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A classical vector method for the curved space–time analysis

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Abstract In flat space, the classical vectors such as a position vector are bilocal ("point for head and point for tail"). The four-dimensional curved-space Schwarzschild metric is mathematically similar to the metric of a sphere surface in three-dimensional flat space, where we can write an incremental displacement vector at a point on surface but cannot write position vectors along the curved surface. Similarly, in curved space, we can write an incremental displacement vector based on the curved-space metric, even if writing a position vector is difficult. We suggest a classical vector method based on this incremental vector which gives all the desired mathematical results including various identities, similar to conventional tensor analysis. We examine whether this mathematical similarity between a curved space and a sphere surface in flat space can also lead to geometrical similarity, but we encounter some difficulties. Therefore, the curved space-time requires discarding bilocal vectors and defining vectors called local vectors. Changing the definition of vectors can overcome the difficulties but introduces new concerns. This Newtonian vector method is an easier mathematical alternative to conventional tensor analysis in curved multidimensional space, and it also illuminates geometrical concerns. We can also establish a relationship between the three-dimensional Lagrangian method and the four-dimensional Geodesic analysis, both giving the same results.

1 Introduction

Classically, there are three important aspects of nature: (a) particles and (b) the empty space around them, and this empty space is filled with a conceptual material called (c) the field, which exerts forces on the particles. The force is a vector. In this article, we try to understand and compare the geometrical pictures of the vectors described in a classical approach and in general relativity. In the flat-space classical analysis, we can write a position vector connecting two points in space. Even in this flat space, we cannot write a curved-path position vector between two points on the surface of a sphere, as the directions of unit vectors change along the path. However, we can always write an incremental displacement (position) vector at any point on the sphere. This incremental vector can give a metric corresponding to the sphere surface, in spherical coordinates. We can also write an incremental displacement vector corresponding to the Schwarzschild metric, similar to the incremental displacement vector on the surface of a sphere. We can establish a mathematical similarity of the four-dimensional curved-space metric with the metric for the curved surface of the sphere in a flat space. We shall obtain results identical to those from the tensor analysis in general relativity [1]. We examine whether this mathematical similarity can also lead to a similarity between the classical geometrical pictures. The scheme of this analysis is as follows: (a) initially, we show that the classical vector analysis gives all the desired results, identical to the tensor analysis based on scalar components of tensors; (b) to list all the geometrical difficulties associated with the classical picture; (c) to examine how general relativity can overcome these difficulties by defining a new version of vectors; (d) to present a table giving concerns, if any, about this new definition based on our analysis.

Note that we are attempting to draw a classical geometrical picture of vectors in the multidimensional curved space. Finally, Sec. 5.2 discusses the significance of the vector method.

2 Geometrical aspects of the space

In this section, we revisit the Cartesian and spherical coordinate systems in flat space. We attempt to relate the

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mathematical conditions obtained from the geometry of flat space to those of curved space.

2.1 Geometrical requirements for the Christoffel symbol symmetry

The Cartesian coordinate system is a "flat-space, linear coordinate system." Flat space allows us to write a position vector connecting any two points in space because the directions of unit vectors remain the same along the linear path connecting the two points. The Cartesian coordinates are orthogonal to each other, and the position vector connecting the origin with a point in space is given by $\vec{s} = x\,\hat{x} + y\,\hat{y} + z\,\hat{z}$. This is a bilocal vector connecting two points in space, in the "point for the head and point for the tail" format. The incremental displacement vector is $\mathrm{d}\vec{s} = \hat{x}\,\mathrm{d}x + \hat{y}\,\mathrm{d}y + \hat{z}\,\mathrm{d}z = \vec{\varepsilon}_i\,\mathrm{d}x^i$, where $\vec{\varepsilon}_i$ denotes the basis vectors. This system is suitable for studying a line segment or a cube. The three-dimensional Cartesian coordinate metric is $\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 = \vec{\varepsilon}_i \cdot \vec{\varepsilon}_j\,\mathrm{d}x^i \mathrm{d}x^j$.

It is often easier to study spherical shapes, such as a sphere, in the spherical coordinate system. This represents a change in the mathematical and geometrical approach, although the physical nature of space itself is not expected to change. We can call this coordinate system a "flat-space, curved coordinate system." We can write a position vector using the radial coordinate, as the direction of the radial unit vector remains unchanged: $\vec{s} = r\,\hat{r}$. We define a covariant basis vector $\vec{\varepsilon}_r = \hat{r}$. It is difficult to specify a precise point in space with this expression, since the direction of the basis vector is a function of position. However, our interest lies in the incremental displacement vector. Because the space is flat in both coordinate systems, we can compare the geometrical pictures of the two to write the spherical unit vectors $\hat{r}$, $\hat{\theta}$, and $\hat{\phi}$ in terms of the unit vectors of the Cartesian coordinates, such as $\hat{r} = \hat{x}\sin\theta\cos\phi + \hat{y}\sin\theta\sin\phi + \hat{z}\cos\theta$. These relations connect the geometries of the two coordinate systems with mathematics, allowing any mathematical expression to be transformed from one coordinate system to another.

In the spherical coordinate system, an infinitesimally small increment ($\mathrm{d}s \to 0$) of this position vector can be written by differentiating the position vector:

$$\begin{aligned}\mathrm{d}\vec{s} &= \vec{\varepsilon}_r\,dr + r \times \left(\frac{\partial \vec{\varepsilon}_r}{\partial r}\,\mathrm{d}r + \frac{\partial \vec{\varepsilon}_r}{\partial \theta}\,\mathrm{d}\theta + \frac{\partial \vec{\varepsilon}_r}{\partial \phi}\,\mathrm{d}\phi\right) \\ &= \vec{\varepsilon}_r\,\mathrm{d}r + \vec{\varepsilon}_\theta\,\mathrm{d}\theta + \vec{\varepsilon}_\phi\,\mathrm{d}\phi. \end{aligned} \tag{1}$$

The geometry of the coordinate system gives the other basis vectors, and we can write:

$$\begin{aligned}&\frac{\partial \vec{\varepsilon}_r}{\partial r} = 0, \quad r\left(\frac{\partial \vec{\varepsilon}_r}{\partial \theta}\right) = \vec{\varepsilon}_\theta = r\hat{\theta}, \\ &r\left(\frac{\partial \vec{\varepsilon}_r}{\partial \phi}\right) = \vec{\varepsilon}_\phi = r\sin\theta\,\hat{\phi}. \end{aligned} \tag{2}$$

Hence, the incremental vector is

$$\mathrm{d}\vec{s} = \hat{r}\,\mathrm{d}r + r\,\mathrm{d}\theta\,\hat{\theta} + r\sin\theta\,\mathrm{d}\phi\,\hat{\phi}. \tag{3}$$

The three-dimensional flat-space metric then follows as $\mathrm{d}s^2 = \mathrm{d}\vec{s}\cdot\mathrm{d}\vec{s} = \mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\phi^2$. Suppose $A(r_1, \theta_1)$ and $B(r_2, \theta_2)$ are two very close positions in space, and the difference between the position vectors at B and A is $\mathrm{d}\vec{s}$. We can travel along a straight line from A to B; alternatively, we may first move along the $\hat{r}$ direction and then along the $\hat{\theta}$ direction, or first along $\hat{\theta}$ and then $\hat{r}$, to reach the same position B. In a flat space, the displacement vector $\mathrm{d}\vec{s}$ should be identical irrespective of the path taken. This condition expresses the *path independence* property of the incremental displacement vector:

$$\frac{\partial^2 \vec{s}}{\partial r\,\partial \theta} = \frac{\partial^2 \vec{s}}{\partial \theta\,\partial r}. \tag{4}$$

We can write an incremental displacement vector in a generalized coordinate system as $\mathrm{d}\vec{s} = \vec{\varepsilon}_i\,\mathrm{d}x^i$, and the corresponding metric as $\mathrm{d}s^2 = \vec{\varepsilon}_i \cdot \vec{\varepsilon}_j\,\mathrm{d}x^i\mathrm{d}x^j = g_{ij}\,\mathrm{d}x^i\mathrm{d}x^j$, where the components of the metric are given by $g_{ij} = \vec{\varepsilon}_i \cdot \vec{\varepsilon}_j$. The path independence of the incremental displacement vector can then be expressed as

$$\frac{\partial^2 \vec{s}}{\partial x^j\,\partial x^i} = \frac{\partial^2 \vec{s}}{\partial x^i\,\partial x^j}. \tag{5}$$

This condition leads directly to the symmetry of the basis vectors [2].

$$\frac{\partial \vec{\varepsilon}_i}{\partial x^j} = \frac{\partial \vec{\varepsilon}_j}{\partial x^i}. \tag{6}$$

In a flat space, the condition (6) is supported by the geometry of the coordinate systems. Eq. (6) expresses the symmetry of the Christoffel symbols ($\Gamma^\beta_{ij} = \Gamma^\beta_{ji}$) with respect to their lower two indices, assuming a torsion-free connection.

$$\frac{\partial \vec{\varepsilon}_i}{\partial x^j} = \Gamma^\beta_{ij}\,\vec{\varepsilon}_i = \Gamma^\beta_{ji}\,\vec{\varepsilon}_i = \frac{\partial \vec{\varepsilon}_j}{\partial x^i}. \tag{7}$$

Based on these geometrical properties, we can derive the general formula for the Christoffel symbols:

$$\Gamma^k_{ij} = \tfrac{1}{2}\, g^{k\alpha}\left(\frac{\partial g_{i\alpha}}{\partial x^j} + \frac{\partial g_{j\alpha}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^\alpha}\right). \tag{8}$$

Since the metric components are zero for $i \neq j$, this reduces to

$$\Gamma^k_{ij} = \frac{g^{kk}}{2}\left(\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k}\right), \quad \text{(no sum on } k\text{)}. \tag{9}$$

The derivation of the Christoffel symbol formula (8) is based on the assumption that the symmetry of the basis vectors, Eq. (6), is satisfied. This can be verified by expressing the metric components as $g_{ij} = \vec{\varepsilon}_i \cdot \vec{\varepsilon}_j$ in Eq. (8). Therefore, whenever we use the Christoffel symbols, it is implicitly presumed that the symmetry of the basis vectors (6) holds true. Now, consider the surface of a sphere. We can draw a curved line segment connecting two points on the sphere surface; however, we cannot write a single mathematical expression for a curved position vector connecting these two points, since the directions of the unit vectors change along the path. If the two points on the sphere surface are very close to each other, we can nevertheless write an incremental displacement vector at that location as $\mathrm{d}\vec{s} = r\,\mathrm{d}\theta\,\hat{\theta} + r\sin\theta\,\mathrm{d}\phi\,\hat{\phi}$. The corresponding metric at that point is $\mathrm{d}s^2 = r^2\,\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\phi^2$. As $\mathrm{d}\vec{s}$ is infinitesimally small, the small surface around it can be treated as locally flat. We obtain the same value of $\mathrm{d}\vec{s}$ whether we first move along the $\hat{\theta}$ direction and then along $\hat{\phi}$, or vice versa, between two nearby points on the sphere surface. Therefore, this surface incremental displacement vector satisfies the path-independence property of Eq. (5). Even though writing a position vector on the sphere surface is difficult, the symmetry conditions (5) and (6) are both satisfied by the incremental displacement vector.

As the distance between the two points is very small, this infinitesimally small incremental vector $\mathrm{d}\vec{s}$ can be mathematically termed a *local vector*. This local vector can be represented by the coordinates of any one of the two points, say (r_1, θ_1, ϕ_1), on the sphere surface. As the small surface around this point can be mathematically treated as flat, the small vector $\mathrm{d}\vec{s}$ on the surface is equivalent to a small tangent vector in the tangent space at that point. The path-independence property (Eq. (5)) of the vector $\mathrm{d}\vec{s}$ on the sphere surface is important for providing geometrical support to the Christoffel-symbol symmetry in Eq. (6). We now proceed to examine the concept and geometry of the curved space.

2.2 Unit vectors and their derivatives in four-dimensional curved space

The classical total energy H for a planet moving around a large mass in a conservative gravitational field, in a three-dimensional flat space, can be written in spherical coordinates as follows. We can treat the mass of the planet as a single unit, since it does not affect the shape of the orbit. The total energy consists of kinetic energy T and potential energy U:

$$\begin{aligned} H &= T + U = K \\ &= \tfrac{1}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2\right) - \frac{a^2 M}{r}. \end{aligned} \tag{10}$$

Here M is the mass of the central body and r is the radial distance. The Lagrangian analysis ($L = T - U$) of this total-energy equation yields the Newtonian planetary orbits [3]. We can also write a corresponding four-dimensional metric that produces the same Newtonian orbits under a geodesic analysis.

$$\mathrm{d}s^2 = \mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\phi^2 + e^{-\mu}\mathrm{d}t^2. \tag{11}$$

Here $e^\mu = 1 - 2M/r$, and we have introduced an additional fourth coordinate such that $\dot{t} = \frac{\mathrm{d}t}{\mathrm{d}s} = a\,e^\mu$. The space remains flat; we have merely extended it by adding a temporal dimension that yields the same expressions for planetary orbits, which are relations between the distance r (or $u = 1/r$) and the azimuthal angle ϕ.

We now study the curved-space, four-dimensional Schwarzschild metric [4]:

$$\mathrm{d}s^2 = -e^\lambda\mathrm{d}r^2 - r^2\mathrm{d}\theta^2 - r^2\sin^2\theta\,\mathrm{d}\phi^2 + e^\mu\mathrm{d}t^2. \tag{12}$$

There is an additional coefficient e^λ that appears in the radial component. It now becomes impossible to write any position vector connecting two points in space, even using the radial coordinate (Sec. 2.3). This coordinate system can therefore be called a *curved-space, curved-coordinate system*. Mathematically, this metric can be treated analogously to the metric for the surface of a sphere in flat space (Sec. 2.1). Similarly, the incremental displacement vector (satisfying $\mathrm{d}s^2 = \mathrm{d}\vec{s}\cdot\mathrm{d}\vec{s}$) corresponding to the Schwarzschild metric can be written as

$$\begin{aligned} \mathrm{d}\vec{s} &= \vec{\varepsilon}_r\,\mathrm{d}r + \vec{\varepsilon}_\theta\,\mathrm{d}\theta + \vec{\varepsilon}_\phi\,\mathrm{d}\phi + \vec{\varepsilon}_t\,\mathrm{d}t \\ &= i\,e^{\lambda/2}\,\hat{r}\mathrm{d}r + ir\,\hat{\theta}\,\mathrm{d}\theta + ir\sin\theta\,\hat{\phi}\,\mathrm{d}\phi + e^{\mu/2}\,\hat{t}\,\mathrm{d}t. \end{aligned} \tag{13}$$

Modifying the flat-space three-dimensional metric g_{jk} to the four-dimensional curved-coordinate metric g^m_{jk} alters both the basis and the unit vectors. For simplicity, we denote this modified metric g^m_{jk} as g_{jk}. It is assumed that μ and λ are functions of r. Imaginary factors have been introduced only so that $i^2 = -1$, with the understanding that all unit vectors are real. These imaginary terms vanish when evaluating the curvature-tensor components. The modified metric coefficients are $g_{11} = -e^{\lambda}$, $g_{22} = -r^2$, $g_{33} = -r^2 \sin^2\theta$, and $g_{44} = e^{\mu}$. The Christoffel symbols can then be calculated using Eq. (9). The modified covariant and contravariant basis vectors are

$$\vec{\varepsilon}_r = i\, e^{\lambda/2}\,\hat{r}, \qquad \vec{\varepsilon}_\theta = ir\,\hat{\theta},$$
$$\vec{\varepsilon}_\phi = ir\sin\theta\,\hat{\phi}, \qquad \vec{\varepsilon}_t = e^{\mu/2}\,\hat{t}, \tag{14}$$

and

$$\vec{\varepsilon}^{\,r} = -e^{-\lambda/2} i\,\hat{r}, \qquad \vec{\varepsilon}^{\,\theta} = -\frac{i}{r}\,\hat{\theta},$$
$$\vec{\varepsilon}^{\,\phi} = -\frac{i}{r\sin\theta}\,\hat{\phi}, \qquad \vec{\varepsilon}^{\,t} = e^{-\mu/2}\,\hat{t}. \tag{15}$$

The unit vectors are assumed orthogonal, such that $\hat{r}\cdot\hat{r} = \hat{\theta}\cdot\hat{\theta} = \hat{\phi}\cdot\hat{\phi} = \hat{t}\cdot\hat{t} = 1$, and $\hat{r}\cdot\hat{\theta} = \hat{\theta}\cdot\hat{\phi} = \hat{r}\cdot\hat{t} = 0$. We can now obtain the derivatives of the curved-space unit vectors using the Christoffel symbols :

$$\frac{\partial\vec{\varepsilon}_t}{\partial t} = \frac{\partial}{\partial t}\left(e^{\mu/2}\hat{t}\right) = \Gamma^1_{44}\,\vec{\varepsilon}_t = \frac{1}{2}e^{\mu-\lambda}\mu'\,\vec{\varepsilon}_r$$
$$= \frac{1}{2}e^{\mu-\lambda}\mu'\left(ie^{\lambda/2}\hat{r}\right), \tag{16a}$$

$$\frac{\partial\hat{t}}{\partial t} = \frac{i}{2}\,\mu'\,e^{(\mu-\lambda)/2}\,\hat{r}. \tag{16b}$$

Table 1 compares the derivatives of the unit vectors for the three-dimensional flat space and the four-dimensional curved space. The derivation of the Christoffel-symbol formula (Eq. (8)) is based on the symmetry of basis vectors given by Eq. (6). This presumption continues automatically in curved space since we use the Christoffel symbols. We verify this symmetry using values from Table 1.

$$\frac{\partial\vec{\varepsilon}_r}{\partial\theta} = \frac{\partial}{\partial\theta}\left(ie^{\lambda/2}\hat{r}\right) = ie^{\lambda/2}\frac{\partial\hat{r}}{\partial\theta} = \frac{\vec{\varepsilon}_\theta}{r} = \frac{ir\,\hat{\theta}}{r}$$
$$= \Gamma^\theta_{r\theta}\,\vec{\varepsilon}_\theta = \frac{\partial\,(ir\hat{\theta})}{\partial r} = \frac{\partial\vec{\varepsilon}_\theta}{\partial r}. \tag{17}$$

However, it is difficult to propose a clear geometrical picture that supports this symmetry in a four-dimensional curved space (Sec. 2.3). This difficulty leads to contradictory results, as noted in the comments column of Table 1:

1. Row 5: $\hat{\theta}$ is not a function of r.
2. Row 6: $\hat{\theta}$ is a function of r since $\lambda = \lambda(r)$.

We can now identify the first difficulty associated with the geometrical framework of the curved space.

Difficulty 1. It is inconsistent to obtain contradictory derivatives for the same unit vectors (Table 1) associated with the curved-space metric. Consequently, there is no purely geometrical justification for assuming the path independence (Eqs. (5) and (6)) of the incremental displacement vector $d\vec{s}$ defined by Eq. (13) in the four-dimensional curved space (see also Difficulty 3). This marks the key distinction between the geometry of the curved space defined by the Schwarzschild metric and that of a spherical surface embedded in a flat space (Sec. 2.1).

2.3 Geometry of the four-dimensional curved space

We now study the space defined by the Schwarzschild metric to see whether a position vector can be written in the curved space. In analogy with Sec. 2.1, we suggest the spatial position vector $\vec{s} = r\,\vec{\varepsilon}_r = r\,i\,e^{\lambda/2}\,\hat{r}$. Restricting to the three spatial coordinates and differentiating $\vec{s}$, using the Schwarzschild Christoffel symbols or the unit–vector derivatives from Table 1:

$$d\vec{s} = \vec{\varepsilon}_r dr + r\left(\frac{\partial\vec{\varepsilon}_r}{\partial r}\,dr + \frac{\partial\vec{\varepsilon}_r}{\partial\theta}\,d\theta + \frac{\partial\vec{\varepsilon}_r}{\partial\phi}\,d\phi\right)$$
$$= \left(1 + \frac{r\lambda'}{2}\right)\vec{\varepsilon}_r dr + \vec{\varepsilon}_\theta\,d\theta + \vec{\varepsilon}_\phi\,d\phi. \tag{18}$$

Then, using $r(\partial\vec{\varepsilon}_r/\partial\theta) = \vec{\varepsilon}_\theta = i\,r\,\hat{\theta}$ and $r(\partial\vec{\varepsilon}_r/\partial\phi) = \vec{\varepsilon}_\phi = i\,r\sin\theta\,\hat{\phi}$ give the same angular basis vectors as required in Sec. 2.2. However, by inserting the entries from Table 1, one finds that this incremental displacement vector is not always path independent:

$$\frac{\partial^2\vec{s}}{\partial r\,\partial\theta} \neq \frac{\partial^2\vec{s}}{\partial\theta\,\partial r}, \tag{19}$$

Also, the incremental vector (18) obtained by differentiating the position vector is not the desired incremental vector (13) corresponding to the Schwarzschild metric. The unit vectors are assumed to be orthogonal (Sec. 2.2) and therefore, the coordinates should be independent. But, a direct verification of the dot and cross product rules is not possible. We can verify the dot and cross product rules through differentiation using the derivatives in Table 1:

$$(a)\quad \frac{\partial(\hat{r}\cdot\hat{r})}{\partial\theta} = 2\,\hat{r}\cdot\frac{\partial\hat{r}}{\partial\theta} = 2\,\hat{r}\cdot e^{-\lambda/2}\hat{\theta} = 0. \tag{20}$$

$$(b)\quad \frac{\partial(\hat{\theta}\times\hat{\phi})}{\partial\phi} = \frac{\partial\hat{\theta}}{\partial\phi}\times\hat{\phi} + \hat{\theta}\times\frac{\partial\hat{\phi}}{\partial\phi} = \sin\theta\, e^{-\lambda/2}\,\hat{\phi} = \frac{\partial\hat{r}}{\partial\phi}. \tag{21}$$

The dot and cross product rules are satisfied by the curved-space unit vectors $\hat{r}$, $\hat{\theta}$, and $\hat{\phi}$, similar to the unit vectors in a three-dimensional flat-space analysis. We now introduce the fourth coordinate t and suggest a position vector in the curved-space coordinate system as $\vec{s} = r\,\vec{\varepsilon}_r + t\,\vec{\varepsilon}_t = i\,r\,e^{\lambda/2}\,\hat{r} + t\,e^{\mu/2}\,\hat{t}$. Differentiating $\vec{s}$, we obtain the incremental vector

$$\mathrm{d}\vec{s} = \left(\vec{\varepsilon}_r + r\,\frac{\partial\vec{\varepsilon}_r}{\partial r} + t\,\frac{\partial\vec{\varepsilon}_t}{\partial r}\right)\mathrm{d}r + \vec{\varepsilon}_\theta\,\mathrm{d}\theta + \vec{\varepsilon}_\phi\,\mathrm{d}\phi + \left(\vec{\varepsilon}_t + r\,\frac{\partial\vec{\varepsilon}_r}{\partial t} + t\,\frac{\partial\vec{\varepsilon}_t}{\partial t}\right)\mathrm{d}t. \tag{22}$$

Again, this vector $\mathrm{d}\vec{s}$ does not satisfy the path-independence rule for all possible combinations of j and k:

$$\frac{\partial^2\vec{s}}{\partial x^j\,\partial x^k} \neq \frac{\partial^2\vec{s}}{\partial x^k\,\partial x^j}. \tag{23}$$

The incremental vector in Eq. (22) is not the desired incremental vector corresponding to the Schwarzschild metric given in Eq. (13). It is difficult to write any position vector $\vec{s}$ that, upon differentiation, yields the required incremental displacement vector of Eq. (13). As discussed in Sec. 2.2, the path independence of the incremental displacement vector of Eq. (13) is only a mathematical assumption. The analysis of the curved-space metric is mathematically similar to that of a spherical surface in the flat-space spherical coordinate system (Sec. 2.1). The vector $\vec{s}$ cannot be defined, but we can still write the vector $\mathrm{d}\vec{s}$. The differences then arise in the geometrical interpretation. We now examine the cross and dot-product rules.

$$(c)\quad \frac{\partial(\hat{r}\cdot\hat{t})}{\partial t} = \frac{\partial\hat{r}}{\partial t}\cdot\hat{t} + \hat{r}\cdot\frac{\partial\hat{t}}{\partial t} = -\frac{i\mu'}{2}\,e^{(\mu-\lambda)/2}\hat{t}\cdot\hat{t} + \frac{i\mu'}{2}\,e^{(\mu-\lambda)/2}\hat{r}\cdot\hat{r} = 0. \tag{24}$$

$$(d)\quad \frac{\partial(\hat{t}\times\hat{\theta})}{\partial t} = \frac{\partial\hat{t}}{\partial t}\times\hat{\theta} + \hat{t}\times\frac{\partial\hat{\theta}}{\partial t} = \tfrac{1}{2}\mu'\,e^{(\mu-\lambda)/2}\,i\big(\hat{r}\times\hat{\theta}\big) = \tfrac{i}{2}\mu'\,e^{(\mu-\lambda)/2}\,\hat{\phi}. \tag{25}$$

$$(e)\quad \frac{\partial(\hat{\phi}\times\hat{r})}{\partial t} = \frac{\partial\hat{\phi}}{\partial t}\times\hat{r} + \hat{\phi}\times\frac{\partial\hat{r}}{\partial t} = -\tfrac{1}{2}\mu'\,e^{(\mu-\lambda)/2}\,i\big(\hat{\phi}\times\hat{t}\big) = 0. \tag{26}$$

We can now write four additional difficulties with the geometrical framework of a curved space:

Difficulty 2. The derivatives of the unit vectors $\hat{r}$, $\hat{\theta}$, and $\hat{\phi}$ satisfy the cross-product rules, but these derivatives should be taken only with respect to r, θ, or ϕ. When the time-like unit vector $\hat{t}$ or the coordinate t is introduced, the cross-product rules fail. From Eq. (e) it is evident that the differentiation of a cross product of two spatial unit vectors also fails when differentiating with respect to the fourth coordinate t.

Difficulty 3. Table 1 was obtained using the Christoffel symbols in Eq. (9), not by differentiating the unit vectors. Because the flat-space unit vectors are modified in the curved space, it is not possible to write explicit expressions for $\hat{r}$, $\hat{\theta}$, $\hat{\phi}$, and $\hat{t}$ that reproduce the derivatives in Table 1 directly by differentiation, owing to the contradictions noted in the comments column of that table. These coordinates can still be treated as orthogonal (Sec. 2.2), since dot products vanish [Eqs. (a), (c)], but explicit unit-vector forms corresponding to them cannot be written even locally. Hence, a clear geometrical picture of the infinitesimal local vector $\mathrm{d}\vec{s}$ of Eq. (13) cannot be drawn (see Difficulty 1).

Difficulty 4. Explicit expressions for the unit vectors are required to write any physical vector such as the force vector. Since these expressions cannot be defined consistently, writing any bilocal ("point-for-head and point-for-tail") vector is not possible in the curved-space geometry.

Difficulty 5. Due to Difficulty 3, it is not possible to write relationships between the unit vectors of the curved space curved coordinate system and the unit vectors of flat space, spherical or Cartesian coordinate systems (see Sec. 2.1) by comparing their geometrical pictures. Hence, it is not possible to transform either Eq. (13) or the Schwarzschild metric Eq. (12) from curved space, curved coordinate system to flat space spherical or Cartesian coordinate systems. It is not possible to suggest a correspondence between flat space spherical coordinate system and curved space curved coordinate system.

Table 1 Partial derivatives of unit vectors.

Three-dimensional flat space	Four-dimensional curved space	Comments
$\frac{\partial \hat{r}}{\partial r} = 0$	$\frac{\partial \hat{r}}{\partial r} = 0$	If zero in flat space, it remains zero in curved space.
$\frac{\partial \hat{r}}{\partial \theta} = \hat{\theta}$	$\frac{\partial \hat{r}}{\partial \theta} = e^{-\lambda/2}\hat{\theta}$	
$\frac{\partial \hat{r}}{\partial \phi} = \sin\theta\,\hat{\phi}$	$\frac{\partial \hat{r}}{\partial \phi} = \sin\theta\, e^{-\lambda/2}\hat{\phi}$	
$\frac{\partial \hat{r}}{\partial t} = 0^{a}$	$\frac{\partial \hat{r}}{\partial t} = -\,i\,\frac{\mu'}{2}e^{(\mu-\lambda)/2}\hat{t}$	
$\frac{\partial \hat{\theta}}{\partial r} = 0$	$\frac{\partial \hat{\theta}}{\partial r} = 0$	$\hat{\theta}$ is not a function of r.
$\frac{\partial \hat{\theta}}{\partial \theta} = -\hat{r}$	$\frac{\partial \hat{\theta}}{\partial \theta} = -e^{-\lambda/2}\hat{r}$	$\hat{\theta}$ is a function of r. as λ is a function of r.
$\frac{\partial \hat{\theta}}{\partial \phi} = \cos\theta\,\hat{\phi}$	$\frac{\partial \hat{\theta}}{\partial \phi} = \cos\theta\,\hat{\phi}$	
$\frac{\partial \hat{\theta}}{\partial t} = 0^{a}$	$\frac{\partial \hat{\theta}}{\partial t} = 0$	
$\frac{\partial \hat{\phi}}{\partial r} = 0$	$\frac{\partial \hat{\phi}}{\partial r} = 0$	$\hat{\phi}$ independent of r.
$\frac{\partial \hat{\phi}}{\partial \theta} = 0$	$\frac{\partial \hat{\phi}}{\partial \theta} = 0$	
$\frac{\partial \hat{\phi}}{\partial \phi} = -\sin\theta\,\hat{r} - \cos\theta\,\hat{\theta}$	$\frac{\partial \hat{\phi}}{\partial \phi} = -\sin\theta\, e^{-\lambda/2}\hat{r} - \cos\theta\,\hat{\theta}$	$\hat{\phi}$ is a function of r. as λ is a function of r.
$\frac{\partial \hat{\phi}}{\partial t} = 0^{a}$	$\frac{\partial \hat{\phi}}{\partial t} = 0$	
	$\frac{\partial \hat{t}}{\partial r} = 0$	$\hat{t}$ independent of r.
	$\frac{\partial \hat{t}}{\partial \theta} = 0$	$\hat{t}$ independent of θ.
	$\frac{\partial \hat{t}}{\partial \phi} = 0$	$\hat{t}$ independent of ϕ.
	$\frac{\partial \hat{t}}{\partial r} = i\,\frac{\mu'}{2}\,e^{(\mu-\lambda)/2}\,\hat{r}$	$\hat{t}$ is a function of r due to μ and λ.

[a]The coordinate t is not used in a three-dimensional analysis.

3 A comparison between the tensor and the vector methods

The Schwarzschild metric is given in Eq. (12). In the standard tensor formulation, one seeks to determine the functions λ and μ and the relation between them. This analysis proceeds through the evaluation of the Ricci tensor, the Einstein tensor, and the use of the Bianchi identities. Equivalently, one can obtain the required tensor equations by taking gradients of the metric or of the basis vectors. In the following, we briefly outline the standard tensor analysis in Sec. 3.1, and then present a corresponding vector formulation in Secs. 3.2 and 3.3, which provides an alternative and more geometrically intuitive interpretation of the same results.

3.1 Tensor analysis: Riemann–Christoffel tensor

In a classical analysis, the force vector can be written as $\vec{F} = F_i\,\vec{\varepsilon}^i$. Let us now consider an arbitrary vector $\vec{A} = A_\alpha\,\vec{\varepsilon}^\alpha$ in a four-dimensional space. The gradient of a vector, using the summation convention, can be expressed as

$$\nabla\vec{A} = \frac{\partial\left(A_\alpha\vec{\varepsilon}^\alpha\right)}{\partial x^j}\,\vec{\varepsilon}^j = \sum_{\alpha=1}^{4}\sum_{j=1}^{4}\frac{\partial\left(A_\alpha\vec{\varepsilon}^\alpha\right)}{\partial x^j}\,\vec{\varepsilon}^j$$
$$= \sum_{\alpha=1}^{4}\sum_{j=1}^{4}\frac{\partial A_\alpha}{\partial x^j}\,\vec{\varepsilon}^\alpha\vec{\varepsilon}^j + A_\alpha\,\frac{\partial\vec{\varepsilon}^\alpha}{\partial x^j}\vec{\varepsilon}^j. \tag{27}$$

Defining the gradient with respect to a specific index j,

$$\nabla_j\vec{A} = \sum_{\alpha=1}^{4}\frac{\partial\left(A_\alpha\vec{\varepsilon}^\alpha\right)}{\partial x^j}\,\vec{\varepsilon}^j$$
$$= \sum_{\alpha=1}^{4}\frac{\partial A_\alpha}{\partial x^j}\,\vec{\varepsilon}^\alpha\vec{\varepsilon}^j + A_\alpha\frac{\partial\vec{\varepsilon}_\alpha}{\partial x^j}\vec{\varepsilon}^j, \tag{28}$$

we obtain (no sum on j)

$$\nabla_j\vec{A} = \frac{\partial(A_\alpha\vec{\varepsilon}^\alpha)}{\partial x^j}\vec{\varepsilon}^j = \frac{\partial A_\alpha}{\partial x^j}\,\vec{\varepsilon}^\alpha\vec{\varepsilon}^j + A_\alpha\,\frac{\partial\vec{\varepsilon}^\alpha}{\partial x^j}\vec{\varepsilon}^j,$$
$$= \left(\frac{\partial A_\alpha}{\partial x^j} - A_\beta\,\Gamma^\beta_{j\alpha}\right)\vec{\varepsilon}^\alpha\vec{\varepsilon}^j. \tag{29}$$

Taking a further gradient with respect to a specific index k, we define the *Riemann curvature tensor* as the measure of non-commutativity of successive covariant derivatives:

$$(\nabla_k\nabla_j - \nabla_j\nabla_k)\vec{A}$$
$$= A_\beta\left(\frac{\partial\Gamma^\beta_{\alpha k}}{\partial x^j} - \frac{\partial\Gamma^\beta_{\alpha j}}{\partial x^k} + \Gamma^\beta_{\gamma j}\Gamma^\gamma_{\alpha k} - \Gamma^\beta_{\gamma k}\Gamma^\gamma_{\alpha j}\right)\vec{\varepsilon}^\alpha\vec{\varepsilon}^j\vec{\varepsilon}^k$$
$$= A_\beta\;R^\beta{}_{\alpha jk}\,\vec{\varepsilon}^\alpha\vec{\varepsilon}^j\vec{\varepsilon}^k. \tag{30}$$

$$(\nabla_k\nabla_j - \nabla_j\nabla_k)\vec{A} = R^\beta{}_{\alpha jk}\,A_\beta\,\varepsilon^\alpha\,\varepsilon^j\,\varepsilon^k, \tag{31}$$

where

$$R^\beta{}_{\alpha jk} = \frac{\partial\Gamma^\beta_{\alpha k}}{\partial x^j} - \frac{\partial\Gamma^\beta_{\alpha j}}{\partial x^k} + \Gamma^\beta_{\gamma j}\Gamma^\gamma_{\alpha k} - \Gamma^\beta_{\gamma k}\Gamma^\gamma_{\alpha j}. \tag{32}$$

Thus, the curvature tensor can also be written as the difference between the covariant double derivatives of a vector:

$$A_{\alpha,jk} - A_{\alpha,kj} = A_\beta\,R^\beta_{\alpha jk}$$
$$= A_\beta\left(\frac{\partial\Gamma^\beta_{k\alpha}}{\partial x^j} - \frac{\partial\Gamma^\beta_{j\alpha}}{\partial x^k} + \Gamma^\beta_{j\gamma}\Gamma^\gamma_{k\alpha} - \Gamma^\beta_{k\gamma}\Gamma^\gamma_{j\alpha}\right) \tag{33}$$

The quantity $R^\beta{}_{\alpha jk}$ is the curvature (Riemann) tensor. We can also express the difference of the double covariant derivatives of a basis vector as

$$(\nabla_k\nabla_j - \nabla_j\nabla_k)\vec{\varepsilon}^\alpha = R^\alpha_{ijk}\vec{\varepsilon}^i\vec{\varepsilon}^j\vec{\varepsilon}^k. \tag{34}$$

Any physical quantity can be expressed either as a scalar or a vector. We will give the vector analysis corresponding to this work.

3.2 Vector analysis: Gradient and partial derivatives of a vector

Consider a vector $\vec{A}(x^j, x^k) = A_\alpha\,\vec{\varepsilon}^\alpha$ defined in space. We first move by a small distance Δx^j from the origin O to a nearby point A. Taking the gradient of this vector with respect to the specific index j, we obtain the corresponding expression for the vector at the new position:

$$\vec{A}_{0\to j}(x^j + \Delta x^j, x^k)$$
$$= A_\alpha\vec{\varepsilon}^\alpha + (\nabla_j\vec{A})\cdot\vec{\varepsilon}_j\,\Delta x^j$$
$$= A_\alpha\,\vec{\varepsilon}^\alpha + \frac{\partial(A_\alpha\,\vec{\varepsilon}^\alpha)}{\partial x^j}\vec{\varepsilon}^j\cdot\vec{\varepsilon}_j\,\Delta x^j, \tag{35}$$

We then travel from point A a further distance Δx^k to reach the point B.

$$\vec{A}(x^j + \Delta x^j,\, x^k + \Delta x^k) = \Big[A_\alpha\,\vec{\varepsilon}^\alpha + (\nabla_j\vec{A})\cdot\vec{\varepsilon}_j\,\Delta x^j\Big]$$
$$+\;\nabla_k\Big[A_\alpha\,\vec{\varepsilon}^\alpha + (\nabla_j\vec{A})\cdot\vec{\varepsilon}_j\,\Delta x^j\Big]\cdot\vec{\varepsilon}_k\,\Delta x^k, \tag{36}$$

Expanding the derivatives explicitly,

$$\vec{A}(x^j + \Delta x^j,\, x^k + \Delta x^k) = A_\alpha\,\vec{\varepsilon}^\alpha + \frac{\partial(A_\alpha\,\vec{\varepsilon}^\alpha)}{\partial x^j}\cdot\Delta x^j$$
$$+\;\frac{\partial}{\partial x^k}\left(A_\alpha\,\vec{\varepsilon}^\alpha + \frac{\partial(A_\alpha\,\vec{\varepsilon}^\alpha)}{\partial x^j}\cdot\Delta x^j\right)\cdot\Delta x^k. \tag{37}$$

The incremental vector between the two points B and O is $\Delta\vec{A} = \Delta\vec{A}_{O\to j\to k} = \vec{A}(x^j+\Delta x^j,\, x^k+\Delta x^k) - \vec{A}(x^j,\, x^k)$. We could also reach the same final coordinates by travelling along another route: first moving by Δx^k from O to P, and then by Δx^j from P to Q. The end point B of the first route and end point Q of the second route should be same in a flat space. However, the two routes shall not reach the same point in space if the space is curved. Then, the difference in the incremental vectors will be non-zero and can be written in terms of the difference of the two partial double derivatives of the vector.

$$\Delta(\Delta\vec{A}) = (\vec{A}_{O\to j\to k} - \vec{A}_O) - (\vec{A}_{O\to k\to j} - \vec{A}_O), \tag{38}$$

$$\begin{aligned}\Delta(\Delta\vec{A}) &= \left(\frac{\partial^2\vec{A}}{\partial x^k\partial x^j} - \frac{\partial^2\vec{A}}{\partial x^j\partial x^k}\right)\Delta x^j\Delta x^k \\ &= A_\alpha\left(\frac{\partial^2\vec{\varepsilon}^\alpha}{\partial x^k\partial x^j} - \frac{\partial^2\vec{\varepsilon}^\alpha}{\partial x^j\partial x^k}\right)\Delta x^j\Delta x^k.\end{aligned} \tag{39}$$

The term in the parentheses can be written as the difference between the partial double derivatives of $\vec{\varepsilon}^\alpha$:

$$\frac{\partial\vec{\varepsilon}^\alpha}{\partial x^j} = -\Gamma^\alpha_{j\beta}\,\vec{\varepsilon}^\beta, \tag{40}$$

$$\vec{\varepsilon}^\alpha_{/jk} = \frac{\partial^2\vec{\varepsilon}^\alpha}{\partial x^k\partial x^j} = -\frac{\partial\Gamma^\alpha_{j\beta}}{\partial x^k}\,\vec{\varepsilon}^\beta + \Gamma^\alpha_{j\gamma}\,\Gamma^\gamma_{k\beta}\,\vec{\varepsilon}^\beta, \tag{41}$$

$$\begin{aligned}\vec{\varepsilon}^\alpha_{/jk} - \vec{\varepsilon}^\alpha_{/kj} &= \frac{\partial^2\vec{\varepsilon}^\alpha}{\partial x^k\partial x^j} - \frac{\partial^2\vec{\varepsilon}^\alpha}{\partial x^j\partial x^k} \\ &= \left(\frac{\partial\Gamma^\alpha_{k\beta}}{\partial x^j} - \frac{\partial\Gamma^\alpha_{j\beta}}{\partial x^k} + \Gamma^\alpha_{j\gamma}\Gamma^\gamma_{k\beta} - \Gamma^\alpha_{k\gamma}\Gamma^\gamma_{j\beta}\right)\vec{\varepsilon}^\beta \\ &= R^\alpha_{\beta jk}\,\vec{\varepsilon}^\beta.\end{aligned} \tag{42}$$

The scalar coefficient $R^\alpha_{\beta jk}$ of the difference between the two partial double derivatives of $\vec{\varepsilon}^\alpha$, as given by Eq. (42), is the same as the coefficient that appears in the difference of the covariant double gradients in Eq. (34). Even though the partial double derivatives $\partial^2\vec{A}/\partial x^k\partial x^j$ and the covariant double derivatives $A_{\alpha,jk}$ of a vector $\vec{A}$ are not the same, the coefficients of their differences are identical (Eqs. (33) and (42)). The space is curved whenever $R^\beta{}_{\alpha jk} \neq 0$. From Eq. (39), this implies that in a curved space the incremental vector is not path independent, and therefore $\Delta(\Delta\vec{A}) \neq 0$.

As this incremental vector $\Delta\vec{A}$ is infinitesimally small, it can be expressed using only the coordinates of the first point and may therefore be regarded as a local vector at that point.

3.3 Vector analysis: Bianchi identity

We can calculate the difference between two partial triple derivative terms, $A_{/jkl} - A_{/jlk}$, as follows:

$$\begin{aligned}&\frac{\partial^3\vec{A}}{\partial x^l\partial x^k\partial x^j} - \frac{\partial^3\vec{A}}{\partial x^k\partial x^l\partial x^j} \\ &\quad= \frac{\partial A_\alpha}{\partial x^j}\left(\frac{\partial^2\vec{\varepsilon}^\alpha}{\partial x^l\partial x^k} - \frac{\partial^2\vec{\varepsilon}^\alpha}{\partial x^k\partial x^l}\right) \\ &\quad+ A_\alpha\left(\frac{\partial^2}{\partial x^l\partial x^k} - \frac{\partial^2}{\partial x^k\partial x^l}\right)\frac{\partial\vec{\varepsilon}^\alpha}{\partial x^j},\end{aligned} \tag{43}$$

$$\begin{aligned}&A_{/jkl} - A_{/jlk} \\ &\quad= \frac{\partial A_\alpha}{\partial x^j}\,R^\alpha_{ikl}\,\vec{\varepsilon}^i - A_\alpha\,\Gamma^\alpha_{j\beta}\,R^\beta_{ikl}\,\vec{\varepsilon}^i.\end{aligned} \tag{44}$$

We can write two additional equations by performing cyclic permutations of the indices j, k, and l, and then adding the three resulting expressions.

$$\begin{aligned}&(A_{/jkl} - A_{/jlk}) + (A_{/klj} - A_{/kjl}) + (A_{/ljk} - A_{/lkj}) \\ &= \left(\frac{\partial A_\alpha}{\partial x^j}\,R^\alpha_{ikl} + \frac{\partial A_\alpha}{\partial x^k}\,R^\alpha_{ilj} + \frac{\partial A_\alpha}{\partial x^l}\,R^\alpha_{ijk}\right)\vec{\varepsilon}^i \\ &\quad - A_\alpha\left(\Gamma^\alpha_{j\beta}\,R^\beta{}_{ikl} + \Gamma^\alpha_{k\beta}\,R^\beta{}_{ilj} + \Gamma^\alpha_{l\beta}\,R^\beta{}_{ijk}\right)\vec{\varepsilon}^i.\end{aligned} \tag{45}$$

Similarly, we can solve the following combination:

$$\begin{aligned}&\vec{A}_{/jkl} - \vec{A}_{/kjl} = \\ &\qquad \frac{\partial^3\vec{A}}{\partial x^l\partial x^k\partial x^j} - \frac{\partial^3\vec{A}}{\partial x^l\partial x^j\partial x^k} \\ &\qquad = \left(\frac{\partial A_\alpha}{\partial x^l}\right)R^\alpha_{ijk}\,\vec{\varepsilon}^i + A_\alpha\,\frac{\partial}{\partial x^l}\left(R^\alpha_{ijk}\vec{\varepsilon}^i\right).\end{aligned} \tag{46}$$

We write two more terms by changing the indices in a cyclic manner and adding them:

$$\begin{aligned}&(A_{/jkl} - A_{/kjl}) + (A_{/klj} - A_{/lkj}) + (A_{/ljk} - A_{/jlk}) \\ &\quad= \left(\frac{\partial A_\alpha}{\partial x^l}R^\alpha_{ijk} + \frac{\partial A_\alpha}{\partial x^j}R^\alpha_{ikl} + \frac{\partial A_\alpha}{\partial x^k}R^\alpha_{ilj}\right)\vec{\varepsilon}^l \\ &\quad+ A_\alpha\left(\frac{\partial\left(\vec{\varepsilon}^i R^\alpha_{ijk}\right)}{\partial x^l} + \frac{\partial\left(\vec{\varepsilon}^i R^\alpha_{ikl}\right)}{\partial x^j} + \frac{\partial\left(\vec{\varepsilon}^i R^\alpha_{ilj}\right)}{\partial x^k}\right)\end{aligned} \tag{47}$$

We subtract Eq. (45) from (47). The left hand side terms of both the equations are same.

$$A_\alpha \Bigg[\frac{\partial R^\alpha_{ijk}}{\partial x^l} - R^\alpha_{\beta jk}\,\Gamma^\beta_{li} + R^\beta_{ijk}\,\Gamma^\alpha_{l\beta} + \frac{\partial R^\alpha_{ikl}}{\partial x^j} - R^\alpha_{\beta kl}\,\Gamma^\beta_{ji} + R^\beta_{ikl}\,\Gamma^\alpha_{j\beta} + \frac{\partial R^\alpha_{ilj}}{\partial x^k} - R^\alpha_{\beta lj}\,\Gamma^\beta_{ki} + R^\beta_{ilj}\,\Gamma^\alpha_{k\beta} \Bigg] \vec{\varepsilon}^{\,i} = 0. \tag{48}$$

The Eq. (48) must hold for any arbitrary vector A_α, and therefore the expression inside the square brackets must vanish for every specific choice of α and i. The coefficient of the basis vector inside the square brackets in Eq. (48) is precisely the Bianchi identity. This identity was obtained by subtracting the same triple–derivative terms in Eq. (45) from those in Eq. (47); hence, the result must always be zero. The derivation does not rely on any additional geometrical or physical assumptions and therefore remains valid for any metric in any number of dimensions. A further contraction of the Bianchi identity yields the Einstein tensor equation.

$$\left(R^j_l - \frac{1}{2}\,\delta^j_l\, R \right)_{,j} = G^j_{l\,,j} = 0. \tag{49}$$

As the Bianchi identity is satisfied by any metric, the covariant derivative of the Einstein tensor G^j_l is always zero, irrespective of the form of the metric or the number of dimensions. We then assume that $G^j_l = 0$ outside the mass. According to general relativity, G^j_l is proportional to the stress–energy tensor, and therefore it vanishes in vacuum. A further contraction yields the Ricci tensor condition in vacuum, $R_{jl} = 0$.

3.4 Space suggested by Ricci tensor condition

We now examine the Schwarzschild metric given in Eq. (12). Using the basis vectors introduced in Sec. 2.2, we compute the Ricci tensor components by applying Eq. (42) together with the curved–space partial derivatives of the unit vectors listed in Table 1.

$$R_{kj}\,\vec{\varepsilon}^{\,k} = R^\alpha_{kj\alpha}\,\vec{\varepsilon}^{\,k} = \left(\frac{\partial^2 \vec{\varepsilon}^{\,\alpha}}{\partial x^\alpha\,\partial x^j} - \frac{\partial^2 \vec{\varepsilon}^{\,\alpha}}{\partial x^j\,\partial x^\alpha} \right). \tag{50}$$

Thus,

$$R^2_{112}\,\vec{\varepsilon}^{\,1} = R^\theta_{r r\theta}\,\vec{\varepsilon}^{\,r} = \left(\frac{\partial^2 \vec{\varepsilon}^{\,\theta}}{\partial\theta\,\partial r} - \frac{\partial^2 \vec{\varepsilon}^{\,\theta}}{\partial r\,\partial\theta} \right). \tag{51}$$

$$\frac{\partial \vec{\varepsilon}^{\,\theta}}{\partial\theta} = \frac{\partial}{\partial\theta}\left(\frac{-i\,\hat{\theta}}{r} \right) = \frac{-i}{r}\left(\frac{\partial\hat{\theta}}{\partial\theta} \right) = \left(\frac{i\,e^{-\lambda/2}}{r} \right)\hat{r}, \tag{52}$$

$$\frac{\partial^2 \vec{\varepsilon}^{\,\theta}}{\partial r\partial\theta} = \frac{\partial}{\partial r}\left(\frac{i\,e^{-\lambda/2}}{r}\,\hat{r} \right) = -ie^{\lambda/2}\hat{r}\left(\frac{1}{r^2} + \frac{\lambda'}{2r} \right) = \vec{\varepsilon}^{\,r}\left(\frac{1}{r^2} + \frac{\lambda'}{2r} \right). \tag{53}$$

Then, calculate $R^\theta_{rr\theta}\,\vec{\varepsilon}^{\,r}$. Similarly, calculate $R^\phi_{rr\phi}\,\vec{\varepsilon}^{\,r}$, $R^t_{rrt}\,\vec{\varepsilon}^{\,r}$, and $R^r_{rrr}\,\vec{\varepsilon}^{\,r} = 0$. Then, from Eq. (51):

$$R_{rr}\,\vec{\varepsilon}^{\,r} = \left(R^r_{rrr} + R^\theta_{rr\theta} + R^\phi_{rr\phi} + R^t_{rrt} \right)\vec{\varepsilon}^{\,r} = \left(\frac{\mu''}{2} - \frac{\mu'\lambda'}{4} + \frac{\mu'^2}{4} - \frac{\lambda'}{r} \right)\vec{\varepsilon}^{\,r}. \tag{54}$$

The vector analysis (Eqs. (42) and (50)) yields the same curvature tensor and Ricci tensor components as the standard tensor analysis based on the scalar components of the tensor [4]. The Ricci tensor condition $R_{rr} = 0$ gives the coefficients of the metric as $\mu = -\lambda$ and $e^\mu = 1 - \frac{2M}{r}$.

In a flat space, every curvature tensor component is zero. The Ricci tensor condition $\sum_\alpha R^\alpha{}_{kj\alpha} = 0$ suggests that the sum of the curvature tensor components, under this contraction, must vanish. Such a space may be described as *approximately flat*. We may write the unit–vector derivatives in this approximately flat space by setting $\lambda = -\mu$ in Table 1. However, even in such a local coordinate system, it remains difficult to write explicit unit–vector expressions (*Difficulty 3*, Sec. 2.3); consequently, constructing a geometrical picture of any classical vector (including the infinitesimal displacement vector $d\vec{s}$) is not possible. An approximately flat space is not equivalent to a flat space in the immediate neighbourhood of $d\vec{s}$ unless $\lambda = -\mu = 0$.

We now examine the behaviour of incremental vectors in the curved space.

4 Path independence property of the incremental vectors

In this section, we compare the path–independence properties of the incremental displacement vector and the incremental vector of any general vector, both in flat space and in curved space.

(a) Flat–space scenario.
The covariant derivative of a general vector $\vec{A} = A^\alpha\,\vec{\varepsilon}_\alpha$ gives a general incremental vector:

$$d\vec{A} = dA^{\alpha}\,\vec{\varepsilon}_{\alpha} + A^{\alpha}\,\frac{\partial\vec{\varepsilon}_{\alpha}}{\partial x^{j}}\,dx^{j}$$
$$= \left(dA^{\alpha} + A^{\beta}\,\Gamma^{\alpha}_{\beta j}\,dx^{j}\right)\vec{\varepsilon}_{\alpha}. \tag{55}$$

In a flat space, this incremental vector of a general vector is path independent, since all curvature tensor components vanish. From Eq. (39), we therefore obtain $\Delta(\Delta\vec{A}) = 0$.

In a three-dimensional flat space (Sec. 2.1), we can write a position vector as, $\vec{A} = \vec{s} = x^{\alpha}\,\vec{\varepsilon}_{\alpha} = r\,\vec{\varepsilon}_{r} = r\,\hat{r}$. Hence, the incremental displacement vector is $d\vec{s} = \hat{r}\,dr + \hat{\theta}\,r\,d\theta + \hat{\phi}\,r\sin\theta\,d\phi$. Using the flat–space unit–vector derivatives from Table 1, one verifies that the mixed partial derivatives are symmetric (Eq. (5)). Therefore, the incremental displacement vector is path independent, $\Delta(\Delta\vec{s}) = 0$.

In a flat space, both the incremental vector of a general vector and the incremental displacement vector are obtained in the same way, namely by taking covariant derivatives of their original vectors ($\vec{A}$ and $\vec{s}$). Hence both incremental vectors are path independent.

(b) Curved-space scenario.
In a curved space, some of the curvature tensor components $R^{\alpha}_{\beta jk}$ (Secs. 3.1 and 3.2) are non-zero. Therefore, the incremental vector of a general vector is not always path independent, $\Delta(\Delta\vec{A}) \neq 0$.

We cannot write any meaningful position vector $\vec{A} = \vec{s}$ in a curved space (Sec. 2.3), and the incremental displacement vector corresponding to the Schwarzschild metric must instead be defined by Eq. (13). The symmetry of the basis vectors has been verified in Eq. (17) using the curved-space unit–vector derivatives in Table 1. Thus, based on Eqs. (6) and (5), we conclude that the incremental displacement vector is always *assumed* to be path independent, $\Delta(\Delta\vec{s}) = 0$. This analysis assumes vanishing torsion (Sec. 2.1).

Based on this discussion, we write the last difficulty with the curved space analysis.

Difficulty 6. The general incremental vector $d\vec{A}$ is *not* always path independent, but the incremental displacement vector $d\vec{s}$ is always path independent. Thus, the two incremental vectors behave differently in the same curved space.

Finally, we summarize the main concerns associated with the four-dimensional curved space:

- It is difficult to write explicit unit–vector expressions or draw a geometrical picture of the coordinate system, as discussed in Secs. 2.2 and 2.3.
- The general incremental vector $d\vec{A}$ is not always path independent, while the incremental displacement vector $d\vec{s}$ is always path independent. Thus the two incremental vectors behave differently in the same curved space.

The Schwarzschild metric contains the coefficient $e^{\mu} = 1 - 2M/r$, where r is the distance from the central mass (as used in deriving planetary orbits). However, writing a position vector $\vec{r}$ in a curved space is not possible. The tensor analysis is therefore a mathematical procedure involving relations between scalar components of tensors, rather than a construction based on classical geometric vectors.

5 Bilocal and Local vectors and significance of vector method

5.1 Geometrical difficulties with the bilocal vectors and concerns with the local vectors

The classical vector analysis adopted in this article gives all the desired results required by the general relativity. A mathematical analysis of the curved space metric in a classical manner can be treated quite similar to the curved surface of a sphere. But, any attempt to draw a similar classical picture raises some serious geometrical concerns and we have listed six important difficulties in the article. Note that, in this analysis till now, we have not given the specific difference between the three spatial dimensions and the fourth dimension. We have treated the fourth dimension very similar to the three spatial dimensions.

In initial paragraph of Sec. 2.2, we saw that we can introduce an additional fourth coordinate in a three dimensional flat space metric which gives same Newtonian planetary orbits in a four dimensional geodesic analysis as obtained by introducing potential in the three dimensional flat space Lagrangian analysis. We can also write a total energy equation for a body in motion under a conservative field, corresponding to the Schwarzschild metric. This mathematical correspondence is limited to obtaining the same planetary orbits from both the general relativistic and classical approaches. We know that physical interpretations of both approaches are different.

In a classical approach, the fourth coordinate can be introduced through an arbitrary variable, for example $\dot{t} = a\,e^{-\mu}$. Using this definition, we may write the corresponding modified total–energy equation by assuming $\mu = -\lambda$ and $e^{\mu} = 1 - \frac{2M}{r}$.

$$H^{m} = \frac{e^{-\mu}\,\dot{r}^{2} + r^{2}\,\dot{\theta}^{2} + r^{2}\sin^{2}\theta\,\dot{\phi}^{2} - e^{\mu}\,\dot{t}^{2}}{2} = -\frac{1}{2}. \tag{56}$$

By inserting the expression for $\dot{t}$ into Eq. (56), we obtain a purely three–dimensional equation. Here, H^{m}, T^{m}, and U^{m} denote the modified total energy, kinetic energy, and potential energy, respectively, in the curved space.

$$H^m = \frac{e^{-\mu}\,\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2}{2} - \frac{a^2 e^{-\mu}}{2}$$
$$= T^m + U^m = -\frac{1}{2}. \qquad (57)$$

The Lagrangian analysis of this three–dimensional total–energy equation [3] yields the three components of the force along the spatial coordinates. Solving these equations reproduces the same planetary orbits (for $\theta = 90°$) as those obtained from the general–relativistic geodesic analysis of the Schwarzschild metric. However, this analysis in curved space encounters significant geometrical difficulties. It is hard to provide geometrical support for the symmetry of the basis vectors discussed in Sec. 2.2 (Difficulty 1), even though such symmetry is also a necessary requirement of the Lagrangian formulation. As shown in Sec. 2.3, it is difficult to write explicit unit–vector expressions in curved space. We may mathematically assume that the coordinates are orthogonal (as in Sec. 2.2), but writing the force vector becomes problematic because the unit–vector expressions themselves cannot be constructed.

Thus, the classical analysis up to this point may give the impression that the four–dimensional Schwarzschild metric is mainly a mathematical structure that reproduces the correct planetary orbits, while lacking an appropriate underlying geometrical interpretation.

We now examine how general relativity provides remedies for these geometrical difficulties, remedies that require fundamental changes in the underlying geometrical concepts. In general relativity, the fourth coordinate is identified with time, and every point in the four–dimensional space–time manifold corresponds to a physical event. Consequently, the conventional notion of a vector must be modified in curved space–time (see Ref. [1]). The familiar interpretation of vectors as arrows connecting two points (a "bilocal" picture with a head and a tail) must be abandoned. In curved space–time, such a bilocal construction is not well defined. Instead, vectors must be treated as *purely local* objects. They cannot be translated physically from one point to another, and each vector is attached to a specific event on the manifold.

There is therefore no position vector on a general curved manifold. Points are labelled only by their coordinates and are not themselves vectors. Rather, vectors are elements of the tangent space at each event, and the basis vectors are given by the coordinate partial derivatives along the coordinate lines:

$$\vec{\varepsilon}_i \equiv \frac{\partial}{\partial x^i}. \qquad (58)$$

Modifying the three-dimensional flat space metric to the four-dimensional curved space-time metric modifies geometry and also the interpretations of the physical phenomenon. Completely discarding the bilocal vectors, can overcome some of the difficulties listed in the article but this also raises some new concerns. Therefore, drawing the classical orbits showing the mass bodies rotating with certain velocities can still remain difficult. In a classical Newtonian picture, both bodies rotate around the center of mass by exerting forces on each other and we have to incorporate the reduced mass in calculations. This factor becomes important if the mass of the planet is not small compared with the central mass. Analysis of such situation will be quite complicated in a geodesic analysis and we normally study situations where central mass is very large. We shall list the difficulties with the classical vectors and briefly discuss how the general relativity tries to overcome these difficulties by defining a new version of vectors called the local vectors in Table 2. This modification raises some new concerns which are also briefly discussed in the Table 2.

Before proceeding, we recall the distinction between bilocal and local vectors as used in our analysis. The classical vectors $\vec{s}$ and $\vec{A}$ are *bilocal* vectors: they point from one point in space to another and therefore depend simultaneously on two spatial locations. Such bilocal vectors are not admissible in curved space–time (Sec. 2.3). In contrast, the infinitesimally small classical incremental vectors $d\vec{s}$ and $d\vec{A}$ can be treated mathematically as *local* vectors, since they are represented entirely by the coordinates of a single point in space (Secs. 2.1, 2.2, and 3.2). These local vectors reside in the tangent space associated with that point and do not require a bilocal construction.

5.2 Significance of the vector method

Finally, we discuss the significance of the vector method used in Sec. 3.2, Sec. 3.3 and Sec. 3.4. An element of arc $d\vec{s}$ can be written as $ds^2 = g_{ij}\,dx^i dx^j$. This tensor is called a metric tensor because all the metric properties of space should be completely determined by this tensor. Success of a curved space multidimensional analysis very much depends upon the intelligent choice of the metric tensor and this may not easily enlighten us about the geometrical aspects of the physical problem. However, an incremental displacement vector $\vec{ds} = \vec{\varepsilon}_i\,dx^i$ corresponding to this multidimensional metric can be written in terms of basis vectors such that, $g_{ij} = \vec{\varepsilon}_i \cdot \vec{\varepsilon}_j$. The basis vector symmetry condition (6) is true for zero torsion condition. Suitable conditions will have to be written if we wish to study nonzero torsion situations. In Section 3, we have shown that the Riemann-Christoffel tensor is related to the second partial derivatives while the Bianchi identity is related to the third partial derivatives of these basis vectors. Describing the classical vectors encounters various geometrical difficulties (Sec. 2.2, Sec. 2.3, Sec. 4). The standard general relativity has tried to overcome these

Table 2. The problems, the remedies, and the concerns with the remedies

Difficulties with a classical bilocal vector in a curved space	**Remedy in GR: redefine vectors as local vectors in spacetime**	**Concerns regarding the local-vector formulation and the geometry of spacetime**
Difficulty in defining a coordinate system (Sec. 2.3): In flat space a global coordinate system can be chosen with any convenient origin to describe positions, motions, and vectors. In curved space a global coordinate system is not possible.	In curved spacetime vectors are defined as *local*. One can only attempt to define a *local coordinate system* attached to each event.	In four-dimensional curved space one cannot write unit-vector expressions corresponding to coordinates even in a local frame. Thus writing classical vector expressions becomes difficult.
Difficulty in writing classical vector expressions (Sec. 2.3): Unit vectors cannot be written, and hence classical vectors (displacement, velocity, force) cannot be expressed. A Lagrangian force-based analysis becomes inappropriate for planetary orbits.	Planets do not orbit because of forces acting on them. Instead, one writes geodesic equations; geodesics around a central mass describe planetary motion.	In the Schwarzschild geometry, r, θ, ϕ are used with the same scalar meaning as in flat space when describing orbits, but the curved-space unit vectors cannot be related to flat-space unit vectors. No geometrical picture of vectors along (r, θ, ϕ) is possible. Even the basis vectors of Eq. (58) cannot be given classical unit-vector representations.
Difficulty with path independence (Secs. 4 and 2.2): The general incremental vector $d\vec{A}$ is *not* always path independent, whereas the incremental displacement vector $d\vec{s}$ *is* always path independent. Thus, the two behave differently in the same curved space.	The fourth coordinate is defined as time. There are *no bilocal vectors* in curved spacetime; GR uses only local vectors attached to events. Hence the notion of "path independence of vectors" loses physical meaning.	Time is the fourth coordinate to maintain consistency with special relativity. Local vectors such as $d\vec{s}$ and $d\vec{A}$ can be defined because the second point is infinitesimally close. In flat space or on a sphere, an infinitesimal patch is nearly flat and derivatives behave symmetrically. But in curved space, the tangent space cannot be assumed flat even locally near $d\vec{s}$. Thus the assumption that $d\vec{s}$ and $d\vec{A}$ behave differently has no clear geometrical explanation.
Difficulty with symmetry of Christoffel symbols (Sec. 2.2): There is no geometrical justification for path independence of $d\vec{s}$ or symmetry of basis vectors. Thus symmetric Christoffel symbols appear as a mathematical assumption.	The symmetry of Christoffel symbols is not derived from path independence. In GR the basis vectors are defined as partial derivatives along the coordinate lines (Eq. (58)). They commute because partial derivatives commute.	Partial second derivatives of a scalar are symmetric; for vectors only in flat space. Basis vectors in curved space commute only if the space is locally flat. But even in Ricci-flat space (Sec. 3.4) one cannot write unit vectors. Thus local flatness may not guarantee commuting basis vectors. Hence symmetric Christoffel symbols are a mathematical assumption with limited geometric support.
Difficulty with cross product (Sec. 2.3): Cross products fail when the fourth coordinate is introduced.	Orthogonal cross product is meaningful only in 3D; not in 4D. The fourth coordinate is time; a cross product involving time has no geometrical meaning.	The three spatial unit vectors obey dot and cross product rules under differentiation in curved space; the time-like unit vector obeys only the dot-product rule. GR discards bilocal vectors; therefore classical geometric meaning of dot and cross products between two local vectors becomes unclear.

difficulties by (Sec. 5.1) redefining the vector [1]as a local vector (Eq. (58)) but this definition also raises some new concerns. But, note that, the vector method, based on the classical incremental vectors gives all the desired mathematical results including various identities, similar to the conventional tensor analysis based on scalar components of tensors. A vector analysis is inherently superior to a scalar analysis as a vector has both magnitude and direction. This vector method, based on a classical geometrical picture, is an easier mathematical alternative to the tensor analysis in the curved multidimensional space and also throws light on the geometrical complications, if any, in describing such curved space. This method is generic in nature as it can be applied to any metric of any dimensions. We can also establish relationship between the three-dimensional Lagrangian method and the four-dimensional Geodesic analysis, both giving the same results.

6 Conclusion

This article has initially attempted to study the geometrical representation of vectors in a curved space. The Schwarzschild metric represents a four-dimensional curved-space geometry. This metric can be mathematically compared with the metric of the surface of a sphere embedded in a three-dimensional flat space, where we can write an incremental displacement vector at a point but cannot write a position vector linking two distinct points on the surface. Likewise, in the four-dimensional curved space it is difficult to write a position vector, but we can write an incremental displacement vector based on the four-dimensional metric. This incremental-vector-based classical approach reproduces exactly the same results as the tensor analysis of general relativity, but it faces several geometrical difficulties. We have discussed six such difficulties, which may create the impression that curved space-time lacks appropriate geometrical support. This article has examined why a new definition of vector is required in curved space-time and how it helps overcome these geometrical difficulties.

In general relativity, the fourth dimension is designated as time, so each point in space is associated with a specific event. The temptation to regard vectors as arrows linking two events has to be discarded [1]. The bilocal (head–tail) version of a vector must be replaced by a purely local version. There is no position vector on a manifold, and vectors cannot be physically transported as arrows; each vector must be attached to a specific local event. The difficulties encountered by classical (bilocal) vectors become the basis for assigning the peculiar characteristics of vectors in curved space-time. We can redefine the geometrical picture only because the fourth dimension represents time and is distinct from the spatial dimensions. Using this new definition, one may attempt to draw the geometrical picture of planetary motion in four-dimensional space-time. However, the new definition also raises additional geometrical concerns, especially when one tries to visualize vectors. It is accepted that we can not write bilocal vectors such as a position vector $\vec{s}$ or a general vector $\vec{A}$ in the curved space-time. But we realize that their incremental vectors $\vec{ds}$ and $\vec{dA}$ exhibit different path-independence properties in the same curved space and it is not possible to provide any geometrical justification for this behaviour. Such, characteristics are evident only from a vector analysis.

However, we can still adopt the vector method, which is based on the classical incremental vectors to get all the desired mathematical results including various identities, similar to the conventional tensor analysis. This geometrical vector method can work as an easier mathematical alternative to the tensor analysis in the curved multidimensional space and also throws light on the geometrical concerns, if any, in describing such curved space. This Newtonian analysis is generic in nature as it can be applied to any metric of any dimensions. We can also establish relationship between the three-dimensional Lagrangian method and the four-dimensional Geodesic analysis, both giving the same results.

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