_3(\varepsilon)F(t)+\frac{\bar{C}_1(\varepsilon)}{\min\{L^{-s},n\}^{m_1}}\int_0^t\left(\int_\Omega\rho_n\left|u_n^l\right|^2dx+2\int_0^t\left\|\Delta u_n^l\right\|_2^2d\tau\right)^{m_1}d\tau$$
$$+\frac{\bar{C}_2(\varepsilon)}{\min\{L^{-s},n\}^{m_2}}\int_0^t\left(\int_\Omega\rho_n\left|u_n^l\right|^2dx+2\int_0^t\left\|\Delta u_n^l\right\|_2^2d\tau\right)^{m_2}d\tau$$

$$=C_3(\varepsilon)F(t)+\bar{C}_3(\varepsilon)\int_0^t[(F(\tau))^{m_1}+(F(\tau))^{m_2}]d\tau, \tag{53}$$

which implies

$$F(t)\leq\frac{F(0)}{1-C_3(\varepsilon)}+\frac{\bar{C}_3(\varepsilon)}{1-C_3(\varepsilon)}\int_0^t[F(\tau)^{m_1}+F(\tau)^{m_2}]d\tau. \tag{54}$$

where

$$C_3(\varepsilon)=C_1(\varepsilon)+C_2(\varepsilon), \tag{55}$$

$$\bar{C}_3(\varepsilon)=\max\left\{\frac{2^{m_1}\bar{C}_1(\varepsilon)}{\min\{L^{-s},n\}^{m_1}},\frac{2^{m_2}\bar{C}_2(\varepsilon)}{\min\{L^{-s},n\}^{m_2}}\right\}. \tag{56}$$

By Lemma 2, there exists a $T>0$ such that

$$F(t)\leq\Phi^{-1}\left[\Phi\left(\frac{F(0)}{1-C_3(\varepsilon)}\right)+\frac{\bar{C}_3(\varepsilon)}{1-C_3(\varepsilon)}t\right]. \tag{57}$$

where $\Phi(r_1)=\int_0^{r_1}\frac{d\tau}{(F(\tau))^{m_1}+(F(\tau))^{m_2}}$, $r_1>0$. Therefore, we have $F(t)\leq C_T$, which means

$$\frac{1}{2}\int_\Omega\rho_n\left|u_n^l\right|^2dx+\int_0^t\left\|\Delta u_n^l\right\|_2^2d\tau\leq C_T,\ \forall n,l\in\mathbb{N}_+. \tag{58}$$

Multiplying the first equation of (42) by $\frac{d}{dt}c_{ni}^l(t)$, summing over $i=1,2,\cdots,l$, and integrating (45) over $(0,t)$, we obtain

$$J\left(u_{n0}^l\right)=J\left(u_n^l\right)+\int_0^t\int_\Omega\rho_n\left|u_{n\tau}^l\right|^2dxd\tau. \tag{59}$$

By the continuity of $J(u(t))$ and Eq. (42), we can know that there exists a $\hat{C}>0$ such that

$$J\left(u_{n0}^l\right)\leq\hat{C},\ \forall n,l\in\mathbb{N}_+. \tag{60}$$

From Eqs (19), (59) and Eqs (60), we derive

$$\begin{aligned}
&\hat{C}\geq J\left(u_{n0}^l\right)\geq J\left(u_n^l\right)\\
=&\frac{1}{2}\left\|\Delta u_n^l\right\|_2^2-\frac{1}{p}\int_\Omega\left|\nabla u_n^l\right|^p\log\left|\nabla u_n^l\right|dx\\
&+\frac{1}{p^2}\left\|\nabla u_n^l\right\|_p^p-\frac{1}{q+1}\left\|u_n^l\right\|_{q+1}^{q+1}\\
\geq&\left[\frac{1}{2}-\frac{C_1(\varepsilon)}{p}-\frac{C_2(\varepsilon)}{q+1}\right]\left\|\Delta u_n^l\right\|_2^2-\frac{\bar{C}_1(\varepsilon)}{p}\left\|u_n^l\right\|_2^{2m_1}\\
&+\frac{1}{p^2}\left\|\nabla u_n^l\right\|_p^p-\frac{\bar{C}_2(\varepsilon)}{q+1}\left\|u_n^l\right\|_2^{2m_2}\\
\geq&\left[\frac{1}{2}-\frac{C_1(\varepsilon)}{p}-\frac{C_2(\varepsilon)}{q+1}\right]\left\|\Delta u_n^l\right\|_2^2\\
&+\frac{1}{p^2}\left\|\nabla u_n^l\right\|_p^p-\frac{2^{m_1}\bar{C}_1(\varepsilon)}{p\min\{L^{-s},n\}^{m_1}}(F(t))^{m_1}\\
&-\frac{2^{m_2}\bar{C}_2(\varepsilon)}{(q+1)\min\{L^{-s},n\}^{m_2}}(F(t))^{m_2}.
\end{aligned} \tag{61}$$

Combining Eqs (58) and (61), we obtain

$$\left\|\rho_n^{\frac{1}{2}} u_{nt}^l\right\|_{L^2\left(0,T;L^2(\Omega)\right)} \leq C,\ \forall n,l \in \mathbb{N}_+, \tag{62}$$

$$\left\|u_n^l\right\|_{L^\infty\left(0,T;H_0^2(\Omega)\right)} \leq C,\ \forall n,l \in \mathbb{N}_+, \tag{63}$$

$$\left\|u_n^l\right\|_{L^\infty\left(0,T;W_0^{1,p}(\Omega)\right)} \leq C,\ \forall n,l \in \mathbb{N}_+, \tag{64}$$

$$\left\|u_n^l\right\|_{L^\infty\left(0,T;L^{q+1}(\Omega)\right)} \leq C,\ \forall n,l \in \mathbb{N}_+. \tag{65}$$

Step 3: Pass to the limit Since $u_n^l \in C^1\left([0,T];H_N^2(\Omega)\right)$ and Eq. (63), then $\left\{u_n^l\right\}$ is a uniformly bounded and equicontinuous sequence. Thus, by the Arzela-Ascoli theorem and Eqs (62)-(65), there exist a function u and a subsequence of $\left\{u_n^l\right\}_{n,l=1}^{+\infty}$, which we still denote by $\left\{u_n^l\right\}_{n,l=1}^{+\infty}$, for convenience, such that

$$u_{nt}^l \to u_t \ strongly\ in\ L^2\left(0,T;L^2(\Omega)\right), \tag{66}$$

$$u_n^l \to u\ weakly\ star\ in\ L^\infty\left(0,T;H_0^2(\Omega)\right), \tag{67}$$

$$u_n^l \to u\ weakly\ star\ in\ L^\infty\left(0,T;W_0^{1,p}(\Omega)\right), \tag{68}$$

$$u_n^l \to u\ weakly\ star\ in\ L^\infty\left(0,T;L^{q+1}(\Omega)\right). \tag{69}$$

Combining Eqs (66)-(69) and the Aubin-Lions-Simon theorem, we have

$$u_n^l \to u, \nabla u_n^l \to \nabla u\ strongly\ in\ C\left([0,T];L^2(\Omega)\right), \tag{70}$$

which implies $u_n^l \to u, \nabla u_n^l \to \nabla u,\ a.e.\ \Omega \times (0,T)$. Therefore,

$$\left|\nabla u_n^l\right|^{p-2}\nabla u_n^l \log\left|\nabla u_n^l\right| \to |\nabla u|^{p-2}\nabla u \log|\nabla u|,\quad a.e.\ \Omega\times(0,T). \tag{71}$$

Additionally, we define

$$\Omega_1 := \{x\in\Omega \mid |u_n^l(x)| \leq 1\},\quad \Omega_2 := \{x\in\Omega \mid |u_n^l(x)| > 1\}. \tag{72}$$

Then, from Lemma 1, it follows that there exists a constant $\rho_2 = 1$ such that

$$\begin{aligned}
&\int_\Omega \left(\left|\nabla u_n^l\right|^{p-2}\nabla u_n^l \log\left|\nabla u_n^l\right|\right)^{p'} dx \\
=&\int_{\Omega_1} \left(\left|\nabla u_n^l\right|^{p-2}\nabla u_n^l \log\left|\nabla u_n^l\right|\right)^{p'} dx \\
&+\int_{\Omega_2} \left(\left|\nabla u_n^l\right|^{p-2}\nabla u_n^l \log\left|\nabla u_n^l\right|\right)^{p'} dx \\
\leq&[\mathrm{e}(p-1)]^{-p'}|\Omega| + \frac{1}{e\rho_2}\int_{\Omega_2}\left(\left|\nabla u_n^l\right|^{p-1+\rho_2}\right)^{p'} dx \\
=&[\mathrm{e}(p-1)]^{-p'}|\Omega| + e^{-p'}\int_{\Omega_2}\left(\left|\nabla u_n^l\right|^{p}\right)^{p'} dx \leq C.
\end{aligned} \tag{73}$$

Hence, by Eqs (71) and (73), we have

$$\left|\nabla u_n^l\right|^{p-2}\nabla u_n^l \log\left|\nabla u_n^l\right| \to |\nabla u|^{p-2}\nabla u \log|\nabla u|\ weakly\ star\ in\ L^\infty\left(0,T;L^{p'}(\Omega)\right). \tag{74}$$

In order to show the limit u in Eqs (66)-(69) and (74) is a weak solution to problem (1), we proceed as follows. Fix a positive integer $k \in \mathbb{N}_+$ satisfying $k \leq l$. For the given smooth function $\left\{d_{ni}^l(t)\right\}_{i=1}^k$, we choose a function $\vartheta(x,t) = \sum_{i=1}^k d_{ni}^l(t)$ where $\vartheta_i \in C^1\left([0,T],H_N^2(\Omega)\right)$. Multiplying the first equation of Eq. (42) by $d_{ni}^l(t)$, summing for i from 1 to k and then integrate it with respect to t, we obtain

$$\begin{aligned}
&\int_0^T \left(\rho_n u_{nt}^l, \vartheta\right) + \left(\Delta u_n^l, \Delta\vartheta\right) \\
&\quad -\left(\left|\nabla u_n^l\right|^{p-2}\nabla u_n^l \log\left|\nabla u_n^l\right|, \nabla\vartheta\right) dt \\
&\quad = \int_0^T \left(\left|u_n^l\right|^q - \frac{1}{|\Omega|}\int_\Omega \left|u_n^l\right|^q dx, \vartheta\right) dt.
\end{aligned} \tag{75}$$

Taking $n \to +\infty, l \to +\infty$ in Eq. (75), and using $\lim_{n\to+\infty}\rho_n = |x|^{-s}$ and the convergence results established above, we conclude that

$$\int_0^T \left(|x|^{-s}u_t,\vartheta\right)+(\Delta u,\Delta\vartheta) - \left(|\nabla u|^{p-2}\nabla u\log|\nabla u|,\nabla\vartheta\right)dt = \int_0^T\left(|u|^q-\frac{1}{|\Omega|}\int_\Omega |u|^q dx,\vartheta\right)dt. \tag{76}$$

Since ϑ is dense in $L^2\left([0,T],H_N^2(\Omega)\right)$, Eq. (76) holds for all $\vartheta\in L^2\left([0,T],H_N^2(\Omega)\right)$. Therefore, u is a global weak solution to Eq. (1).

Step 4: Uniqueness Assume that there are different functions $\bar u(x,t)$, $\bar v(x,t)$ as bounded weak solutions to problem (1) with the initial condition $\bar u(x,0)=\bar v(x,0)=u_0\in H_N^2(\Omega)$. Then, we have

$$\left(|x|^{-s}\bar u_t,\vartheta\right)+(\Delta\bar u,\Delta\vartheta)-\left(|\nabla\bar u|^{p-2}\nabla\bar u\log|\nabla\bar u|,\nabla\vartheta\right) = \left(|\bar u|^q-\frac{1}{|\Omega|}\int_\Omega|\bar u|^q dx,\vartheta\right), \tag{77}$$

$$\left(|x|^{-s}\bar v_t,\vartheta\right)+(\Delta\bar v,\Delta\vartheta)-\left(|\nabla\bar v|^{p-2}\nabla\bar v\log|\nabla\bar v|,\nabla\vartheta\right) = \left(|\bar v|^q-\frac{1}{|\Omega|}\int_\Omega|\bar v|^q dx,\vartheta\right), \tag{78}$$

for any $\vartheta\in H_N^2(\Omega)$. Then, we have

$$\left(|x|^{-s}(\bar u-\bar v)_t,\vartheta\right)+(\Delta(\bar u-\bar v),\Delta\vartheta) - \left(|\nabla\bar u|^{p-2}\nabla\bar u\log|\nabla\bar u|-|\nabla\bar v|^{p-2}\nabla\bar v\log|\nabla\bar v|,\nabla\vartheta\right) = \left(|\bar u|^q-\frac{1}{|\Omega|}\int_\Omega|\bar u|^q dx-(|\bar v|^q-\frac{1}{|\Omega|}\int_\Omega|\bar v|^q dx),\vartheta\right). \tag{79}$$

We choose the test function in (79) as

$$\vartheta(\tau)=\begin{cases}\bar u(\tau)-\bar v(\tau), & \text{if } \tau\in[0,t],\\ 0, & \text{if } \tau\in(t,T).\end{cases} \tag{80}$$

which implies that Eq. (79) can be rewritten as

$$\left(|x|^{-s}(\bar u-\bar v)_t,\bar u-\bar v\right)+(\Delta(\bar u-\bar v),\Delta(\bar u-\bar v)) - \left(|\nabla\bar u|^{p-2}\nabla\bar u\log|\nabla\bar u|-|\nabla\bar v|^{p-2}\nabla\bar v\log|\nabla\bar v|,\nabla(\bar u-\bar v)\right) = \left(|\bar u|^q-\frac{1}{|\Omega|}\int_\Omega|\bar u|^q dx-\left(|\bar v|^q-\frac{1}{|\Omega|}\int_\Omega|\bar v|^q dx\right),\bar u-\bar v\right). \tag{81}$$

Integrating over $(0,t)$, we obtain

$$\frac12\left\|\frac{\bar u-\bar v}{|x|^{\frac{s}{2}}}\right\|_2^2+\int_0^t\int_\Omega|\Delta(\bar u-\bar v)|^2dxd\tau = \int_0^t\int_\Omega\left(|\nabla\bar u|^{p-2}\nabla\bar u\log|\nabla\bar u|-|\nabla\bar v|^{p-2}\nabla\bar v\log|\nabla\bar v|\right)\times(\nabla\bar u-\nabla\bar v)\,dxd\tau + \int_0^t\int_\Omega\left[|\bar u|^q-\frac{1}{|\Omega|}\int_\Omega|\bar u|^q dx-\left(|\bar v|^q-\frac{1}{|\Omega|}\int_\Omega|\bar v|^q dx\right)\right]\times(\bar u-\bar v)\,dxd\tau. \tag{82}$$

Define

$$Q_1(\tau):=|\tau|^{p-2}\tau\log|\tau|, \tag{83}$$

$$Q_2(\tau):=|\tau|^q-\frac{1}{|\Omega|}\int_\Omega|\tau|^q dx. \tag{84}$$

By the Lipschitz continuity of $Q_j:\mathbb{R}_+\to\mathbb{R}_+$ for all $j=1,2$, we derive

$$\begin{aligned}&\frac12\left\|\frac{\bar u-\bar v}{|x|^{\frac{s}{2}}}\right\|_2^2+\int_0^t\int_\Omega|\Delta(\bar u-\bar v)|^2dxd\tau\\ &=\int_0^t\int_\Omega\left(Q_1(\nabla\bar u)-Q_1(\nabla\bar v)\right)(\nabla\bar u-\nabla\bar v)\,dxd\tau\\ &\quad+\int_0^t\int_\Omega\left[Q_2(\bar u)-Q_2(\bar v)\right](\bar u-\bar v)\,dxd\tau\\ &\le C_4\int_0^t\int_\Omega|\nabla\bar u-\nabla\bar v|^2dxd\tau+C_5\int_0^t\int_\Omega|\bar u-\bar v|^2dxd\tau.\end{aligned} \tag{85}$$

Taking $\omega=\bar u-\bar v$, from Nirenberg's inequality and Young's inequality with ε, we obtain

$$\begin{aligned}&\frac12\left\|\frac{\omega}{|x|^{\frac{s}{2}}}\right\|_2^2+\int_0^t\int_\Omega|\Delta\omega|^2dxd\tau\\ &\le C_4\int_0^t\int_\Omega|\nabla\omega|^2dxd\tau+C_5\int_0^t\int_\Omega|\omega|^2dxd\tau\\ &\le C_4C_6^2\int_0^t\|\Delta\omega\|_2\|\omega\|_2d\tau+C_5\int_0^t\int_\Omega|\omega|^2dxd\tau\\ &\le\varepsilon C_4C_6^2\int_0^t\|\Delta\omega\|_2^2d\tau+\left[(4\varepsilon)^{-1}C_4C_6^2+C_5\right]\int_0^t\|\omega\|_2^2d\tau.\end{aligned} \tag{86}$$

Let $\varepsilon=\left(C_4C_6^2\right)^{-1}$ and $C_7=\frac{C_4^2C_6^4}{4}+C_5$. Then, we derive

$$\begin{aligned}\frac{1}{2}\left\|\frac{\omega}{|x|^{\frac{s}{2}}}\right\|_2^2 \le & C_7\int_0^t \|\omega\|_2^2 d\tau \\ = & C_7\int_0^t\int_\Omega |x|^s\cdot\frac{|\omega|^2}{|x|^s}dxd\tau \\ \le & C_7 L^s\int_0^t\left\|\frac{\omega}{|x|^{\frac{s}{2}}}\right\|_2^2 d\tau. \end{aligned} \tag{87}$$

By Gronwall's inequality, we have

$$\left\|\frac{\omega}{|x|^{\frac{s}{2}}}\right\|_2 = \left\|\frac{\bar{u}-\bar{v}}{|x|^{\frac{s}{2}}}\right\|_2 = 0, \quad \text{i.e., } \bar{u}=\bar{v}. \tag{88}$$

which contradicts the assumption. Thus, Eq. (1) admits a weak solution in $\Omega\times[0,T]$. Next, we consider Eq. (1) in $\Omega\times[(k-1)T,kT]$ for all $k=2,3,\cdots$. Finally, we conclude that Eq. (1) admits a global weak solution.

The proof is completed.

4 Blow-up in finite time

In this section, we not only derive new threshold results for the finite-time blow-up of solutions with initial value at arbitrary energy levels, but also determine the upper bounds for the blow-up time under appropriate conditions.

Theorem 2 *Let $2<p<q+1$. If $J(u_0)<0$, then the solution to problem (1) blows up in finite time t_*. Moreover, the upper bound of t_* is estimated as*

$$t_* \le \frac{\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2}{p(2-p)J(u_0)}. \tag{89}$$

Proof . We define a function

$$\gamma(t) := -J(u(t)). \tag{90}$$

It is straightforward to observe that $S(0)=\frac{1}{2}\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2>0$ and $\gamma(0)=-J(u_0)>0$. Since the weak solution to problem (1) satisfies the following energy equality

$$J(u_0)=J(u)+\int_0^t\left\|\frac{u_\tau(\tau)}{|x|^{\frac{s}{2}}}\right\|_2^2 d\tau, \tag{91}$$

we obtain

$$\begin{aligned}\gamma'(t) &= -\frac{d}{dt}J(u(t)) = \frac{d}{dt}\int_0^t\left\|\frac{u_\tau(\tau)}{|x|^{\frac{s}{2}}}\right\|_2^2 d\tau - \frac{d}{dt}J(u_0) \\ &= \left\|\frac{u_t(t)}{|x|^{\frac{s}{2}}}\right\|_2^2 \ge 0. \end{aligned} \tag{92}$$

From Eq. (92), it follows that

$$0<\gamma(0)\le\gamma(t), \quad \forall t\in[0,t_*). \tag{93}$$

Based on Eqs (20), (21) and the condition $2<p<q+1$, we derive

$$\begin{aligned} S'(t) &= \int_\Omega \frac{u(t)u_t(t)}{|x|^s}dx = -I(u(t)) \\ &= -pJ(u(t)) + \frac{p-2}{2}\|\Delta u(t)\|_2^2 \\ &+ \frac{1}{p}\|\nabla u(t)\|_p^p + \frac{q+1-p}{q+1}\|u(t)\|_{q+1}^{q+1} \ge -pJ(u(t)). \end{aligned} \tag{94}$$

which implies

$$S'(t)\ge -pJ(u(t)) = p\gamma(t)>0, \ \forall t\le[0,t_*). \tag{95}$$

Using Eqs (92), (95) and the Cauchy-Schwartz inequality, we obtain

$$\begin{aligned} S(t)\gamma'(t) &= \frac{1}{2}\left\|\frac{u(t)}{|x|^{\frac{s}{2}}}\right\|_2^2\left\|\frac{u_t(t)}{|x|^{\frac{s}{2}}}\right\|_2^2 \\ &\ge \frac{1}{2}\left(\int_\Omega\frac{u(t)u_t(t)}{|x|^s}dx\right)^2 \ge \frac{p}{2}S'(t)\gamma(t). \end{aligned} \tag{96}$$

Thus, via direct calculation using Eq. (96), we have

$$\begin{aligned}&\left[S^{-\frac{p}{2}}(t)\gamma(t)\right]' = \\ &\qquad S^{-\frac{p}{2}-1}(t)\left[S(t)\gamma'(t)-\tfrac{p}{2}S'(t)\gamma(t)\right]\ge 0. \end{aligned} \tag{97}$$

It follows that

$$0<S^{-\frac{p}{2}}(0)\gamma(0)\leq S^{-\frac{p}{2}}(t)\gamma(t)$$
$$\leq\frac{2}{p(2-p)}\left(S^{-\frac{p}{2}+1}(t)\right)', \tag{98}$$

and since $\frac{2}{p(2-p)}<0$, we deduce

$$\left(S^{-\frac{p}{2}+1}(t)\right)'\leq\frac{p(2-p)}{2}S^{-\frac{p}{2}}(0)\,\gamma(0). \tag{99}$$

Integrating over $(0,t)$ yields

$$S^{-\frac{p}{2}+1}(t)\leq\frac{p(2-p)}{2}S^{-\frac{p}{2}}(0)\,\gamma(0)t+S^{-\frac{p}{2}+1}(0). \tag{100}$$

Taking he limi $t\to t_*$ in Eq. (100) to obtain

$$\lim_{t\to t_*^-}S^{-\frac{p}{2}+1}(t)=\frac{1}{\lim_{t\to t_*^-}S^{\frac{p}{2}-1}(t)}=0. \tag{101}$$

which implies

$$\lim_{t\to t_*^-}S(t)=\lim_{t\to t_*^-}\frac{1}{2}\left\|\frac{u(t)}{|x|^{s/2}}\right\|_2^2=+\infty. \tag{102}$$

The proof is completed.

Theorem 3 *Let $2<p<q+1$. If $0\leq J(u_0)<\frac{p-2}{2pC_H}\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2$, then the solution to problem (1) blows up in finite time t_*, and the upper bound of t_* is estimated as*

$$t_*\leq\frac{8p\left\|\frac{u_0}{|x|^{s/2}}\right\|_2^2}{(p-2)^2\left[\frac{(p-2)}{C_H}\left\|\frac{u_0}{|x|^{s/2}}\right\|_2^2-2pJ(u_0)\right]}. \tag{103}$$

Proof . Suppose that u is a solution of the Eq. (1) with initial data u_0. We observe that

$$\int_0^t\left\|\frac{u_\tau(\tau)}{|x|^{\frac{s}{2}}}\right\|_2d\tau\geq\left\|\int_0^t\frac{u_\tau(\tau)}{|x|^{\frac{s}{2}}}d\tau\right\|_2$$
$$=\left\|\frac{u(t)}{|x|^{\frac{s}{2}}}-\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2\geq\left\|\frac{u(t)}{|x|^{\frac{s}{2}}}\right\|_2-\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2,\quad\forall t\in[0,+\infty). \tag{104}$$

Though a direct calculation, we obtain

$$\left\|\frac{u(t)}{|x|^{\frac{s}{2}}}\right\|_2\leq\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2+\int_0^t\left\|\frac{u_\tau(\tau)}{|x|^{\frac{s}{2}}}\right\|_2d\tau$$
$$\leq\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2+t^{\frac{1}{2}}\left(\int_0^t\left\|\frac{u_\tau(\tau)}{|x|^{\frac{s}{2}}}\right\|_2^2d\tau\right)^{\frac{1}{2}}$$
$$=\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2+t^{\frac{1}{2}}\left(J(u_0)-J(u(t))\right)^{\frac{1}{2}}. \tag{105}$$

Suppose u is a global weak solution of Eq. (1). We know $J(u(t))\geq 0$ for all $t\in[0,+\infty)$. If this were false, there would exist $t_0\in(0,+\infty)$ such that $J(u(t_0))<0$. Following the proof of Theorem 2, we conclude that u blows up in finite time, contradicting our assumption. Therefore, from Eq. (105), we obtain

$$\left\|\frac{u(t)}{|x|^{\frac{s}{2}}}\right\|_2\leq\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2+t^{\frac{1}{2}}\left(J(u_0)-J(u(t))\right)^{\frac{1}{2}}$$
$$\leq\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2+t^{\frac{1}{2}}\left(J(u_0)\right)^{\frac{1}{2}},\quad\forall t\in[0,+\infty). \tag{106}$$

Furthermore, from the definition of $S(t)$ and using Eqs (20), (21), (33) with the condition $2<p<q+1$, we derive

$$S'(t)=-pJ(u(t))+\frac{p-2}{2}\|\Delta u(t)\|_2^2$$
$$+\frac{1}{p}\|\nabla u(t)\|_p^p+\frac{q+1-p}{q+1}\|u(t)\|_{q+1}^{q+1}$$
$$\geq-pJ(u(t))+\frac{p-2}{2C_H}\left\|\frac{u(t)}{|x|^{\frac{s}{2}}}\right\|_2^2$$
$$=\frac{p-2}{C_H}\left[S(t)-\frac{pC_H}{p-2}J(u(t))\right]. \tag{107}$$

From Eq. (92) , it follows that

$$\frac{d}{dt}J(u(t))=-\left\|\frac{u_t(t)}{|x|^{\frac{s}{2}}}\right\|_2^2\leq 0. \tag{108}$$

Then, from Eqs (107) and (108), it yields

$$\frac{d}{dt}\left(S(t)-\frac{pC_H}{p-2}J(u(t))\right)\geq\frac{d}{dt}S(t)$$
$$\geq\frac{p-2}{C_H}\left[S(t)-\frac{pC_H}{p-2}J(u(t))\right]. \tag{109}$$

Let $\varphi(t) = S(t) - \frac{pC_H}{p-2} J(u(t))$, from Eq. (109), we have

$$\frac{d}{dt}\varphi(t) \geq \frac{p-2}{C_H}\varphi(t), \ \forall t \in [0,+\infty). \tag{110}$$

Integrating Eq. (110) over $(0,t)$, we obtain

$$\varphi(t) \geq \mathrm{e}^{\frac{p-2}{C_H}t}\varphi(0), \quad \forall t \in [0,+\infty). \tag{111}$$

namely,

$$\left\|\frac{u(t)}{|x|^{\frac{s}{2}}}\right\|_2^2 \geq \frac{2pC_H}{p-2} J(u(t)) + 2\mathrm{e}^{\frac{p-2}{C_H}t}\varphi(0), \quad \forall t \in [0,+\infty). \tag{112}$$

Since $0 \leq J(u(t)) \leq J(u_0)$, it follows from Eq. (112) that

$$\left\|\frac{u(t)}{|x|^{\frac{s}{2}}}\right\|_2^2 \geq 2\mathrm{e}^{\frac{p-2}{C_H}t}\varphi(0). \tag{113}$$

which implies

$$\left\|\frac{u(t)}{|x|^{\frac{s}{2}}}\right\|_2 \geq (2\varphi(0))^{\frac{1}{2}}\mathrm{e}^{\frac{p-2}{2C_H}t} = \left(\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2 - \frac{2pC_H}{p-2}J(u_0)\right)^{\frac{1}{2}}\mathrm{e}^{\frac{p-2}{2C_H}t}, \ \forall t \in [0,+\infty), \tag{114}$$

this contradicts Eq. (106) for sufficiently large t. Thus, $t_* < +\infty$.

Next, we derive the upper bound of t_*. Since the weak solution to problem (1) satisfies the energy equality

$$J(u_0) = J(u(t)) + \int_0^t \left\|\frac{u_\tau(\tau)}{|x|^{\frac{s}{2}}}\right\|_2^2 d\tau. \tag{115}$$

Combining this with Eqs (21), (23), (33) under the conditions $2 < p < q+1$ and $0 \leq J(u_0) < \frac{p-2}{2pC_H}\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2$, we obtain

$$\begin{aligned} I(u_0) &= pJ(u_0) - \frac{p-2}{2}\|\Delta u_0\|_2^2 - \frac{1}{p}\|\nabla u_0\|_p^p \\ &\quad - \frac{q+1-p}{q+1}\|u_0\|_{q+1}^{q+1} = p\left[J(u_0) - \frac{p-2}{2pC_H}\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2\right] \\ &\quad - \frac{p-2}{2}\left[\|\Delta u_0\|_2^2 - \frac{1}{C_H}\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2\right] \\ &\quad - \frac{1}{p}\|\nabla u_0\|_p^p - \frac{q+1-p}{q+1}\|u_0\|_{q+1}^{q+1} < 0. \end{aligned} \tag{116}$$

We assert that $I(u(t)) < 0$ for all $t \in [0,t_*)$. Otherwise, there exists a $\tilde{t} \in (0,t_*)$ such that $I(u(\tilde{t})) = 0$, and $I(u(t)) < 0$ for all $t \in [0,\tilde{t})$. Combining with Eq. (95), we have $S'(t) = -I(u(t)) > 0$ for all $t \in [0,\tilde{t})$. It means that $S(t)$ is strictly increasing on $[0,\tilde{t})$, and consequently,

$$0 \leq J(u_0) < \frac{p-2}{2pC_H}\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2 < \frac{p-2}{2pC_H}\left\|\frac{u(\tilde{t})}{|x|^{\frac{s}{2}}}\right\|_2^2. \tag{117}$$

Furthermore, from Eqs (21) and (33), we derive

$$\begin{aligned} J(u_0) \geq J(u(\tilde{t})) &= \frac{p-2}{2p}\|\Delta u(\tilde{t})\|_2^2 + \frac{1}{p^2}\|\nabla u(\tilde{t})\|_p^p \\ &\quad + \frac{q+1-p}{q+1}\|u(\tilde{t})\|_{q+1}^{q+1} + \frac{1}{p}I(u(\tilde{t})) \\ &\geq \frac{p-2}{2p}\|\Delta u(\tilde{t})\|_2^2 \geq \frac{p-2}{2pC_H}\left\|\frac{u(\tilde{t})}{|x|^{\frac{s}{2}}}\right\|_2^2, \end{aligned} \tag{118}$$

which contradicts Eq. (117). Therefore, $I(u(t)) < 0$ for all $t \in [0,t_*)$, and consequently $S'(t) = -I(u(t)) > 0$ for all $t \in [0,t_*)$. This implies that $S(t)$ is strictly increasing on $[0,t_*)$.

In order to derive the upper bound of t_*, we define a function

$$\Phi(t) := \int_0^t \left\|\frac{u(\tau)}{|x|^{\frac{s}{2}}}\right\|_2^2 d\tau + (\hat{T} - t)\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2 + \sigma(t+\kappa)^2, \quad \forall t \in [0,\hat{T}], \tag{119}$$

where $\hat{T} \in (0,t_*)$, $\sigma > 0$ and $\kappa > 0$ are sufficiently small. From Eq. (119), we obtain

$$\Phi'(t)=\left\|\frac{u(t)}{|x|^{\frac{s}{2}}}\right\|_2^2-\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2+2\sigma(t+\kappa)$$
$$=\int_0^t\frac{d}{d\tau}\left\|\frac{u(\tau)}{|x|^{\frac{s}{2}}}\right\|_2^2d\tau+2\sigma(t+\kappa)$$
$$=2\int_0^t\int_\Omega\frac{u(\tau)u_\tau(\tau)}{|x|^s}dxd\tau+2\sigma(t+\kappa). \tag{120}$$

It is evident from Eqs (119) and (120) that

$$\Phi(0)=\hat{T}\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2+\sigma\kappa^2>0,\quad \Phi'(0)=2\sigma\kappa>0. \tag{121}$$

By Eqs (21), (33) and (119), combined with $2<p<q+1$ and $\sigma>0$, we derive

$$\Phi''(t)=2\int_\Omega\frac{u(t)u_t(t)}{|x|^s}dx+2\sigma$$
$$=-2I(u(t))+2\sigma$$
$$=-2pJ(u(t))+(p-2)\|\Delta u(t)\|_2^2+\frac{2}{p}\|\nabla u(t)\|_p^p+\frac{2(q+1-p)}{q+1}\|u(t)\|_{q+1}^{q+1}+2\sigma$$
$$=-2pJ(u_0)+2p\int_0^t\left\|\frac{u_\tau(\tau)}{|x|^{\frac{s}{2}}}\right\|_2^2d\tau+(p-2)\|\Delta u(t)\|_2^2+\frac{2}{p}\|\nabla u(t)\|_p^p+\frac{2(q+1-p)}{q+1}\|u(t)\|_{q+1}^{q+1}+2\sigma$$
$$\geq-2pJ(u_0)+2p\int_0^t\left\|\frac{u_\tau(\tau)}{|x|^{\frac{s}{2}}}\right\|_2^2d\tau+\frac{p-2}{C_H}\left\|\frac{u(t)}{|x|^{\frac{s}{2}}}\right\|_2^2,\forall t\in\left[0,\hat{T}\right]. \tag{122}$$

Recalling $S(t)$ is strictly increasing on $[0,t_*)$, we know $\Phi'(t)>0$, which implies $\Phi(t)$ is strictly increasing on $[0,t_*)$. Through a direct calculation, we obtain

$$-\frac{p}{2}\left[\Phi'(t)\right]^2=-2p\left[\int_0^t\int_\Omega\frac{u(\tau)u_\tau(\tau)}{|x|^s}dxd\tau+\sigma(t+\kappa)\right]^2$$
$$\geq-2p\left[\left(\int_0^t\left\|\frac{u(\tau)}{|x|^{\frac{s}{2}}}\right\|_2^2d\tau\right)^{\frac{1}{2}}\left(\int_0^t\left\|\frac{u_\tau(\tau)}{|x|^{\frac{s}{2}}}\right\|_2^2d\tau\right)^{\frac{1}{2}}+\sigma(t+\kappa)\right]^2$$
$$\geq-2p\left[\int_0^t\left\|\frac{u(\tau)}{|x|^{\frac{s}{2}}}\right\|_2^2d\tau+\sigma(t+\kappa)^2\right]\left[\int_0^t\left\|\frac{u_\tau(\tau)}{|x|^{\frac{s}{2}}}\right\|_2^2d\tau+\sigma\right]$$
$$\geq-2p\Phi(t)\left[\int_0^t\left\|\frac{u_\tau(\tau)}{|x|^{\frac{s}{2}}}\right\|_2^2d\tau+\sigma\right]. \tag{123}$$

Then, it follows from Eqs (119)-(123) that

$$\Phi(t)\Phi''(t)-\frac{p}{2}\left(\Phi'(t)\right)^2$$
$$\geq\Phi(t)\left[-2pJ(u_0)+2p\int_0^t\left\|\frac{u_\tau(\tau)}{|x|^{\frac{s}{2}}}\right\|_2^2d\tau+\frac{p-2}{C_H}\left\|\frac{u(t)}{|x|^{\frac{s}{2}}}\right\|_2^2\right]-2p\Phi(t)\left[\int_0^t\left\|\frac{u_\tau(\tau)}{|x|^{\frac{s}{2}}}\right\|_2^2d\tau+\sigma\right]$$
$$=2p\Phi(t)\left[\left(\frac{p-2}{2pC_H}\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2-J(u_0)\right)-\sigma\right]\geq0, \tag{124}$$

where $\sigma\in\left(0,\frac{p-2}{2pC_H}\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2-J(u_0)\right]$. Let $\Gamma(t)=\Phi^{-\frac{p}{2}+1}(t)$ for any $t\in\left[0,\hat{T}\right]$. By a direct calculation, we obtain

$$\Gamma'(t)=\left(-\frac{p}{2}+1\right)\Phi^{-\frac{p}{2}}(t)\Phi'(t), \tag{125}$$

$$\Gamma''(t)=\left(-\frac{p}{2}+1\right)\Phi^{-\frac{p}{2}-1}(t)\times\left[\Phi(t)\Phi''(t)-\frac{p}{2}\left(\Phi'(t)\right)^2\right]. \tag{126}$$

By Eqs (124), (126) and the condition $p>2$, we have $\Gamma''(t)\leq0$, which implies that $\Gamma(t)$ is a concave function on $\left[0,\hat{T}\right]$, and $\Gamma'(t)$ is a decreasing function on $\left[0,\hat{T}\right]$. Therefore,

$$\Gamma(\hat{T})\leq\Gamma(0),\quad\forall t\in[0,\hat{T}]. \tag{127}$$

By integrating the above inequality over $\left[0,\hat{T}\right]$, we obtain

$$\Gamma\left(\hat{T}\right) \leq \Gamma(0) + \Gamma'(0)\hat{T}. \tag{128}$$

Since $\Phi(0) > 0$ and $\Phi(t)$ is strictly increasing on $\left[0,\hat{T}\right]$, it follows that

$$\Gamma(0) = \Phi^{-\frac{p}{2}+1}(0) > 0, \tag{129}$$

$$\Gamma\left(\hat{T}\right) = \Phi^{-\frac{p}{2}+1}\left(\hat{T}\right) > 0, \tag{130}$$

for any $\hat{T} \in (0,t_*)$. Additionally, by the conditions $\Phi'(0) > 0$, $p > 2$ and using Eq. (125), we have

$$\begin{aligned}\Gamma'(0) &= \left(-\frac{p}{2}+1\right)\Phi^{-\frac{p}{2}}(0)\,\Phi'(0) \\ &= \left(-\frac{p}{2}+1\right)\Gamma(0)\frac{\Phi'(0)}{\Phi(0)} < 0.\end{aligned} \tag{131}$$

From Eqs(128)-(131), combined with Eqs (119) and (120), we derive

$$\begin{aligned}\hat{T} &\leq \frac{\Gamma(0)-\Gamma\left(\hat{T}\right)}{-\Gamma'(0)} \leq \frac{\Gamma(0)}{-\Gamma'(0)} = \frac{2\Phi(0)}{(p-2)\,\Phi'(0)} \\ &= \frac{\hat{T}\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2 + \sigma\kappa^2}{(p-2)\,\sigma\kappa} = \frac{\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2}{(p-2)\,\sigma\kappa}\hat{T} + \frac{\kappa}{p-2},\end{aligned} \tag{132}$$

where $\kappa \in \left(\frac{\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2}{(p-2)\sigma}, +\infty\right)$. Taking the limit as $\hat{T} \to t_*$ in Eq. (132), we have

$$t_* \leq \frac{\sigma\kappa^2}{(p-2)\,\sigma\kappa - \left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2}. \tag{133}$$

We define the set

$$M := \left\{(\sigma,\kappa) : \sigma \in \left(0, \frac{p-2}{2pC_H}\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2 - J(u_0)\right], \quad \kappa \in \left(\frac{\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2}{(p-2)\,\sigma}, +\infty\right)\right\}, \tag{134}$$

and the function

$$f(\sigma,\kappa) := \frac{\sigma\kappa^2}{(p-2)\,\sigma\kappa - \left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2}. \tag{135}$$

Direct calculation shows that $f(\sigma,\kappa)$ attains its minimum value at $\kappa = \frac{2\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2}{(p-2)\sigma}$. Thus, from Eqs (133)–(135), we derive

$$\begin{aligned}t_* &\leq \inf_{(\sigma,\kappa)\in M} f(\sigma,\kappa) \\ &= \frac{8p\left\|\frac{u_0}{|x|^{s/2}}\right\|_2^2}{(p-2)^2\left[\frac{(p-2)}{C_H}\left\|\frac{u_0}{|x|^{s/2}}\right\|_2^2 - 2pJ(u_0)\right]}.\end{aligned} \tag{136}$$

The proof is completed.

Corollary 1 *Assume $2 < p < q+1$. Then there exists a weak solution with arbitrarily high initial energy to Eq. (1) that blows up in finite time.*

Proof . Let Ω_1 and Ω_2 be two disjoint open subdomains of Ω. Choose an arbitrary nontrivial function $v \in H_N^2(\Omega_1)$, and for any $R > 0$, there exists a sufficiently large $\chi > 0$ such that

$$\begin{aligned}\left\|\frac{\chi v}{|x|^{s/2}}\right\|_2^2 &= \chi^2\int_\Omega \frac{|v|^2}{|x|^s}\,dx \\ &= \chi^2\int_{\Omega_1}\frac{|v|^2}{|x|^s}\,dx > \frac{2pC_H}{p-2}R.\end{aligned} \tag{137}$$

and for $2 < p < q+1$ and Eq. (19), it follows that

$$\begin{aligned}R - J(\chi v) &= R - \frac{\chi^2}{2}\|\Delta v\|_2^2 + \frac{\chi^p}{p}\int_\Omega |\nabla v|^p \log|\nabla v|\,dx \\ &\quad + \frac{\chi^p}{p}\log\chi\,\|\nabla v\|_p^p - \frac{\chi^p}{p^2}\|\nabla v\|_p^p \\ &\quad + \frac{\chi^{q+1}}{q+1}\|v\|_{q+1}^{q+1} \to +\infty,\end{aligned} \tag{138}$$

as $\chi \to +\infty$. Fix χ and $\omega \in H_N^2(\Omega_2)$ such that $J(\chi v) + J(\omega) = R$. Next, we extend v and ω as follows:

$$v=\begin{cases} v, \ x\in\Omega_1, \\ 0, \ x\in\Omega_2, \end{cases} \qquad \omega=\begin{cases} \omega, \ x\in\Omega_2, \\ 0, \ \ x\in\Omega_1. \end{cases} \tag{139}$$

Let $u_0 := \chi v+\omega$. Then we have

$$\begin{aligned} \left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2 &= \int_\Omega \left|\frac{\chi v+\omega}{|x|^s}\right|^2 dx \\ &\geq \int_{\Omega_1}\left|\frac{\chi v}{|x|^s}\right|^2 dx > \frac{2qC_H}{q-2}R \end{aligned} \tag{140}$$

and

$$\begin{aligned} J(u_0) =&\frac{1}{2}\int_\Omega |\Delta u_0|^2 dx-\frac{1}{p}\int_\Omega |\nabla u_0|^p \log|\nabla u_0|\,dx \\ &+\frac{1}{p^2}\int_\Omega |\nabla u_0|^p dx-\frac{1}{q+1}\int_\Omega |u_0|^{q+1}dx \\ =&\frac{1}{2}\int_{\Omega_1} |\chi\Delta v|^2 dx-\frac{1}{p}\int_{\Omega_1} |\chi\nabla v|^p \log|\chi\nabla v|\,dx \\ &+\frac{1}{p^2}\int_{\Omega_1} |\chi\nabla v|^p dx-\frac{1}{q+1}\int_{\Omega_1} |\chi v|^{q+1}dx \\ &+\frac{1}{2}\int_{\Omega_2} |\Delta \omega|^2 dx-\frac{1}{p}\int_{\Omega_2} |\nabla \omega|^p \log|\nabla \omega|\,dx \\ &+\frac{1}{p^2}\int_{\Omega_2} |\nabla \omega|^p dx-\frac{1}{q+1}\int_{\Omega_2} |\omega|^{q+1}dx \\ =&J(\chi v)+J(\omega) \\ =&R, \end{aligned} \tag{141}$$

which means $J(u_0)=R<\frac{p-2}{2pC_H}\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2$. Therefore, by Theorem 3, we conclude that there exists a weak solution with arbitrarily high initial energy to Eq. (1) that blows up in finite time.

The proof is completed.

5 Non-extinction and extinction in finite time

In this section, by employing energy estimates and specific ordinary differential inequalities, we characterize both non-extinction and extinction phenomena of solutions in finite time, and rigorously quantify their corresponding extinction rates.

Theorem 4 *Assume that $p\geq 2$ and $p>q+1$. If $J(u_0)<0$, then the solution to Eq.* (1) *does not extinct in finite time.*

Proof . Since the weak solution to Eq. (1) satisfies the following energy inequality:

$$J(u_0)=J(u)+\int_0^t \left\|\frac{u_\tau(\tau)}{|x|^{\frac{s}{2}}}\right\|_2^2 d\tau. \tag{142}$$

Furthermore, from the definition of $S(t)$ and using Eqs (20), (21), (23) and (33) under the conditions $|x|<L$, $p\geq 2$ and $p>q+1$, we derive

$$\begin{aligned} S'(t) =& -pJ(u(t))+\frac{p-2}{2}\|\Delta u(t)\|_2^2 \\ &+\frac{1}{p}\|\nabla u(t)\|_p^p+\frac{q+1-p}{q+1}\|u(t)\|_{q+1}^{q+1} \\ =&-pJ(u_0)+p\int_0^t \left\|\frac{u_\tau(\tau)}{|x|^{\frac{s}{2}}}\right\|_2^2 d\tau+\frac{p-2}{2}\|\Delta u(t)\|_2^2 \\ &+\frac{1}{p}\|\nabla u(t)\|_p^p-\frac{p-q-1}{q+1}\|u(t)\|_{q+1}^{q+1} \\ \geq&-pJ(u_0)+p\int_0^t \left\|\frac{u_\tau(\tau)}{|x|^{\frac{s}{2}}}\right\|_2^2 d\tau \\ &-\frac{p-q-1}{q+1}\|u(t)\|_{q+1}^{q+1} \\ \geq&-pJ(u_0)-\frac{p-q-1}{q+1}|\Omega|^{\frac{1-q}{2}}\left(\int_\Omega u^2(t)\,dx\right)^{\frac{q+1}{2}} \\ \geq&-pJ(u_0)-\frac{p-q-1}{q+1}|\Omega|^{\frac{1-q}{2}}(2L^s)^{\frac{q+1}{2}}S^{\frac{q+1}{2}}(t), \end{aligned} \tag{143}$$

which implies

$$\begin{aligned} S'(t)+\frac{p-q-1}{q+1}|\Omega|^{\frac{1-q}{2}}(2L^s)^{\frac{q+1}{2}}S^{\frac{q+1}{2}}(t) \\ \geq -pJ(u_0). \end{aligned} \tag{144}$$

Since $p\geq 2$, $p>q+1$ and $J(u_0)<0$, we have

$$\alpha_1=\frac{p-q-1}{q+1}|\Omega|^{\frac{1-q}{2}}(2L^s)^{\frac{q+1}{2}}>0. \tag{145}$$

Then, it follows from Eqs (144) and (145) that

$$S'(t)+\alpha_1 S^{\frac{q+1}{2}}(t)\geq \beta_1. \tag{146}$$

By Lemma 4, we know

$$S(t)\geq \min\left\{S(0),\left(\frac{\beta_1}{\alpha_1}\right)^{\frac{2}{q+1}}\right\}>0,\ \forall t\in(0,+\infty). \tag{147}$$

Hence, Eq. (147) implies

$$\left\|\frac{u(t)}{|x|^{\frac{s}{2}}}\right\|_2=\sqrt{2S(t)}>0, \quad \forall t\in(0,+\infty). \tag{148}$$

Then, for any $m_3 > 1$, by the interpolation inequality, we obtain

$$\left\| \frac{u(t)}{|x|^{\frac{s}{2}}} \right\|_2 \leq \left\| \frac{u(t)}{|x|^{\frac{s}{2}}} \right\|_{m_3}^{\frac{1}{2}} \left\| \frac{u(t)}{|x|^{\frac{s}{2}}} \right\|_{\frac{m_3}{m_3-1}}^{\frac{1}{2}}. \tag{149}$$

which together with $\left\| \frac{u(t)}{|x|^{\frac{s}{2}}} \right\|_2 > 0$, implies that for any $m_3 > 1$, there does not exist a $t_* > 0$ such that

$$\lim_{t \to t_*} \left\| \frac{u(t)}{|x|^{\frac{s}{2}}} \right\|_{m_3} = 0 \quad \text{or} \quad \lim_{t \to t_*} \left\| \frac{u(t)}{|x|^{\frac{s}{2}}} \right\|_{\frac{m_3}{m_3-1}} = 0. \tag{150}$$

Therefore, the solution to Eq. (1) does not become extinct in finite time.

The proof is completed.

Theorem 5 *Assume that $p < 2$ and $q < 1$. If $[\mathrm{e}(2-p)]^{-1} B^2 < 1$ and*

$$\left\| \frac{u_0}{|x|^{\frac{s}{2}}} \right\|_2 > \left(\frac{D_2}{D_1} \right)^{\frac{1}{1-q}}. \tag{151}$$

then the solution to Eq. (1) becomes extinct in finite time, with the upper bound of the extinction rate is

$$\begin{cases} \left\| \frac{u(t)}{|x|^{\frac{s}{2}}} \right\|_2 \leq \left[\left\| \frac{u_0}{|x|^{\frac{s}{2}}} \right\|_2^{1-q} \right. \\ \quad \left. + \frac{1}{2} \left(D_2 - D_1 \left\| \frac{u_0}{|x|^{\frac{s}{2}}} \right\|_2^{1-q} \right) t \right]^{\frac{1}{1-q}}, \ 0 < t < T_0, \\ \left\| \frac{u(t)}{|x|^{\frac{s}{2}}} \right\|_2 = 0, \qquad t \geq T_0. \end{cases} \tag{152}$$

where

$$T_0 = \frac{2 \left\| \frac{u_0}{|x|^{\frac{s}{2}}} \right\|_2^{1-q}}{D_1 \left\| \frac{u_0}{|x|^{\frac{s}{2}}} \right\|_2^{1-q} - D_2}, \tag{153}$$

$$D_1 = (1-q) \left\{ 1 - [\mathrm{e}(2-p)]^{-1} B^2 \right\} C_H^{-1} \tag{154}$$

$$D_2 = (1-q) |\Omega|^{\frac{1-q}{2}} L^{\frac{s(q+1)}{2}} \tag{155}$$

and B is the optimal embedding constant of $H_0^2(\Omega) \hookrightarrow W_0^{1,2}(\Omega)$.

Proof . Multiplying Eq. (1) by u and integrating over Ω, we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \left\| \frac{u(t)}{|x|^{\frac{s}{2}}} \right\|_2^2 + \|\Delta u(t)\|_2^2 - \int_\Omega |\nabla u(t)|^p \log |\nabla u(t)| \, dx \\ = \int_\Omega |u(t)|^q u(t) \, dx. \end{aligned} \tag{156}$$

For the left-hand side of Eq. (156), we observe from Lemma 1 and Eq. (32) that there exists a positive constant $\rho_3 = 2 - p$ such that

$$\begin{aligned} & S'(t) + \|\Delta u(t)\|_2^2 - \int_\Omega |\nabla u(t)|^p \log |\nabla u(t)| \, dx \\ & \geq S'(t) + \|\Delta u(t)\|_2^2 - (\mathrm{e}\rho_3)^{-1} \|\nabla u(t)\|_{p+\rho_3}^{p+\rho_3} \\ & = S'(t) + \|\Delta u(t)\|_2^2 - [\mathrm{e}(2-p)]^{-1} \|\nabla u(t)\|_2^2 \\ & \geq S'(t) + \|\Delta u(t)\|_2^2 - [\mathrm{e}(2-p)]^{-1} B^2 \|\Delta u(t)\|_2^2 \\ & \geq S'(t) + \left\{ 1 - [\mathrm{e}(2-p)]^{-1} B^2 \right\} C_H^{-1} \left\| \frac{u(t)}{|x|^{\frac{s}{2}}} \right\|_2^2 \\ & = S'(t) + 2 \left\{ 1 - [\mathrm{e}(2-p)]^{-1} B^2 \right\} C_H^{-1} S(t). \end{aligned} \tag{157}$$

Then, Eq. (105) can be represented as

$$\begin{aligned} & S'(t) + 2 \left\{ 1 - [\mathrm{e}(2-p)]^{-1} B^2 \right\} C_H^{-1} S(t) \leq \int_\Omega |u(t)|^{q+1} dx \\ & \leq |\Omega|^{\frac{1-q}{2}} \|u(t)\|_2^{q+1} \\ & \leq 2^{\frac{q+1}{2}} |\Omega|^{\frac{1-q}{2}} L^{\frac{s(q+1)}{2}} S^{\frac{q+1}{2}}(t). \end{aligned} \tag{158}$$

Setting $H(t) = S^{\frac{1-q}{2}}(t)$, we obtain from the above that

$$\begin{aligned} H'(t) = & \frac{1-q}{2} S^{\frac{-1-q}{2}}(t) S'(t) \\ \leq & (1-q) 2^{\frac{q-1}{2}} |\Omega|^{\frac{1-q}{2}} L^{\frac{s(q+1)}{2}} \\ & - (1-q) \left\{ 1 - [\mathrm{e}(2-p)]^{-1} B^2 \right\} C_H^{-1} S^{\frac{1-q}{2}}(t) \\ = & 2^{\frac{q-1}{2}} D_2 - D_1 H(t) = \Lambda(t). \end{aligned} \tag{159}$$

Since $\left\| \frac{u_0}{|x|^{\frac{s}{2}}} \right\|_2 > \left(D_2 D_1^{-1} \right)^{\frac{1}{1-q}}$, we have $\Lambda(0) < 0$. Then there exists a sufficiently small $\tilde{t} > 0$ such that $\Lambda(t) < \frac{\Lambda(0)}{2} < 0$ for all $t \in (0, \tilde{t}]$. This implies that

$$H'(t) \le \frac{\Lambda(0)}{2}. \tag{160}$$

Integrating Eq. (160) over $(0,t)$, we obtain

$$\begin{cases} H(t) \le H(0) + \frac{\Lambda(0)}{2}t, & 0 < t < T_0, \\ H(t) = 0, & t \ge T_0. \end{cases} \tag{161}$$

From the definitions of $H(t)$ and $\Lambda(t)$, we conclude that Eq. (152) holds.

The proof is completed.

Theorem 6 *Assume that $s = 0$, $p = 2$ and $q < 1$. If $J(u_0) \le 0$ and $\|u_0\|_2 < \left(D_4 D_3^{-1}\right)^{\frac{1}{1-q}}$, then the solution to Eq. (1) becomes extinct in finite time, where the lower bound of the extinction rate is*

$$\begin{cases} \|u(t)\|_2 \ge \Big[D_4 D_3^{-1} + \Big(\|u_0\|_2^{1-q} \\ \quad -D_4 D_3^{-1}\Big) e^{D_3 t}\Big]^{\frac{1}{1-q}}, \quad 0 < t < T_1, \\ \|u(t)\|_2 = 0, \quad t \ge T_1. \end{cases} \tag{162}$$

where

$$T_1 = \frac{1}{D_3} \log \left(\frac{D_4 D_3^{-1}}{D_4 D_3^{-1} - \|u_0\|_2^{1-q}} \right) \tag{163}$$

$D_3 = \frac{1-q}{2} B_1^{-2}$, $D_4 = \frac{(q-1)^2}{q+1} |\Omega|^{\frac{1-q}{2}}$, and B_1 is the optimal embedding constant of $W_0^{1,2}(\Omega) \hookrightarrow L^2(\Omega)$.

Proof . We define

$$G(t) := \tfrac{1}{2}\|u(t)\|_2^2 \tag{164}$$

Combined with the conditions $s = 0$, $p = 2$ and $q < 1$, and based on the energy equality

$$J(u_0) = J\big(u(t)\big) + \int_0^t \|u_\tau(\tau)\|_2^2 d\tau \tag{165}$$

we derive from Eqs (20), (21), (23) and $J(u_0) \le 0$ that

$$\begin{aligned} G'(t) &= \int_\Omega u(t) u_t(t) dx \\ &= -I(u(t)) \\ &= -2J(u(t)) + \frac{1}{2}\|\nabla u(t)\|_2^2 + \frac{q-1}{q+1}\|u(t)\|_{q+1}^{q+1} \\ &\ge -2J(u_0) + \frac{B_1^{-2}}{2}\|u(t)\|_2^2 \\ &+ \frac{q-1}{q+1}|\Omega|^{\frac{1-q}{2}}\|u(t)\|_2^{q+1} \\ &\ge B_1^{-2} G(t) + 2^{\frac{q+1}{2}} \frac{q-1}{q+1} |\Omega|^{\frac{1-q}{2}} G^{\frac{q+1}{2}}(t). \end{aligned} \tag{166}$$

Setting $Z(t) = G^{\frac{1-q}{2}}(t)$, we obtain from Eq. (166) that

$$\begin{aligned} Z'(t) &\ge \frac{1-q}{2} G^{-\frac{q+1}{2}}(t) \\ &\times \left[B_1^{-2} G(t) + 2^{\frac{q+1}{2}} \frac{q-1}{q+1} |\Omega|^{\frac{1-q}{2}} G^{\frac{q+1}{2}}(t) \right] \\ &= \frac{1-q}{2} B_1^{-2} G^{\frac{1-q}{2}}(t) - 2^{\frac{q-1}{2}} \frac{(q-1)^2}{q+1} |\Omega|^{\frac{1-q}{2}} \\ &= D_3 Z(t) - 2^{\frac{q-1}{2}} D_4. \end{aligned} \tag{167}$$

Solving this ordinary differential equation yields

$$\begin{cases} Z(t) \ge 2^{\frac{q-1}{2}} D_4 D_3^{-1} + \left(Z(0) - 2^{\frac{q-1}{2}} D_4 D_3^{-1} \right) e^{D_3 t}, \\ \qquad 0 < t < T_1, \\ Z(t) = 0, \quad t \ge T_1, \end{cases} \tag{168}$$

which implies that Eq. (162) holds.

The proof is completed.

Theorem 7 *Assume that $q + 1 < 2 < p$. If $J(u_0) \le 0$ and*

$$\left\| \frac{u_0}{|x|^{\frac{s}{2}}} \right\|_2 < \left(\frac{D_6}{D_5} \right)^{\frac{1}{1-q}}. \tag{169}$$

then the solution to Eq. (1) becomes extinct in finite time, and the lower bound of the extinction rate is

$$\begin{cases} \left\| \frac{u(t)}{|x|^{\frac{s}{2}}} \right\|_2 \ge \left[D_6 D_5^{-1} + \left(\left\| \frac{u_0}{|x|^{\frac{s}{2}}} \right\|_2^{1-q} - D_6 D_5^{-1} \right) e^{D_5 t} \right]^{\frac{1}{1-q}}, \\ \qquad\qquad 0 < t < T_2, \\ \left\| \frac{u(t)}{|x|^{\frac{s}{2}}} \right\|_2 = 0, \qquad t \ge T_2, \end{cases} \tag{170}$$

where

$$T_2 = \frac{1}{D_5} \log \left(\frac{D_6 D_5^{-1}}{D_6 D_5^{-1} - \left\| \frac{u_0}{|x|^{s/2}} \right\|_2^{1-q}} \right), \tag{171}$$

$$D_5 = \frac{(1-q)(p-2)}{2C_H}, \tag{172}$$

$$D_6 = (1-q) L^{\frac{s(q+1)}{2}} |\Omega|^{\frac{1-q}{2}} \cdot \frac{p-q-1}{q+1}. \tag{173}$$

Proof . By a direct calculation, similar to Eq. (166), we obtain

$$\begin{aligned} S'(t) =& -I(u(t)) \\ =& -pJ(u(t)) + \frac{p-2}{2} \|\Delta u(t)\|_2^2 \\ &+ \frac{1}{p} \|\nabla u(t)\|_p^p + \frac{q+1-p}{q+1} \|u(t)\|_{q+1}^{q+1} \\ \geq& -pJ(u_0) + \frac{p-2}{2} \|\Delta u(t)\|_2^2 + \frac{1}{p} \|\nabla u(t)\|_p^p \\ &- \frac{p-q-1}{q+1} \|u(t)\|_{q+1}^{q+1} \\ \geq& \frac{p-2}{2} \|\Delta u(t)\|_2^2 - \frac{p-q-1}{q+1} \|u(t)\|_{q+1}^{q+1}. \end{aligned} \tag{174}$$

Then, by Eqs (33), (174) and $|x| \leq L$, we derive

$$\begin{aligned} S'(t) \geq& \frac{p-2}{2} C_H^{-1} \left\| \frac{u(t)}{|x|^{\frac{s}{2}}} \right\|_2^2 \\ &- |\Omega|^{\frac{1-q}{2}} \frac{p-q-1}{q+1} \left(\int_\Omega u^2(t) dx \right)^{\frac{q+1}{2}} \\ \geq& (p-2) C_H^{-1} S(t) \\ &- L^{\frac{s(q+1)}{2}} |\Omega|^{\frac{1-q}{2}} \frac{p-q-1}{q+1} \left(\int_\Omega \frac{u^2(t)}{|x|^s} dx \right)^{\frac{q+1}{2}} \\ =& (p-2) C_H^{-1} S(t) \\ &- 2^{\frac{q+1}{2}} L^{\frac{s(q+1)}{2}} |\Omega|^{\frac{1-q}{2}} \frac{p-q-1}{q+1} S^{\frac{q+1}{2}}(t) \end{aligned} \tag{175}$$

According to Eq. (175) and $H(t) = S^{\frac{1-q}{2}}(t)$,

$$\begin{aligned} H'(t) \geq& \frac{1-q}{2} S^{-\frac{q+1}{2}}(t) \Big((p-2) C_H^{-1} S(t) \\ &- 2^{\frac{q+1}{2}} L^{\frac{s(q+1)}{2}} |\Omega|^{\frac{1-q}{2}} \frac{p-q-1}{q+1} S^{\frac{q+1}{2}}(t) \Big) \\ =& D_5 H(t) - 2^{\frac{q-1}{2}} D_6. \end{aligned} \tag{176}$$

This ordinary differential equation directly implies that

$$\begin{cases} H(t) \geq 2^{\frac{q-1}{2}} D_6 D_5^{-1} \\ \quad + \left(H(0) - 2^{\frac{q-1}{2}} D_6 D_5^{-1} \right) e^{D_5 t}, & 0 < t < T_2, \\ H(t) = 0, & t \geq T_2. \end{cases} \tag{177}$$

Then, we obtain Eq. (170).

The proof is completed.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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