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Hamiltonian and Diffeomorphism Generators of a Quantum Field in $R \times S^2$ Spacetime

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Abstract Within the ADM formalism, spacetime is foliated by spacelike hypersurfaces, dividing spacetime into space and time. In this formalism, the evolution of fields can be separated into evolution on these hypersurfaces and evolution from one hypersurface to the next. The commutators of these generators form an algebra known as the Dirac Algebra or hypersurface deformation algebra. In this paper, we expand the matter field on the three-dimensional $R \times S^2$ spacetime using the basis of spherical harmonics on S^2. Inspired by the approach of second quantization, we promote the coefficients of this expansion to creation and annihilation operators. We demonstrate that the algebra of the evolution generators, determines the Hamiltonian and diffeomorphism generators on S^2. The tensor character of the matter field is classified based on its Lie derivative with respect to its diffeomorphism generators. This quantization method determines the ordering of creation and annihilation operators in the evolution generators.

1 Introduction

In the ADM formalism, three-dimensional spacetime is foliated into spacelike surfaces. In this paper, we decompose spacetime $\mathscr{M} = R \times S^2$ into spacelike surfaces S^2 and a time axis R. Within this framework, the generators are decomposed into one generator normal to the constant-time surface $\mathscr{H}_\perp$ for evolution in time and two generators parallel to the surface $\mathscr{H}_i$, which are also known as generators of infinitesimal diffeomorphisms [1]. This decomposition leads to the emergence of the Dirac algebra (hypersurface deformation algebra) among the generators [2, 3]:

$$\{\mathscr{H}_\perp(x), \mathscr{H}_\perp(x')\} = \delta_{,i}(x,x')\left(h^{ij}(x)\mathscr{H}_j(x) + h^{ij}(x')\mathscr{H}_j(x')\right), \quad (1)$$

$$\{\mathscr{H}_i(x), \mathscr{H}_\perp(x')\} = \delta_{,i}(x,x')\mathscr{H}_\perp(x), \quad (2)$$

$$\{\mathscr{H}_i(x), \mathscr{H}_j(x')\} = \delta_{,i}(x,x')\mathscr{H}_j(x) + \delta_{,j}(x,x')\mathscr{H}_i(x'). \quad (3)$$

h_{ij} is the spatial part of metric, and its determinant is h. h^{ij} is the inverse of the metric with the condition $h_{ik}h^{kj} = \delta_i^j$. The symbol δ denotes the Dirac delta distribution defined as [4]:

$$\int h^{\frac{1}{2}} d^2x\, F(x)\; \delta(x,x') = F(x'). \quad (4)$$

$\delta_{,j} := \frac{\partial \delta}{\partial x^j}$, and x represents a point in space. $\{,\}$ denotes the Poisson bracket given by

$$\{\phi(x), \pi(x')\} = \delta(x,x'), \quad (5)$$

where ϕ is a matter field and π is its canonical conjugate momentum.

In general relativity, classical trajectories are restricted to a subspace of phase space defined by $\mathscr{H}_\perp = 0$ and $\mathscr{H}_i = 0$. Dirac algebra ensures that generators form first-class constraints. Quantization of the generators is achieved by promoting them to corresponding operators. A classical constraint is implemented in the quantum theory as a restriction on physically allowed wave functions as follows [1, 3]:

$$\hat{\mathscr{H}}_i|\Psi\rangle = 0, \quad \hat{\mathscr{H}}|\Psi\rangle = 0. \quad (6)$$

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The first equation enforces the independence of the wave function $|\Psi\rangle$ under infinitesimal coordinate transformations on the surfaces S^2. The second equation is the Wheeler-DeWitt equation [1]. Solving the Wheeler-DeWitt equation is non-trivial and often requires imposing symmetries on the theory. Additionally, operator ordering ambiguities arise when promoting Hamiltonian constraints to quantum operators [1, 5].

In the Teitelboim approach, diffeomorphism generators are determined *a priori* based on the tensorial properties of the field. For a coordinate transformation $x^\mu \to x^\mu + N^\mu$ consider evolution rule as:

$$\delta F(y) = \int d^2x\, h^{\frac{1}{2}}(x)\{F(y),\, \mathscr{H}_\mu(x)\}N^\mu(x), \tag{7}$$

Where $\mathscr{H}_\mu = (\mathscr{H}_\perp, \mathscr{H}_i)$ and $N_\mu = (N, N_i)$. If F belongs to a class of (weighted) tensor fields, the generators of infinitesimal diffeomorphism can be derived by comparing the result of (7) with its Lie derivative. For example, for a scalar field:

$$\mathscr{H}_i = \phi_{,i}\pi. \tag{8}$$

The Hamiltonian generator $\mathscr{H}_\perp$ is then derived by solving Poisson brackets (1) and (2) as differential equations [6, 7].

This paper proposes a method where all generators, along with the tensorial behavior of quantum fields, are identified through the Dirac algebra. Our approach assumes quantum fields admit an expansion in creation and annihilation operators, analogous to second quantization. We express generators in terms of these operators and demonstrate that this procedure resolves operator ordering ambiguities in generators. Note that we do not perform the second quantization approach. Unlike standard second quantization where one starts with a classical Hamiltonian, here we use Dirac algebra to derive the Hamiltonian directly in the basis of creation and annihilation operators similar to second quantization.

To circumvent mathematical complexities, we analyze the problem in three-dimensional spacetime $R \times S^2$. We expand fields using spherical harmonics $Y_{lm}(\theta,\phi)$ on S^2. This expansion yields a discrete spectrum with countable degrees of freedom. This mathematical simplification is not crucial, in conclusions and discussion we describe how one can extend this method to higher dimensions or different spacetimes.

2 ADM Decomposition

To describe this approach, we first decompose the three-dimensional spacetime with metric $g_{\mu\nu}$ into space plus time. The spacetime is hyperbolic, and on every hyperbolic manifold, there exists a temporal function, such as τ, such that each surface of constant time is a Cauchy surface with metric h_{ij} [8]. Therefore, the manifold can be foliated by these Cauchy surfaces. Every null or timelike curve intersects each Cauchy surface exactly once. Hence, τ can be considered as a representation of time, and the Cauchy surfaces can be regarded as representations of space [9, 10]. The spatial part of the metric, h_{ij}, can be obtained from the ADM metric [11]:

$$g_{\mu\nu} = \begin{pmatrix} -N^2 + N^i N_i & N_j \\ N_k & h_{jk} \end{pmatrix}. \tag{9}$$

N is called the lapse function, and N_i is known as the shift function [1]. The upper index is defined using the relation $N^i = h^{ij}N_j$.

The result of this decomposition is the emergence of Dirac algebra (1)-(3) among the Hamiltonian generators of the theory. This algebra also arises from other ideas such as parametrized systems or the principle of path independence [1–4].

The total Hamiltonian is obtained as a linear combination of these generators using the lapse and shift functions:

$$\mathscr{H} = N\mathscr{H}_\perp + N^i\mathscr{H}_i. \tag{10}$$

2.1 Vacuum solution and matter solution

We assume that there exists a vacuum solution when the matter part vanishes, so the algebra (1)-(3) closes for pure gravity.

The diffeomorphism generators decompose to gravitational and matter parts as:

$$\mathscr{H}_i = \mathscr{H}_i^{\text{matter}} + \mathscr{H}_i^{\text{gravitational}}. \tag{11}$$

If we decompose $\mathscr{H}_\perp$ into gravitational and matter parts:

$$\mathscr{H}_\perp = \mathscr{H}_\perp^{\text{matter}} + \mathscr{H}_\perp^{\text{gravitational}}, \tag{12}$$

the matter part is ultralocal with respect to the metric, meaning that it does not depend on the integral or derivative of the metric field. This result follows from the assumption that the matter part does not include the conjugate momentum of the metric. Another result of this assumption is that in Dirac algebra equations (1) and (3) are closed for matter part [4]:

$$\{\mathscr{H}_\perp^{\text{matter}}(x), \mathscr{H}_\perp^{\text{matter}}(x')\} = \delta_{,i}(x,x')\left(h^{ij}(x)\mathscr{H}_j^{\text{matter}}(x) + h^{ij}(x')\mathscr{H}_j^{\text{matter}}(x')\right), \quad (13)$$

$$\{\mathscr{H}_i^{\text{matter}}(x), \mathscr{H}_j^{\text{matter}}(x')\} = \delta_{,i}(x,x')\mathscr{H}_j^{\text{matter}}(x) + \delta_{,j}(x,x')\mathscr{H}_i^{\text{matter}}(x'). \quad (14)$$

But the remaining part of the algebra, i.e. equation (2), is not closed for matter part. From equation (13) it is obvious that $\mathscr{H}_\perp^{\text{matter}}$ depends on the metric, so for acting diffeomorphism generator on it we need the gravitational diffeomorphism generators as:

$$\{\mathscr{H}_i^{\text{matter}}(x) + \mathscr{H}_i^{\text{gravitational}}(x), \mathscr{H}_\perp^{\text{matter}}(x')\} = \delta_{,i}(x,x')\mathscr{H}_\perp^{\text{matter}}(x). \quad (15)$$

2.2 Tensorial character of the generators

Lie derivative of a function of canonical degrees of freedom like F could be calculated by using diffeomorphism generators:

$$\mathscr{L}_N F(y) = \int d^3x\, N^i(x)\{F(y),\, \mathscr{H}_i(x)\}. \quad (16)$$

Using equations (2) and (3) and by calculating Lie derivative of generators we find out $\mathscr{H}_\perp$ is a scalar density and $\mathscr{H}_i$ is a vector density of weight one under spatial coordinate transformations:

$$x^i \to x^i + N^i(x). \quad (17)$$

The scalar density field is defined by its properties under a general coordinate transformation:

$$\phi(x') = \phi(x)\left|\frac{\partial x}{\partial x'}\right|^\lambda. \quad (18)$$

In this expression, $|\frac{\partial x}{\partial x'}|$ denotes the Jacobi determinant of the transformation. We regard the parameter λ as the weight of the scalar density field. For example, the determinant of the metric h is a scalar density with weight 2. Vector density is also defined by its properties under a general coordinate transformation:

$$V_i'(x') = V_j(x)\left|\frac{\partial x}{\partial x'}\right|^\lambda \frac{\partial x^j}{\partial x'^i}. \quad (19)$$

3 Finding generators in the bases of spherical harmonics Y_{lm}

In this paper, we restrict our study to computing the generators of ϕ while neglecting the gravitational sector. For the remainder of this paper, the superscript "matter" for generators will be dropped unless stated otherwise. Consider a field like ϕ, on the S^2 we can expand this field in the basis of spherical harmonics as:

$$\phi(x) = F(x) + F^\dagger(x), \quad (20)$$

Where $F(x) := f^{lm}Y_{lm}(x)$ and $F^\dagger(x) := f^{\dagger lm}Y^*_{lm}(x)$. Summation on the repeated indexes is assumed. We assume all generators ($\mathscr{H}_\perp$, $\mathscr{H}_i$) and field operators (ϕ, π) to be Hermitian in the quantum theory, implying f^{lm} and $f^{\dagger lm}$ are Hermitian conjugates of each other. The canonical momentum conjugate to this field must also be expanded in the spherical harmonic basis. Applying Dirac quantization, replacing Poisson brackets with commutators of quantum operators, Equation (5) takes the following form:

$$[\phi(x), \pi(y)] = \mathrm{i}\delta(\mathrm{x},\mathrm{y}). \quad (21)$$

If we impose the algebra of creation and annihilation operators as:

$$[f^{lm}, f^{\dagger l'm'}] = \delta^{ll'}\delta^{mm'}, \quad (22a)$$

$$[f^{lm}, f^{l'm'}] = 0, \quad [f^{\dagger lm}, f^{\dagger l'm'}] = 0, \quad (22b)$$

where $\delta^{ll'}$ is the Kronecker delta, the canonical momentum field becomes the following Hermitian operator:

$$\pi = \frac{1}{2\mathrm{i}}(F - F^\dagger). \quad (23)$$

Similar to second quantization, by promoting the coefficients f^{lm} and $f^{\dagger lm}$ to creation and annihilation operators, we define the vacuum state $|\Omega\rangle$ via the annihilation operators:

$$f^{lm}|\Omega\rangle = 0, \quad (24)$$

while excited one-particle states are obtained by acting the creation operators on the vacuum state [12, 13]:

$$f^{\dagger lm}|\Omega\rangle = |l,m\rangle. \tag{25}$$

The Hamiltonian $\mathscr{H}_\perp$ must depend on spatial derivatives of F and $F^\dagger$ to satisfy equations (1)-(2). If we assume no derivative of fields appear in $\mathscr{H}_\perp$, from equation (1) diffeomorphism generators must be zero. then from (15), $\mathscr{H}_\perp$ vanishes.

The only fundamental elements in this theory are:

- Creation and annihilation operators $F, F^\dagger$,
- Spatial derivatives of Creation and annihilation operators like $F_{,i}$ and $F^\dagger_{,ij}$,
- The spatial metric h_{ij} and its inverse h^{ij}.

Recall that equation (15) implies that $\mathscr{H}_\perp$ must be a scalar density [4]. Consequently, terms in $\mathscr{H}_\perp$ containing only single partial derivatives of the degrees of freedom (like $F_{,i}$) are prohibited. The only admissible terms must contain Two partial derivatives contracted with the metric like $h^{ij}F_{,i}F^\dagger_{,j}$.

We exclude second-order derivative dependencies in $\mathscr{H}_\perp$ like $F^\dagger_{,ij}$ because, as shown in [7], they lead to results incompatible with Teitelboim's assumptions [4].

These constraints force the most general least-order derivative dependence to take the form:

$$\begin{aligned}\mathscr{H}_\perp =& \big(AF_{,i}F_{,j} + A^\dagger F^\dagger_{,i}F^\dagger_{,j} + BF^\dagger_{,i}F_{,j} + B^\dagger F_{,i}F^\dagger_{,j}\big)h^{ij} \\ &+ E(\text{non-derivative terms}).\end{aligned} \tag{26}$$

$$[f^{lm}, f^{\dagger l'm'}] = \delta^{ll'}\delta^{mm'}, \tag{27a}$$

$$[f^{lm}, f^{l'm'}] = 0, \quad [f^{\dagger lm}, f^{\dagger l'm'}] = 0. \tag{27b}$$

The coefficients A and B are independent of creation and annihilation operators. E is a Hermitian polynomial of F and $F^\dagger$ that does not depend on their derivatives. As we will show, these coefficients will be determined by the algebra (13) and (14).

By substituting (26) into equation (13), we find:

$$AA^\dagger = B^2 = (B^\dagger)^2. \tag{28}$$

Also we obtain an expression for the diffeomorphism generator. These generators must also satisfy the closed diffeomorphism generators algebra (14). By imposing this condition, we find that the following expressions must commute with the creation and annihilation operators:

$$[f^{\dagger lm}, [f^{\dagger l'm'}, E]], \quad [f^{lm}, [f^{l'm'}, E]], \quad [f^{lm}, [f^{\dagger l'm'}, E]]. \tag{29}$$

We therefore conclude that E takes the following form:

$$E = aF^\dagger F + b(F^\dagger)^2 + b^\dagger F^2 + \mathrm{iy}(\mathrm{F}^\dagger - \mathrm{F}) + \mathrm{G}. \tag{30}$$

In this equation, the coefficients a and y are Hermitian and independent of creation or annihilation operators. G can only depend on F and $F^\dagger$ if it commutes with all derivative-dependent terms in the time evolution generator (26).

By fully computing equation (15), we obtain the following constraints:

$$a = \frac{1}{8A}, \quad b = b^\dagger = \frac{1}{16A}, \tag{31}$$

which shows that A must be Hermitian. The calculation for the quantum field theory generators can be summarized as follows:

$$\begin{aligned}\mathscr{H}_\perp =& \left(\big(A(F_{,i}F_{,j} + F^\dagger_{,i}F_{,j})h^{ij} + \frac{1}{16A}(F^\dagger F - FF)\big) + h.c.\right) \\ &+ \mathrm{iy}(\mathrm{F}^\dagger - \mathrm{F}) + \mathrm{G},\end{aligned} \tag{32}$$

$$\mathscr{H}_j = \frac{1}{4\mathrm{i}}\left(\big((FF)_{,j} + (FF^\dagger_{,j} + F^\dagger_{,j}F)\big) - h.c.\right), \tag{33}$$

where $h.c.$ denotes the Hermitian conjugate. Note that A, y, G can be depended on gravitational degrees of freedom. These results can be rewritten in terms of ϕ and π as:

$$\mathscr{H}_\perp = \frac{1}{4A}\pi^2 + A\phi_{,i}\phi_{,j}h^{ij} + 2y\pi + G, \tag{34}$$

$$\mathscr{H}_i = \frac{1}{2}\pi\phi_{,i} + \frac{1}{2}\phi_{,i}\pi. \tag{35}$$

These results demonstrate that the operator ordering is not arbitrary but emerges systematically from the theoretical calculations.

The Lie derivative of the field is computed using the diffeomorphism generators as:

$$\begin{aligned}\mathscr{L}_N\phi(x) =& \int \frac{h^{\frac{1}{2}}}{\mathrm{i}} d^2x' N^j(x')\left[\phi(x), \mathscr{H}_j(x')\right] \\ =& \phi_{,j}(x)N^j(x),\end{aligned} \tag{36}$$

which describes the field's behavior under infinitesimal coordinate transformations $x^i \to x^i + N^i(x)$. This confirms that ϕ is a scalar field. Similarly, we find its conjugate momentum π transforms as a scalar density of weight 1:

$$\begin{aligned}\mathscr{L}_N \pi(x) = \int \frac{h^{\frac{1}{2}}}{\mathrm{i}} d^2x' N^j(x') \left[\pi(x), \mathscr{H}_j(x')\right] \\ = \left(\pi(x) N^j(x)\right)_{,j}.\end{aligned} \tag{37}$$

4 Conclusions and Discussion

In this paper, inspired by the second quantization approach, we expanded the matter field ϕ and its conjugate momen-tum on the three-dimensional spacetime $R \times S^2$ using the spherical harmonic basis on S^2, promoting the expansion co-efficients to creation and annihilation operators. We demon-strated that the Dirac algebra under certain assumptions such as Hermiticity, determines the Hamiltonian $\mathscr{H}_\perp$ and the dif-feomorphism generators of the theory. By analyzing the dif-feomorphism generators, we established that these assump-tions lead to ϕ being a scalar field. Crucially, this algebra *automatically* fixes the operator ordering in equations (34) and (35) or equivalently, in equations (32) and (33).

The model generalizes to higher dimensions via an analogous procedure. To include the gravitational generators, one first derives the gravitational diffeomorphism generators from:

$$\begin{aligned}[\mathscr{H}_j^{\text{matter}}(x) + \mathscr{H}_j^{\text{gravitational}}(x), \mathscr{H}_\perp^{\text{matter}}(x')] \\ = \mathrm{i}\delta_{\mathrm{,j}}(\mathrm{x},\mathrm{x}')\mathscr{H}_\perp^{\text{matter}}(\mathrm{x}).\end{aligned} \tag{38}$$

The Hamiltonian and diffeomorphism generators are then fully determined by requiring closure of the Dirac algebra (1)-(3) for the gravitational part.

Since our quantization relies only on two key properties of spherical harmonics—orthonormality and completeness—we expect this method to generalize to other spacetimes with suitable complete basis functions.

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