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On nonperturbative and relativistic corrections to the highly resonant hadronic bound states

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Abstract In this article, we suggest a method for calculating the mass spectrum of a bound state with relativistic corrections. In standard calculations of mass spectra that consider the nonperturbative nature of interactions, researchers typically consider only the lowest-order term in v/c, the ratio of a particle's velocity inside the bound state to the speed of light. Using a theoretical framework based on specific mathematical techniques, we go beyond the lowest-order term in v/c approximation and define the structure of relativistic nonperturbative interaction potential by summing an infinite series in powers of v/c. Furthermore, we introduce and characterize a relativistic mass correction formulated through the current–current correlated tensor associated with two charged scalar particles in an external gauge field, where the fields o bey G aussian s tatistics. T his t wo-point correlation function describes the correlation of polarization function components of the current across different spacetime points in a 4D Euclidean spacetime coordinate frame. Under relativistic conditions, this correlation structure contributes to the formation of bound states, which provide a basis for defining the constituent or relativistic mass of the particle in the bound state. Our analysis is carried out within the frameworks of quantum field theory and quantum mechanics.

1 Introduction

The relativistic and nonperturbative effects in nuclear and hadronic interactions are important and interesting subjects that, from the past to the present, have been actively studied [1–5]. Traditionally, nuclear and particle physics focus on the structure and interactions of relativistic systems composed of protons, neutrons, and hadrons [6–9]. These fermionic particles consist of quarks bound together by the strong force, and such systems are described by quantum chromodynamics (QCD). As experimental and theoretical tools have developed, the field has progressed toward hadronic physics, which studies the properties and dynamics of quarks bound and interacting with gluons, including heavy bottomonium [10]. Hadronic physics attempts to determine and understand how quarks and gluons form and bind in bound states, how confinement arises, how relativistic and ultra-relativistic velocities affect a system, and how mass and spin describe the behavior of heavy, highly hadronic resonant states from QCD dynamics. It is generally accepted that traditional potential models focus on nonrelativistic approximations and first-order perturbative expansions in v/c. These models are useful for describing nuclear physics and light hadronic bound states [10]. However, the heavy, highly resonant states and long-range gluonic interactions of bottomonium bound states require corrections beyond the leading order to capture the fine structure and decay behavior that occur when the system is under strong coupling and relativistic dynamics. Bottomonium can be modelled as systems of point-like quarks interacting in a gauge field governed by QCD, according to the framework of quantum field theory (QFT). While potential models have proven successful in describing many properties and characteristics of heavy quarkonium spectra, most approaches are fundamentally limited by their nonrelativistic conventions and neglect of dynamical gluon and relativistic potential effects [11]. Early studies of relativistic corrections in light and heavy hadronic bound states used relativistic potential models such as Dirac-based or Bethe–Salpeter approaches, as well as effective field theories like nonrelativistic quantum chromodynamics (NRQCD) [12, 13], which describe heavy quark–antiquark systems by expanding QCD in the lowest order of v/c, assuming that the particle mass is much larger than the QCD scale (Λ_{QCD}). Researchers have included relativistic corrections and color contributions in the bound states, and sometimes derived bound-state behaviors from potential NRQCD models (pNRQCD), which lead to Schrödinger-like

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equations with the potential derived from QCD, representing both perturbative and nonperturbative effects. In addition, systems using lattice QCD to capture nonperturbative effects have been studied. Therefore, based on recent results and theoretical suggestions for future experiments at the LHC, Super KEKB (Belle II), ALICE, LHCb, CMS, and ATLAS, the proposed relativistic corrections on properties of the exotic highly resonant states of heavy bottomonium (HRSB) can be very important and useful [14–16]. Bottomonium systems have a large mass and might not seem to require investigation of relativistic effects, although their highly resonant behavior makes them an interesting subject for studying relativistic corrections and their effects on the mass and the relativistic nonperturbative interaction potential. Different from the spin interaction potentials, and at high energies, relativistic corrections to the potential and mass must also be taken into account. The presented theoretical approach on the relativistic corrections in understanding nonperturbative relativistic terms of Hamiltonian interactions in strong interactions and defining quark dynamics, decay rates, and properties can be useful. This model for analyzing interaction mechanisms in extreme temperature environments, such as quark–gluon environments, neutron stars, and compact stars, can also be considered. According to the clarification above, we will use combined methods to identify the relativistic corrections on the HRSB bound states. First, the mass spectrum of the HRSB of the bottom–antibottom bound state in the framework of QFT and analogous to quantum mechanics (QM) based on the Green function (GF) in the form of the Feynman functional integral (FPI) and path integral methods [17–19], with a given interaction potential, will be presented. The mass spectrum of the bottom–antibottom bound state within relativistic corrections, based on the long-range behavior of the current–current correlation tensor (polarization function (PF) or polarization tensor) for scalar charged bottom–antibottom quarks in the excited state with a specified orbital quantum number ℓ in a background gauge field $A(x)$, is completely described and explained. The relativistic correction to the interaction behaviors under a given potential was analytically determined. Additionally, the relativistic correction to the mass, which is interpreted as the constituent masses of the bottom–antibottom charged quarks, will be logically calculated, taking into account a specified orbital quantum number ℓ. We can also introduce the effects of the full spin interactions, including spin–spin, spin–orbit, and tensor spin Hamiltonians.

$$H_{\text{int}} = H_0 + H_{\text{pert}} + H_{\text{nonpert}} + H_{SS} + H_{LS} + H_{TT}. \tag{1}$$

Into the bound state mass spectrum equation, one would typically add the Breit–Fermi Hamiltonian. However, since the goal of this paper is to describe the relativistic corrections on mass and the nonperturbative relativistic interaction potential, the Breit–Fermi Hamiltonian has been omitted. Hence, we present an alternative method to describe the properties of HRSB. In this article, equations are written in natural units with $\hbar = c = 1$, and the mass spectrum is calculated in units of GeV. We used computational coding in MATLAB R2021a software, and HRSB $10s$: $\Upsilon(11020)$ mass spectrum calculations were performed in Microsoft Office Excel 2021 software. This paper is organized as follows. In Sect. 2, the determination of the mass spectrum and constituent mass of bottom–antibottom quarks of the HRSB under relativistic corrections within QM and QFT principles is defined. In Sect. 3, the details of the relativistic nonperturbative term of the interaction potential, along with the definition and calculation method based on mathematical physics principles for the HRSB, are described.

2 The mass spectrum including relativistic correction

We present the asymptotic properties of the interaction of two scalar charged particles through the polarization function (PF) and then evaluate functional integrals under a variational technique to analyze the bound state system of these particles [20]. A four-dimensional Euclidean spacetime coordinate frame $x = (\bar{x}, x_0)$ with imaginary time defined as $x_0 = it$, based on field conditions to define the PF and Green function (GF), is considered. We assume that the annihilation channel is neglected, i.e., no particle–antiparticle annihilation channels can be created. Then, we define the solutions using the functional integral formalism and average over the external Gaussian gauge field $A(x)$. The averaging over the field with the polarization function $\Pi(x; y)$ is indicated. To extract the PF in the asymptotic limit, we employ the variational approach, which allows us to describe the effects of relativistic behaviors in the interaction of two charged scalar particles within an external gauge field. The results of these mathematical definitions provide a representation of the nonperturbative interaction term under relativistic conditions in high-energy physics. Now, we obtain formulae that give the mass spectrum of an HRSB of bottom–antibottom as charged scalar particles interacting in an external gauge field $A(x)$. We focus on the mechanism of HRSB formation and the estimation of the contributions from potential and nonpotential interactions. In the standard form, the interaction of a quark in an external gauge field is written with the spinor field equation (Dirac equation).

$$\left[(i\gamma^{\alpha}\partial_{\alpha} - g\gamma^{\alpha}A_{\alpha}(x)) - m\right]\Psi(x) = 0. \tag{2}$$

Owing to the importance of spin effects, g is the coupling constant [18, 20]. As mentioned in the introduction, in this paper, our focus is on nonperturbative relativistic corrections (NRC) of the potential and mass; therefore, the spin interaction is not considered, and we describe the quark behavior as a scalar particle and investigate the constrained interaction with the Klein–Gordon equation.

$$\left[(i\partial_\alpha + gA_\alpha)^2 + m^2\right]\phi(x) = 0. \tag{3}$$

Now, we consider the interaction between the bottom quark with mass m_1 and the antibottom quark with mass m_2, which together form a localized bottomonium bound state in the external gauge field $A(x)$. Considering the concept of a localized and nonlocalized interaction, included in $W(x; y)$, the Green's function (GF) for this system can be written as

$$\Big[\left(i\partial_\alpha + gA_\alpha(x)\right)^2 + m_1^2 + \left(i\partial_\alpha + gA_\alpha(y)\right)^2 + m_2^2 + V(x; y)\Big]\, G(x; y) = \delta^{(4)}(x - y). \tag{4}$$

It is widely recognized that the Green's function (GF) is a special solution to a differential equation characterized by the Dirac delta function $\delta^{(4)}(x - y)$, and it describes how the bottom and antibottom quarks respond to a localized disturbance in the gauge field within the interaction potential $W(x; y)$ [18, 21]. Depending on the localization of the interaction, the local density of scalar The local density of scalar particles can be expressed by a "current" in a nonrelativistic $4D$ Euclidean spacetime coordinate frame $x = (\vec{x}, x_0)$ with imaginary time $x_0 = it$. Euclidean formulations are particularly convenient when the annihilation channel is neglected, i.e. no particle–antiparticle annihilation is included. For the two-point correlator, especially when studying the correlation Green's function, it is advantageous to use scalar densities to construct observables, probe local scalar gauge-field behavior, and simplify bound-state calculations in the absence of annihilation. Hence, one can define the current in correlators and effective sources of bottom–antibottom charged quarks in the form [20, 23]

$$J(x, y) = \Phi^-(x)\, \Phi^+(y). \tag{5}$$

Here $\Phi^+(y)$ is the creation operator and $\Phi^-(x)$ is the annihilation operator. This current $J(x)$ can act as an effective source in the bottomonium bound state and yields the two-point correlation function

$$\langle 0|\widehat{T}(J(x)\, J(y))\,|0\rangle = \langle 0|\widehat{T}\left(\Phi^-(x_1)\Phi^+(y_1)\, \Phi^-(x_2)\Phi^+(y_2)\right)|0\rangle. \tag{6}$$

Here x_i denotes the point where the particle is annihilated and y_i the point where the bottom–antibottom quarks are created, while τ represents the proper time. The operator $\widehat{T}$ is the time–ordering operator, and we write $x = (\vec{x}, x_0)$, $y = (\vec{y}, y_0)$. In QFT and QM, the transformed propagator function (quantum amplitude) of the initial and final states of the bound state in the background gauge field A is connected by the Hamiltonian according to

$$G(x; y \mid A) = \langle x|e^{-i\widehat{H}t}|y\rangle. \tag{7}$$

That is, the amplitude for a particle to evolve from position y to position x in time t, governed by the Hamiltonian $\widehat{H}$. Therefore, the Green's function (GF) based on the two-point correlation function for the bottom–antibottom bound state is written as

$$G(x; y) = \langle 0|T\left(\Phi^-(x_1)\Phi^+(y_1)\, \Phi^-(x_2)\Phi^+(y_2)\right)|0\rangle. \tag{8}$$

The GF defines the amplitude for bottom–antibottom quarks through a four-point function that propagates from y_i to x_i in the bottomonium bound state [20]. The interaction of bottom–antibottom quarks in the external gauge field A is described by averaging all transformed propagators over the external field using the polarization-function (PF) formalism as follows:

$$\begin{aligned}\Pi(x; y) &= \langle J(x_1; y_1)\, J(x_2; y_2)\rangle \\ &= \langle \Phi^-(x_1)\Phi^+(y_1)\, \Phi^-(x_2)\Phi^+(y_2)\rangle \\ &= \left\langle G_1(x_1; y_1 \mid A_\alpha)\, G_2(x_2; y_2 \mid A)\right\rangle_A.\end{aligned} \tag{9}$$

As is commonly known in QFT and QM, one determines the properties of a particle of mass m under all possible configurations of interaction by evaluating the propagator.

$$D(x; y) = \int \left(\frac{dk}{2\pi}\right)^4 \widetilde{D}(k^2)\, e^{ik\cdot(x-y)}. \tag{10}$$

This formalism allows us to define the amplitude for the bottom–antibottom bound state to propagate from a relative separation y and the center-of-mass position x in the form

$$D(x; y) = \int \left(\frac{dp}{2\pi}\right)^4 \left(\frac{dk}{2\pi}\right)^4 \widetilde{D}(p, k)\, e^{ip\cdot x}\, e^{ik\cdot y}. \tag{11}$$

Here x is the center-of-mass coordinate, y is the relative coordinate, and $\widetilde{D}(p,k)$ is the propagator of the bottom–antibottom bound state with total momentum p and internal momentum k. The function $D(x;y)$ can be used to determine the bottomonium bound-state mass M. In QFT, the polarization function decays exponentially based on the spectral representation formalism in Euclidean space, given by

$$\begin{aligned}\Pi(x;y) &= |\langle 0|J(x;y)|n\rangle|^2 \; e^{-xE_n} \\ &= |\langle 0|J(x;y)|n\rangle|^2 \; e^{-xM_n}.\end{aligned} \tag{12}$$

where $\langle 0|$ denotes the vacuum state, $|n\rangle$ is the nth state, E_n is the energy of the nth state (E_n being the energy eigenvalue of the Schrödinger equation, as will be explained in Section 3), and M_n is the corresponding mass. This representation of the polarization function leads to the calculation of the mass spectrum of the bottom–antibottom bound state [20]. Hence, if we consider that both bottom–antibottom quarks are created at the same spacetime point (the origin) and propagate to $y = y_1 - y_2$, and are annihilated at x_1 and x_2 respectively, then the polarization function at final positions, depending on their relative separation $x = x_1 - x_2$, with a normalization constant $C = |\langle 0|J(x)|n\rangle|^2$ reads

$$\Pi(x) = \int \left(\frac{dp}{2\pi}\right)^4 \frac{e^{ip\cdot x}}{p^2 - M^2 + i\varepsilon} = C\, e^{-M|x|}. \tag{13}$$

Now, we consider the path integral formalism because of its powerful theoretical benefits in QM and QFT. The Feynman path integral (FPI) representation provides a useful method for describing bound states by averaging over all possible ways to create a bound state, i.e., both classical and quantum paths. It has become one of the most essential formalisms in which operator methods are rendered practical for quantizing gauge theories under QCD [16, 17]. We consider the interaction between a bottom quark with mass m_1 and an antibottom quark with mass m_2. These states form a localized bottomonium bound state in the external gauge field $A(x)$. The bottomonium bound state contains both localized and nonlocalized interactions $V(x;y)$. To better understand the mathematical formulation of the FPI, we begin by formulating it for a single charged particle in an external gauge field. This property of localized and nonlocalized behavior illustrates the roles of proper time, gauge coupling, and quantum fluctuations. In this section, we extend the formalism to the case of two interacting particles, incorporating both individual gauge interactions and the mutual interaction potential. Therefore we first represent the GF in a 4D Euclidean spacetime coordinate frame for a charged bottom quark in an external gauge field $A(x)$ in the form of the FPI formalism. Let us consider GF satisfies

$$\left[(i\partial_\alpha + A)^2 + m^2\right] G(x;y\,|\,A) = \delta^{(4)}(x;y). \tag{14}$$

And then, introducing variables s and τ, one can represent the GF in the form of the FPI as

$$\begin{aligned}G(x;y\,|\,A) &= \int_0^\infty ds\; e^{-sm^2}\, \widehat{T}_\tau \\ &\quad\times \exp\left[-s\int_0^1 d\tau \left(\tfrac{\partial}{\partial x(\tau)} + A(x(\tau))\right)^2\right] \\ &\quad\times \delta^{(4)}(x-y) \\ &= \int_0^\infty ds\; e^{-sm^2} \int d\varphi(\tau) \\ &\quad\times \exp\left[-\int_0^1 d\tau\, \varphi^2(\tau)\right. \\ &\qquad\left. + 2i\sqrt{s}\int_0^1 d\tau\, \varphi(\tau)A(z')\right] \\ &\quad\times \delta^{(4)}(z).\end{aligned} \tag{15}$$

where

$$z' = x - 2\sqrt{s}\int_0^1 d\tau'\, \varphi(\tau'), \tag{16a}$$

$$z = x - y - 2\sqrt{s}\int_0^1 d\tau\, \varphi(\tau). \tag{16b}$$

where $\varphi(\tau)$ is the path or trajectory of the quark and is a functional integration variable over paths, $x(\tau)$ is the physical trajectory, $\widehat{T}_\tau$ is the proper-time ordering or time-ordering operator, the variable $s : 0 \preceq s < \infty$ is an auxiliary parameter introduced by Schwinger's proper time to define the propagator $1/(p^2 - M^2)$ and it controls the damping factor e^{-sm^2}, while τ, τ' are proper times describing the relative motion of the two particles at different spacetime points with $\tau = \tau_1 - \tau_2$. The proper time is the time interval measured by a clock moving with the particle and it remains invariant in all inertial frames, while the fourth component of spacetime x_0 provides the direction in spacetime in which the particle is moving. In functional integral notation, the parameter s is a computational tool to write path integrals more conveniently; it is not a physical real time in spacetime coordinates and is not an observable. The dimensionless parameters τ and τ' with $0 \preceq (\tau, \tau') \preceq 1$ are the proper times along the given path and parameterize the position along the quark's trajectory $x(\tau)$, and the normalization parameter reads [20, 22].

$$C = \int d\varphi(\tau)\, \exp\left[-\int_0^1 d\tau\, \varphi^2(\tau)\right] \delta^{(4)}(z')$$
$$= \int \left(\frac{dk}{2\pi}\right)^4 \exp\left[-ikx - sk^2\right]$$
$$= \left(\frac{1}{4\pi s}\right)^2 \exp\left(-\frac{x^2}{4s}\right). \tag{17}$$

And then, introducing variables s and τ, one can represent GF in the form of FPI by way of

$$G(x; y \,|\, A) = \int_0^\infty \left(\frac{ds}{4\pi s}\right)^2 \exp\left(-sm^2\right)$$
$$\times \exp\left(-\tfrac{(x-y)^2}{4s}\right) \mathcal{R}(x; y \,|\, A), \tag{18}$$

And the kinetic term of the given path is included in

$$\mathcal{R}(x; y \,|\, A) = \int d\sigma\, J(x; y \,|\, A)$$
$$= \int d\sigma\, \exp\left[i \int_0^1 d\tau\, N(\tau)\, A\big(N'(\tau)\big)\right]. \tag{19}$$

where $x(\tau)$ is the fluctuation of the trajectory in the spacetime coordinate of the bottom quark, defined as a function of the proper time τ under Dirichlet boundary conditions $x(0) = x(s) = 0$, which describe fixed start and end points. The derivative $\dot{x}(\tau)$ denotes the velocity of the fluctuating path along the trajectory of the bottom quark. The auxiliary functions $N(\tau) = x - y - 2\sqrt{s}\,\theta(\tau)$ and $N'(\tau) = x\tau + y(1-\tau) + 2\sqrt{s}\,\theta'(\tau)$ describe the full path of the bottom quark, i.e. the position of the quark at proper time τ along its worldline between the spacetime points x and y [22]. The relation $\int d\sigma = 1$ plays the same role as the normalization of a probability distribution $\int dP = 1$, ensuring that the total measure of integration is normalized to unity. Now, using the Green's function in the form of Eq. (18) together with these new variables, and after some algebra, we consider averaging over the field restricted to two Gaussian correlators.

$$\left\langle \exp\left[i \int dx\, A(x)\, J(x)\right]\right\rangle_A$$
$$= \exp\left[-\tfrac{1}{2} \int dx \int dy\, J(x)\, D(x; y)\, J(y)\right], \tag{20}$$

where the propagator of the gauge field $D(x; y)$ reads

$$D(x; y) = \langle A(y) A(x)\rangle_A = \partial^2 \mathcal{D}_1(x; y) + \delta\, \check{\mathcal{D}}(x; y). \tag{21}$$

Hence, if we consider that both bottom–antibottom quarks are created at the same spacetime point (the origin) and propagate to $y = y_1 - y_2$, and are annihilated at x_1 and x_2 respectively, then the polarization function at the final positions depends on their difference $x = x_1 - x_2$. We can then define the asymptotic behavior ($|x| \to \infty$) of the polarization operator $\Pi(x) = \langle G_1\, G_2\rangle_A$, which describes the interaction between bottom–antibottom charged quarks of masses m_1 and m_2 that form a localized bottomonium bound state in the external gauge field $A(x)$ and interact through a nonlocal potential $V(\mu_1, \mu_2)$, as follows:

$$\Pi(x|A) = \int_0^\infty \int_0^\infty \frac{d\mu_1\, d\mu_2}{(8\pi^2 x)^2}$$
$$\times \exp\left[-\tfrac{|x|}{2} \sum_{i=1}^{2} \left(\tfrac{m_i^2}{\mu_i} + \mu_i\right)\right] V(\mu_1, \mu_2). \tag{22}$$

The function $V(\mu_1, \mu_2)$ resembles the Feynman path integral (FPI) in $4D$ nonrelativistic quantum mechanics for particles with masses μ_1 and μ_2 moving with velocities v_1 and v_2 within nonlocal and local interaction potentials [20, 23]. The total interaction is therefore expressed as $V(\mu_1, \mu_2)$. Thus, based on QFT, m_1 and m_2 are the current masses of these particles, while μ_1 and μ_2 denote the constituent masses, and the corresponding Schrödinger equation takes the form

$$\left(\frac{\hat{p}_1^2}{2\mu_1} + \frac{\hat{p}_2^2}{2\mu_2} + V(\mu_1, \mu_2)\right)\Psi = E(\mu_1, \mu_2)\, \Psi. \tag{23}$$

We will describe this equation in detail at the end of this section and in the next section, where we will derive the relativistic nonperturbative potential term from it. Thus, with the nonrelativistic formulation of the FPI in $4D$ nonrelativistic quantum mechanics, by comparing the functional $V(\mu_1, \mu_2)$ obtained from the relativistic Green's function in Eq. (22), we establish the definition of the variational parameters μ_1 and μ_2 [20, 23, 24]. In this formalism, μ_1 and μ_2 represent the constituent masses of quarks in the bound state, i.e. the relativistic mass of the moving charged quarks. The functional in Eq. (22), based on Eq. (13) in the asymptotic limit, reads

$$M = \lim_{|x|\to\infty} \left(-\frac{1}{|x|}\, \ln(\Pi(x|A))\right). \tag{24}$$

Thus, the functional integral $\Pi(x|A)$ is determined by a saddle-point technique [20, 22, 25] in the FPI representation and gives the lightest intermediate bound-state mass spectrum of bottom–antibottom charged quarks with $m_1 = m_2 = m_b$ and $\mu_1 = \mu_2 = \mu_b$ at the large-distance decay $|x| \to \infty$, within the interaction potential $V(\mu_b, \mu_{\bar{b}})$ in the form of.

$$M = \min_{\mu_b}\left[\frac{m_b^2}{\mu_b} + \mu_b + E(\mu_b)\right], \tag{25}$$

where the constituent mass of the bottom quark in the bound state, i.e. the relativistic mass of the quark inside the bottomonium bound state within the interaction potential, is determined by minimizing Eq. (25) as follows:

$$2\mu_b^2\,\frac{\mathrm{d}E(\mu_b)}{\mathrm{d}\mu_b} + \mu_b^2 - m_b^2 = 0. \tag{26}$$

m_b is the mass of the free bottom quark, the reduced mass of the system is $\mu = \frac{\mu_1\mu_2}{\mu_1+\mu_2} = \frac{\mu_b}{2}$, and $E(\mu)$ is the relativistic energy of the bottom–antibottom bound state within the interaction potential under the external gauge field. Using the reduced mass μ of bottomonium, we can define the mass spectrum M and the constituent mass μ_b of HRSB in the form of

$$M = \left(4m_b^2 - 8\mu\,\frac{dE(\mu)}{d\mu}\right)^{1/2} + \mu\,\frac{dE(\mu)}{d\mu} + E(\mu), \tag{27}$$

and

$$\mu_b = \left(m_b^2 - 2\mu\,\frac{dE(\mu)}{d\mu}\right)^{1/2}. \tag{28}$$

If we represent the Schrödinger equation of two particles with masses μ_1 and μ_2 in the nonrelativistic form, these constituent masses of particles in the bound state represent the relativistic correction on mass in the $4D$ Euclidean spacetime coordinate with the reduced mass defined as $\frac{1}{\mu} = \frac{1}{\mu_1} + \frac{1}{\mu_2}$, and then the term $E(\mu)$ can be considered as the eigenenergy of the equation $H\Psi = E(\mu)\Psi$ that determines the mass spectrum of the bound state under the relativistic correction on mass.

3 Relativistic nonperturbative interaction potential

In the study of highly resonant states of heavy quarkonia, to describe their structure and dynamics, especially HRSB states such as $\Upsilon(10860)$, $\Upsilon(11020)$, and $\Upsilon(10750)$. These states are formed under a strong coupling constant g that becomes large and perturbative QCD breaks down, and hence, including nonperturbative terms of the interaction Hamiltonian is necessary [21]. The interaction between bottom–antibottom quarks is dominated by long-range nonperturbative effects rather than short-range gluon exchange with a string tension constant σ. These effects include vacuum structure, symmetry-breaking signature, nonzero vacuum expectation value of quarks and gluons (quark condensate and gluon condensate in QCD theory), spontaneous chiral symmetry breaking, coupled channel dynamics, gluon condensations, field couplers, and power-suppressed corrections. Thus, the interactions of Hamiltonian terms in HRSB cannot be described by the perturbative behavior of the potential model $H = H_0 + \lambda H_p$. Owing to the importance of explaining the mass displacement, decay width, and resonance structures of hadronic states in experimental data, the usual quark potential models often do not take these phenomena into account. This break requires the use of nonperturbative formalism such as QCD pair-creation mechanism, screened potentials, effective field theories, or the transition matrix (T–Matrix) of a scattering process. These frameworks allow us to describe the mass spectrum, energy eigenvalues, and decay properties of HRSB states, which are completely under the relativistic effects. Therefore, nonperturbative relativistic corrections under nonperturbative potential and mass in the Hamiltonian are necessary to understand the HRSB behaviors. The relativistic mass correction was introduced in the previous section. In this section, we extract the relativistic nonperturbative Hamiltonian of the interaction potential from the functional $V(\mu_b, \mu_{\bar{b}})$ in Eq. (22) of the bottom–antibottom quark bound state polarization function. Eq. (22) presents a GF in the form of FPI when bottom–antibottom quarks with the constituent masses μ_b and $\mu_{\bar{b}}$ interact via a local and nonlocal potential, including inside function.

$$V(\mu_b, \mu_{\bar{b}}) = V_{bb} + V_{\bar{b}\bar{b}} - V_{b\bar{b}} - V_{\bar{b}b}, \tag{29}$$

in a $4D$ Euclidean spacetime coordinate frame with imaginary time $x_0 = it$, where $m_b, m_{\bar{b}}$ are the current masses of bottom–antibottom quarks. The functions $V_{bb}, V_{\bar{b}\bar{b}}$ contain the self-energy diagram interaction potential of the bottom–antibottom quarks with themselves under constituent masses $\mu_b, \mu_{\bar{b}}$. Functions $-V_{b\bar{b}}, -V_{\bar{b}b}$ describe the interaction potential via a dynamical internal gauge field mediating the interaction between the bottom–antibottom quarks of the constituent masses μ_1 and μ_2. Therefore, based on the interaction potential formalism, functions $V_{bb}, V_{\bar{b}\bar{b}}$ correspond to non-potential interactions, while functions $-V_{b\bar{b}}, -V_{\bar{b}b}$ determine the contribution to the mass renormalization of the bottom–antibottom quarks, and correspond to the potential interaction. Thus, we can consider that the functional $V(\mu_1, \mu_2)$ contains all imaginable and achievable types of scattering diagrams or virtual particle exchange diagrams. At this stage, we use the interaction potential relation in the asymptotic behavior of the polarization function, Eq. (22), which depends on the energy eigenvalue in QFT under the spectral representation formalism in the Euclidean space, which is defined as follows

$$\begin{aligned}\lim_{|x|\to\infty} V(\mu_b,\mu_{\bar{b}}) &= C\,\exp\big[-x\,E(\mu_b,\mu_{\bar{b}})\big]\\ &= C\,\exp\big[-x\,E(\mu)\big],\end{aligned} \tag{30}$$

where $E(\mu)$ is the energy eigenvalue of the Schrödinger equation, which depends on μ_b, $\mu_{\bar{b}}$, m_b, $m_{\bar{b}}$, and also on the interaction constant g of the HRSB bound state. Now, we start from the total interaction potential in Eq. (22). The functional $V(\mu_b,\mu_{\bar{b}})$ in the FPI form is

$$\begin{aligned}V(\mu_b,\mu_{\bar{b}}) = &\int\!\!\int N\,dx_{0b}\,dx_{0\bar{b}}\\ &\times\exp\Bigg[-\int d\tau \sum_{i=b,\bar{b}} \frac{\mu_i}{2}\,\dot{z}_i(\tau)^2\Bigg]\\ &\times\exp\Big[V_{bb}+V_{\bar{b}\bar{b}}-V_{b\bar{b}}-V_{\bar{b}b}\Big].\end{aligned} \tag{31}$$

and integration is performed over the $4D$ Euclidean spacetime coordinate frame $x_b=(\bar{x}_b,x_{0b})$, $x_{\bar{b}}=(\bar{x}_{\bar{b}},x_{0\bar{b}})$. Because the trajectories of the bottom–antibottom quarks are considered in the $4D$ spacetime with the imaginary time $x_0=it$, where the particle point is (x,y,z,x_0), this allows us to define the interaction potential that contains corrections associated with nonperturbative, relativistic, and nonlocal characteristics of the correlations and scattering processes. The reason for the $4D$ spacetime choice is that the particle location in the $4D$ Euclidean spacetime frame is $(x,y,z,x_0)\longrightarrow(x,y,z,it)$, and using this coordinate, the vacuum expectation values of time-ordered products of field operators and Green's functions exhibit covariant behavior in nonperturbative regimes. Considering the Wick rotation method of finding a solution in (x,y,z,it) from a solution to a related problem in (x,y,z,x_0), the FPIs appear in the action under the function e^{iS}; with the transformation $x_0\longrightarrow it$, the action S becomes real and the FPI transforms as $\int\mathcal{D}\varphi\,e^{iS}\longrightarrow\int\mathcal{D}\varphi\,e^{-S}$, i.e. the FPI functional becomes exponentially damped, allowing for convergent calculations. Therefore, if integration over the proper times τ_1 and τ_2 in the FPI formalism is neglected, bottomonium bound states reduce to nonrelativistic systems and are described by the Schrödinger equation $\hat{H}\Psi=E(\mu)\Psi$ represented by Eq. (49).

As described in Eq. (12) and Eq. (30), the desired term $E(\mu_b,\mu_{\bar{b}})=E(\mu)$ is the energy eigenvalue of the Schrödinger equation [26]. It is widely recognized that QCD is formulated in $4D$ spacetime, and by nonrelativistic conditions we mean that one can expand the relativistic Lagrangian in powers of v/c or p/m, integrate over high-energy modes, and consider effective theories to describe hadronic bound states. These conditions in $4D$ spacetime describe the behavior of bottom–antibottom quarks with slow velocity ($v/c\sim 0.1-0.3$) under the nonrelativistic limit, where relativistic effects such as time dilation, length contraction, and mass–energy equivalence become negligible. In contrast, due to the heavy mass of bottom–antibottom quarks and their small relative velocity compared to the speed of light, HRSB states display relativistic and nonperturbative effects arising from the larger spatial separation between the bottom–antibottom quarks compared to ground states, such that the bound state extends to the long-distance asymptotic regime of QCD, where the strong coupling constant g becomes large and perturbation theory ceases to apply. Moreover, topological fluctuation effects cannot be described by Feynman diagrams or perturbative expansions. Considering this explanation of HRSB, we now start to derive the nonperturbative relativistic potential from Eq. (31). In QFT and particle physics, nonperturbative relativistic corrections to the interaction Hamiltonian are important and determined using an approximate correction to the lowest order of v/c. In this section, we extract an additional nonperturbative relativistic approach using some practical suggestions from the interaction process and the behavior of the $4D$ spacetime term based on integrating over the proper times τ_b and $\tau_{\bar{b}}$ of the bottom–antibottom charged quarks. In Eq. (31), we divide the fourth spacetime coordinates x_{0b} and $x_{0\bar{b}}$ (where the proper times are τ_b and $\tau_{\bar{b}}$) and define potential terms in three separated parts of the vacuum interaction: (1) a potential term (the nonrelativistic potential of interaction between particles with the gauge field), (2) a nonperturbative term, and (3) a nonperturbative relativistic term, i.e.

$$-V_{b\bar{b}},-V_{\bar{b}b}=V_{\rm nonrel}+V_{\rm nonpert}+V_{\rm rel}. \tag{32}$$

To define the nonperturbative relativistic part of the interaction, we have to integrate over proper time in Eq. (31); hence, we consider the constant of interaction to be small based on the perturbative interaction formalism, and this allows us to integrate over proper time. We consider this form of interaction potential because the standard and useful solution of the functional $V(\mu_b,\mu_{\bar{b}})$ in QFT is absent. Now, we start the representation of the nonperturbative relativistic interaction potential of HRSB. The functional of the total interaction $V(\mu_b,\mu_{\bar{b}})$ contains the components $V_{bb}+V_{\bar{b}\bar{b}}-V_{b\bar{b}}-V_{\bar{b}b}$ that read

$$\begin{aligned}V_{bb}=&\frac{g^2}{2}\int_0^t\int_0^t d\tau_b\,d\tau_{\bar{b}}\\ &\times\dot{z}^{(b)}(\tau_b)\,\mathcal{D}\Big(z^{(b)}(\tau_b)-z^{(b)}(\tau_{\bar{b}})\Big)\,\dot{z}^{(b)}(\tau_{\bar{b}}).\end{aligned} \tag{33}$$

$$V_{\bar{b}\bar{b}} = \frac{g^2}{2}\int_0^t\int_0^t d\tau_b d\tau_{\bar{b}} \times \dot{z}^{(\bar{b})}(\tau_b)\,\mathcal{D}\Big(z^{(\bar{b})}(\tau_b) - z^{(\bar{b})}(\tau_{\bar{b}})\Big)\,\dot{z}^{(\bar{b})}(\tau_{\bar{b}}). \tag{34}$$

$$V_{b\bar{b}} = V_{\bar{b}b} = \frac{g^2}{2}\int_0^t\int_0^t d\tau_b d\tau_{\bar{b}} \times \dot{z}^{(b)}(\tau_b)\,\mathcal{D}\Big(z^{(b)}(\tau_b) - z^{(\bar{b})}(\tau_b)\Big)\,\dot{z}^{(\bar{b})}(\tau_{\bar{b}}). \tag{35}$$

where function $z^{(b)}(\tau_b)$ and $z^{(\bar{b})}(\tau_{\bar{b}})$ describe how the trajectory contributes to the propagator structure under relativistic corrections and are defined by $z^{(b)}(\tau_b) = (x_b - x_{\bar{b}})\tau_b + x_{\bar{b}} - 2B(\tau_b)\sqrt{s}$ and $z^{(\bar{b})}(\tau_{\bar{b}}) = (x_b - x_{\bar{b}})\tau_{\bar{b}} + x_{\bar{b}} - 2B(\tau_{\bar{b}})\sqrt{s}$, where parameter s is an additional relativistic correction term, and $B(\tau_b)$, $B(\tau_{\bar{b}})$ are normalization constants (for details, see equations in Section 2). By comparing the functional $V(\mu_1,\mu_2)$ obtained from the relativistic Green's function, Eq. (12) with the nonrelativistic formulation of the Feynman path integral in $4D$ nonrelativistic QM (NQM), we establish the definition of key parameters μ_1,μ_2 that are introduced in Section 3 as the constituent masses of bottom–antibottom quarks in the bottomonium highly resonant state. Next, we determine the NRC structure of the interaction potential in the Hamiltonian. The interaction between the constituent bottom–antibottom quarks with masses $\mu_b,\mu_{\bar{b}}$ is mediated by the exchange of gauge fields, where particles do not interact directly but rather exchange strong–field gauge bosons, resulting in the high–energy physics limit. Hence, the Green's function of the two–point correlation function can be written in momentum space in the form of

$$\widetilde{D}(q^2+s^2) = \int d\eta\,\exp\left[-\eta\,(q^2+s^2)\right]. \tag{36}$$

and using $\mathcal{D}(x)$ together with (4), then after integrating over dq, the functional of interaction terms with $x_0 \sim (\tau_b - \tau_{\bar{b}}) \to x_0 = \alpha(\tau_b - \tau_{\bar{b}})$ reads

$$\begin{aligned} V_{bb} = V'_{bb} + V''_{bb} &= \frac{g^2}{2}\int_0^t\int_0^t d\tau_b d\tau_{\bar{b}} \\ &\times \int_{-\infty}^{\infty}\frac{ds}{2\pi}\int_{-\infty}^{\infty}\frac{dq}{(2\pi)^3}\int_0^{\infty} d\eta\, e^{-\eta(q^2+s)} \\ &\times \exp\Big[-i\,q\cdot\big(\bar{x}_b(\tau_b) - \bar{x}_b(\tau_{\bar{b}})\big)\Big] \\ &\times \exp\Big[-is\,(\bar{x}_{0b}(\tau_b) - \bar{x}_{0b}(\tau_{\bar{b}})) + is\tau\Big]\,\Theta_{bb} \\ &= \frac{g^2}{2}\int_0^t\int_0^t d\tau_b d\tau_{\bar{b}}\int_{-\infty}^{\infty}\frac{ds}{2\pi}\int_0^{\infty}\frac{d\eta}{8(\sqrt{\pi}\,\eta)^3}\,e^{-\frac{\bar{x}}{4\eta}+is\tau} \\ &\times \sum_{k=0}^{\infty}\sum_{n=0}^{k}\frac{(-1)^{n+k}}{n!\,(k-n)!}\,\eta^n\,(x_0)^{k-n}\,(is)^{n+k}\,\Theta_{bb}, \end{aligned} \tag{37}$$

$$\begin{aligned} V_{\bar{b}\bar{b}} = V'_{\bar{b}\bar{b}} + V''_{\bar{b}\bar{b}} &= \frac{g^2}{2}\int_0^t\int_0^t d\tau_b d\tau_{\bar{b}} \\ &\times \int_{-\infty}^{\infty}\frac{ds}{2\pi}\int_{-\infty}^{\infty}\frac{dq}{(2\pi)^3}\int_0^{\infty} d\eta\, e^{-\eta(q^2+s)} \\ &\times \exp\Big[-i\,q\cdot\big(\bar{x}_{\bar{b}}(\tau_b) - \bar{x}_{\bar{b}}(\tau_{\bar{b}})\big)\Big] \\ &\times \exp\Big[-is\,(\bar{x}_{0\bar{b}}(\tau_b) - \bar{x}_{0\bar{b}}(\tau_{\bar{b}})) + is\tau\Big]\,\Theta_{\bar{b}\bar{b}} \\ &= \frac{g^2}{2}\int_0^t\int_0^t d\tau_b d\tau_{\bar{b}}\int_{-\infty}^{\infty}\frac{ds}{2\pi}\int_0^{\infty}\frac{d\eta}{8(\sqrt{\pi}\,\eta)^3}\,e^{-\frac{\bar{x}}{4\eta}+is\tau} \\ &\times \sum_{k=0}^{\infty}\sum_{n=0}^{k}\frac{(-1)^{n+k}}{n!\,(k-n)!}\,\eta^n\,(x_0)^{k-n}\,(is)^{n+k}\,\Theta_{\bar{b}\bar{b}}, \end{aligned} \tag{38}$$

$$\begin{aligned} V_{b\bar{b}} = V'_{b\bar{b}} + V''_{b\bar{b}} &= \frac{g^2}{2}\int_0^t\int_0^t d\tau_b d\tau_{\bar{b}} \\ &\times \int_{-\infty}^{\infty}\frac{ds}{2\pi}\int_{-\infty}^{\infty}\frac{dq}{(2\pi)^3}\int_0^{\infty} d\eta\, e^{-\eta(q^2+s)} \\ &\times \exp\Big[-i\,q\cdot\big(\bar{x}_b(\tau_b) - \bar{x}_{\bar{b}}(\tau_{\bar{b}})\big)\Big] \\ &\times \exp\Big[-is\,(\bar{x}_{0b}(\tau_b) - \bar{x}_{0\bar{b}}(\tau_{\bar{b}})) + is\tau\Big]\,\Theta_{\bar{b}\bar{b}} \\ &= \frac{g^2}{2}\int_0^t\int_0^t d\tau_b d\tau_{\bar{b}}\int_{-\infty}^{\infty}\frac{ds}{2\pi}\int_0^{\infty}\frac{d\eta}{8(\sqrt{\pi}\,\eta)^3}\,e^{-\frac{\bar{x}}{4\eta}+is\tau} \\ &\times \sum_{k=0}^{\infty}\sum_{n=0}^{k}\frac{(-1)^{n+k}}{n!\,(k-n)!}\,\eta^n\,(x_0)^{k-n}\,(is)^{n+k}\,\Theta_{b\bar{b}}, \end{aligned} \tag{39}$$

where

$$\Theta_{bb} = 1 + n\big(\dot{\bar{x}}_b(\tau_b) - \dot{\bar{x}}_b(\tau'_b)\big) + \dot{\bar{x}}_b(\tau_b)\,\dot{\bar{x}}_b(\tau'_b), \tag{40}$$

$$\Theta_{\bar{b}\bar{b}} = 1 + n\big(\dot{\bar{x}}_{\bar{b}}(\tau_b) - \dot{\bar{x}}_{\bar{b}}(\tau'_b)\big) + \dot{\bar{x}}_{\bar{b}}(\tau_b)\,\dot{\bar{x}}_{\bar{b}}(\tau'_b), \tag{41}$$

$$\Theta_{b\bar{b}} = 1 + n\big(\dot{\bar{x}}_b(\tau_b) - \dot{\bar{x}}_{\bar{b}}(\tau'_b)\big) + \dot{\bar{x}}_b(\tau_b)\,\dot{\bar{x}}_{\bar{b}}(\tau'_b). \tag{42}$$

where $\tau = \tau_1 - \tau_2$ is the proper time of the relative motion of the constituent particles and includes velocity-dependent terms, which are common in relativistic treatments of interactions. Now, integrating over ds and $d\eta$ from Eqs (37)-(39) within the QCD formalism in the study of bottom–antibottom quarks, we define the diagonal interaction component V'_B as the one–gluon exchange effect that determines the mass renormalization coefficient arising from self–energy corrections $(V'_{bb} + V'_{\bar{b}\bar{b}})$ and the interaction term $(V'_{b\bar{b}})$ as follows

$$\begin{aligned} V'_B &= V'_{bb} + V'_{\bar{b}\bar{b}} + V'_{b\bar{b}} \\ &= \left[\frac{g^2}{8\pi} \int_0^t \int_0^t d\tau_b d\tau'_b \, \frac{\delta(\tau_b - \tau_{\bar{b}})}{\left|\bar{x}_b(\tau_b) - \bar{x}_b(\tau_b)\right|} \right] \\ &\quad + \left[\frac{g^2}{8\pi} \int_0^t \int_0^t d\tau_{\bar{b}} d\tau_{\bar{b}} \, \frac{\delta(\tau_{\bar{b}} - \tau_{\bar{b}})}{\left|\bar{x}_{\bar{b}}(\tau_b) - \bar{x}_{\bar{b}}(\tau_{\bar{b}})\right|} \right] \\ &\quad + \left[-\frac{g^2}{4\pi} \int_0^t \int_0^t d\tau_b d\tau_{\bar{b}} \, \frac{\delta(\tau_b - \tau_{\bar{b}})}{\left|\bar{x}_b(\tau_b) - \bar{x}_{\bar{b}}(\tau_{\bar{b}})\right|} \right] \\ &= -\frac{g^2}{4\pi} \int_0^t d\tau \left[\frac{1}{\left|\bar{x}_b(\tau) - \bar{x}_{\bar{b}}(\tau)\right|} - \int \frac{dq}{q^2} \right]. \end{aligned} \tag{43}$$

and the second term of the functional interaction potential is $V''_B = V''_{bb} + V''_{\bar{b}\bar{b}} + V''_{b\bar{b}}$. To integrate this functional, we first consider the following conditions: (1) introduce an auxiliary variable ε, which is a mathematical tool to describe the motion of bottom–antibottom quarks in the center–of–mass coordinate frame $R(\tau)$, assumed to be at rest such that $\frac{dR(\tau)}{d\tau} = 0$, and (2) the speeds of relative motion of the bottom–antibottom quarks are

$$\begin{cases} v_b(\tau) = \dfrac{d\bar{x}_b}{d\tau} = \text{const}, \\ v_{\bar{b}}(\tau) = \dfrac{d\bar{x}_{\bar{b}}}{d\tau} = \text{const}, \end{cases} \tag{44}$$

and the auxiliary variable ε shows the evolution of proper time $\tau \to \tau + \varepsilon$. By setting the auxiliary variable ε to $\varepsilon = 0$, and after integrating over the functionals V''_{bb}, $V''_{\bar{b}\bar{b}}$, and $V''_{b\bar{b}}$ while considering the center of mass in the form of

$$\begin{cases} R(\tau + \varepsilon) = \dfrac{\mu_{\bar{b}}}{\mu_b + \mu_{\bar{b}}} - \bar{x}_b(\tau + \varepsilon), \\ R(\tau) = \dfrac{\mu_b}{\mu_b + \mu_{\bar{b}}} - \bar{x}_{\bar{b}}(\tau). \end{cases} \tag{45}$$

then, the second term of functional interactions potential is as follows

$$\begin{aligned} V''_{bb} &= \frac{g^2}{8\pi} \int d\tau \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \\ &\quad \times \sum_{n=0}^{k} \frac{(-1)^n \, k!}{n! \, (k-n)!} \, \frac{1}{2^{2n}\sqrt{\pi}} \\ &\quad \times \int_0^{\infty} u \, du^{z-1} e^{-u} \, \frac{d^{\,k+n}}{d\tau^{\,k+n}} \Big[\left|\bar{x}_b(\tau + \varepsilon) - \bar{x}_b(\tau)\right|^{2n-1} \\ &\quad \times \left(\bar{x}_{0b}(\tau + \varepsilon) - \bar{x}_{0b}(\tau)\right)^{k-n} \Big], \end{aligned} \tag{46}$$

$$\begin{aligned} V''_{\bar{b}\bar{b}} &= \frac{g^2}{8\pi} \int d\tau \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \\ &\quad \times \sum_{n=0}^{k} \frac{(-1)^n \, k!}{n! \, (k-n)!} \, \frac{1}{2^{2n}\sqrt{\pi}} \\ &\quad \times \int_0^{\infty} u \, du^{z-1} e^{-u} \, \frac{d^{\,k+n}}{d\tau^{\,k+n}} \Big[\left|\bar{x}_{\bar{b}}(\tau + \varepsilon) - \bar{x}_{\bar{b}}(\tau)\right|^{2n-1} \\ &\quad \times \left(\bar{x}_{0\bar{b}}(\tau + \varepsilon) - \bar{x}_{0\bar{b}}(\tau)\right)^{k-n} \Big], \end{aligned} \tag{47}$$

$$\begin{aligned} V''_{b\bar{b}} &= \frac{g^2}{8\pi} \int d\tau \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \\ &\quad \times \sum_{n=0}^{k} \frac{(-1)^n \, k!}{n! \, (k-n)!} \, \frac{1}{2^{2n}\sqrt{\pi}} \\ &\quad \times \int_0^{\infty} u \, du^{z-1} e^{-u} \, \frac{d^{\,k+n}}{d\tau^{\,k+n}} \Big[\left|\bar{x}_b(\tau + \varepsilon) - \bar{x}_{\bar{b}}(\tau)\right|^{2n-1} \\ &\quad \times \left(\bar{x}_{0b}(\tau + \varepsilon) - \bar{x}_{0\bar{b}}(\tau)\right)^{k-n} \Big]. \end{aligned} \tag{48}$$

In the context of mathematical physics in the series expansion matching formalism, when the Green's function (GF) is defined as a double sum over n and k, it is useful to consider the case $k = n$ to define only diagonal terms in the expansion. This technique is often applied in theoretical physics to simplify expressions. By focusing on the order $k = n$ and choosing the asymptotic behavior of the bottom–antibottom bound state at distance $|x| \to \infty$, and setting $\varepsilon = 0$, Eqs (46)-(48) are obtained, which show that the diagonal terms vanish, $V''_{bb} = V''_{\bar{b}\bar{b}} = 0$, while the cross term $V''_{b\bar{b}}$ of the second part of the interaction potential of bottom–antibottom quarks in the center–of–mass coordinate frame $R(\tau)$ has the following form

$$\begin{aligned} V''_B &= -\frac{g^2}{4\pi} \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \, \frac{\Gamma\left(\frac{1}{2} - k\right)}{\pi^{1/2} \, 2^{2k}} \\ &\quad \times \int_0^t d\tau \, \frac{d^{\,2k}}{d\tau^{\,2k}} \Big[\left|\bar{x}_b(\tau) - \bar{x}_{\bar{b}}(\tau)\right|^{2k-1} \Big]. \end{aligned} \tag{49}$$

After integrating Eq. (49) over the parameter τ, which represents the proper–time relative motion of the constituent bottom–antibottom quarks in the bound state of the HRSB, we define

$$V''_B = -\frac{g^2}{4\pi} \sum_{k=1}^{\infty} \frac{(-1)^k}{\pi^{1/2} \, (2k)!} \, \frac{d^{\,2k}}{d\tau^{\,2k}} \Big(|\bar{x}(\tau)|^{\,2k-1} \Big), \tag{50}$$

and then

$$V_B'' = -\frac{g^2}{4\pi}\,\mathcal{L}(\tau). \tag{51}$$

Now, to determine the standard form of the nonperturbative relativistic correction term $\mathcal{L}(\tau)$ of bottom–antibottom quarks in the HRSB, we define

$$\bar{\partial}_\mu(\tau) \; = \; \frac{d^{2k}}{d\tau^{2k}}\left(|\bar{x}(\tau)|^{2k-1}\right), \tag{52}$$

and consider that the relative velocity of particles is constant, namely $v(\tau) = \frac{|\dot{\bar{x}}(\tau)|}{d\tau} = \text{const}$ so that $\dot{v} = \frac{dv(\tau)}{d\tau} = 0$, and represent the velocity by the unit vector $n = \frac{\bar{x}(\tau)}{|\bar{x}(\tau)|}$ in the form of

$$v(\tau) = \frac{d\bar{x}(\tau)}{d\tau} \quad \longrightarrow \quad \frac{d|\bar{x}(\tau)|}{d\tau} = \frac{\bar{x}(\tau)}{|\bar{x}(\tau)|}\cdot\frac{d\bar{x}(\tau)}{d\tau} = n\,v. \tag{53}$$

On the other hand, we can use one of the canonical noncommutative relations in QM, which reflects the Heisenberg uncertainty principle of position and momentum, $[\bar{x}, p] = i$, and within this expression define a noncommutative relation $[\bar{x}, v]^2 = \hat{L}^2/|\bar{x}(\tau)|^3$. This notation is particularly interesting and useful in central potential problems under relativistic conditions, where the angular momentum $\hat{L}$ plays an important role. Now, one can define $\mathcal{L}(\tau)$ for the values $k = 1, 2, 3, \ldots$ as follows:

$$\begin{aligned}
k = 1: &\quad \frac{d^2|\bar{x}(\tau)|}{d\tau^2} = \frac{[\bar{x}(\tau), v(\tau)]^2}{|\bar{x}(\tau)|^3} = \frac{\hat{L}^2}{|\bar{x}(\tau)|^3},\\
k = 2: &\quad \frac{d^4|\bar{x}(\tau)|^3}{d\tau^4} = \frac{[\bar{x}(\tau), v(\tau)]^4}{|\bar{x}(\tau)|^5} = \frac{9\hat{L}^2}{|\bar{x}(\tau)|^5},\\
k = 3: &\quad \frac{d^6|\bar{x}(\tau)|^5}{d\tau^6} = \frac{[\bar{x}(\tau), v(\tau)]^6}{|\bar{x}(\tau)|^7} = \frac{9.25\,\hat{L}^2}{|\bar{x}(\tau)|^5}. \quad \ldots
\end{aligned} \tag{54}$$

Then, for $k = n$, it reads

$$\begin{aligned}
\bar{\partial}(\tau)\big|_{k=n} &= \frac{d^{2k}|\bar{x}(\tau)|^{2k-1}}{dk} = \frac{[\bar{x}(\tau), v(\tau)]^{2k}}{|\bar{x}(\tau)|^{2k+1}}\\
&= \frac{\left[\frac{(2k-1)!}{2^k(k-1)!}\right]^2 \hat{L}^{2k}}{|\bar{x}(\tau)|^{2k+1}}.
\end{aligned} \tag{55}$$

To define the standard form of the NRC, some calculations and mathematical substitutions in Eq. (55) are required, and using

$$\begin{aligned}
\left[\frac{(2k-1)!}{2^{(k-1)}(k-1)!}\right]^2 &= \left[\frac{\Gamma(2k)}{2^{(k-1)}\Gamma(k)}\right]^2\\
&= \left[\frac{2^{(2k-1)}\,\Gamma(k+0.5)}{\pi^{1/2}\,2^{k-1}}\right]^2.
\end{aligned} \tag{56}$$

Then, we represent the new commutator form based on momentum and mass

$$m[\bar{x}, v] = [\bar{x}, p], \tag{57}$$

and consider the angular momentum operator's eigenvalue $\hat{L}^2 = \ell(\ell+1)$, then the canonical noncommutative relation $[\bar{x}, v]^2 = \hat{L}^2$ becomes

$$[\bar{x}, v]^2 \; = \; \frac{\ell(\ell+1)}{m^2\,|\bar{x}(\tau)|^3}. \tag{58}$$

Now, the standard form of the nonperturbative relativistic correction of the interaction potential in the HRSB as follows

$$\begin{aligned}
V_B'' &= -\frac{g^2}{4\pi}\,\mathcal{L}(\tau)\\
&= -\frac{g^2}{4\pi}\sum_{k=1}^{\infty}\frac{(-1)^k}{\pi^{1/2}(2k)!}\left[\frac{\Gamma(2k)}{2^k(k-1)\Gamma(k)}\right]^2\frac{\hat{L}^{2k}}{|\bar{x}(\tau)|^{2k+1}}\\
&= -\frac{g^2}{4\pi}\sum_{k=1}^{\infty}\frac{(-1)^k}{(2k)!}\left[\frac{2(2k-1)\Gamma(k+0.5)}{\pi^{1/2}\,2^k(k-1)!}\right]^2\frac{\hat{L}^{2k}}{|\bar{x}(\tau)|^{2k+1}}\\
&= -\frac{g^2}{4\pi}\sum_{k=1}^{\infty}\frac{(-1)^k}{k!}\frac{\Gamma(k+0.5)}{\pi^{1/2}|\bar{x}(\tau)|}\left(\frac{\hat{L}}{4\mu\,|\bar{x}(\tau)|}\right)^{2k}\\
&= -\frac{g^2}{4\pi}\sum_{k=1}^{\infty}\frac{(-1)^k}{k!}\frac{1}{\pi^{1/2}|\bar{x}(\tau)|}\left(\int_0^\infty du\,u^{k-1/2}e^{-u}\right)\\
&\quad\times\left(\frac{\hat{L}}{4\mu\,|\bar{x}(\tau)|}\right)^{2k}.
\end{aligned} \tag{59}$$

After some mathematical simplification of the gamma integral function under $\left(\frac{\hat{L}}{4\mu|\bar{x}(\tau)|}\right)^{2k}$, Eq. (59) reads

$$V_B'' = -\frac{g^2}{4\pi}\sum_{k=1}^{\infty}\frac{(-1)^k}{k!\,\pi^{1/2}\,|\bar{x}(\tau)|} \times \left(\int_0^\infty du\, u^{k-1/2}e^{-u}\right)\left(\frac{\hat{L}}{4\mu|\bar{x}(\tau)|}\right)^{2k}$$
$$= -\frac{g^2}{4\pi}\int_0^\infty \frac{1}{\pi^{1/2}\,|\bar{x}(\tau)|}\,u^{-1/2}e^{-u} \times \left[\exp\left(-\frac{\hat{L}}{(4\mu|\bar{x}(\tau)|)}\right)^2 - 1\right]du. \tag{60}$$

using the exponential function, Taylor series, and properties under the operator formalism of the gamma integral function

$$e^{-x\left(\frac{\hat{L}}{4\mu|\bar{x}(\tau)|}\right)^2} = \sum_{k=0}^{\infty}\frac{x^k}{k!}\left(\frac{\hat{L}}{4\mu|\bar{x}(\tau)|}\right)^{2k}, \tag{61}$$

and integral evaluation of the gamma function

$$\int_0^\infty e^{-u}\,u^{1/2}\left(\frac{\hat{L}}{4\mu|\bar{x}(\tau)|}\right)^2 du = \frac{\pi^{1/2}}{\sqrt{\left(\frac{\hat{L}}{4\mu|\bar{x}(\tau)|}\right)^2}}. \tag{62}$$

we define

$$V_B'' = -\frac{g^2}{4\pi^{\frac{3}{2}}|\bar{x}(\tau)|}\left(\frac{\pi^{1/2}}{\sqrt{1+\left(\frac{\hat{L}}{4\mu|\bar{x}(\tau)|}\right)^2}} - \pi^{1/2}\right). \tag{63}$$

and then, the nonperturbative relativistic correction term of the interaction is defined in the standard form

$$V_B'' = -\frac{g^2}{4\pi|\bar{x}(\tau)|}\left(\frac{1}{\sqrt{1+\left(\frac{\hat{L}}{4\mu|\bar{x}(\tau)|}\right)^2}} - 1\right). \tag{64}$$

This term can be called the nonperturbative relativistic Hamiltonian of interaction, i.e., $H_{\text{nonpert.}} = V_B''$. At the end of the theoretical calculations presented above and the final result of Eq. (64), we derived NRC to the interaction potential of bottom–antibottom quarks in the HRSB bound state based on the relativistic nature of bottomonium behaviors when the bottom–antibottom quarks move relative to each other at $v(\tau) = \text{const}$. The NRC (V_B'') term to the Hamiltonian of interaction can be neglected in the nonrelativistic limit [27], and all spin interactions can be added to the total Hamiltonian [28, 29]. We present the asymptotic properties of the interaction of two scalar charged particles through PF using a variational technique.

4 Conclusion

In this theoretical research, the polarization correlation function of the highly resonant bound state of the bottom–antibottom quarks (bottomonium) was investigated, focusing on QFT and QM theories and the functional path integral formalism to define the relativistic correction on mass and on the interaction potential, which is considered a nonperturbative term of the total Hamiltonian. To characterize these corrections, we introduced a two-point polarization correlation function in the $4D$ Euclidean spacetime coordinate frame within the bottom–antibottom bound state. This formalism analytically describes the mass spectrum of the highly resonant bound state. The creation mechanism of the constituent mass of bottom–antibottom quarks forming the bound state was explained. The constituent masses of bottom–antibottom quarks differ from those of their free state masses. As is known, in the interaction gauge field the gluon acquires mass when it becomes part of a bound-state formation during the interaction of bottom–antibottom quarks with the background and external fields. Then, using the concept of proper time in $4D$ Euclidean spacetime coordinates, the relative motion of the constituent bottom–antibottom quarks was defined. Assuming that bottom–antibottom quarks move relative to each other, corrections to the interaction Hamiltonian associated with relativistic interaction were extracted from the functional integral part of the polarization correlation function. Both approaches consistently yield the relativistic corrections and behaviors that allow us to use the Schrödinger equation to determine the mass spectrum and energy eigenvalue of the highly resonant bound state. The calculation of relativistic corrections can be generalized for all hadronic and non-hadronic bound states. According to the approach presented in this paper, the computational results of the mass spectrum and constituent mass of the bottomonium highly resonant states, such as $\Upsilon(11020)$ with $m_b = 4.823$ GeV under the relativistic correction on mass [31], are defined as follows:

Table 1 Comparison of theoretical and experimental results for the highly resonant bottomonium state.

$M_{\text{cl.}}$	$M_{\text{rel.}}$	μ_b	M_{Th} [30]	M_{Exp} [31]
10.829	11.052	5.051	11.021	11.020

The calculation result based on this research includes the nonrelativistic mass ($M_{\text{cl.}}$), the mass with the relativistic correction on mass ($M_{\text{rel.}}$), and the constituent mass (μ_b). The results are acceptable in comparison with the nonrelativistic result, the experimental value, and the theoretical relativistic results obtained from other computational methods.

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