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Some new considerations about the ν-function

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Abstract In this work, we construct a new family of coherent states associated with generalized hypergeometric functions. These hypergeometric coherent states are introduced through an appropriate choice of weight factors in the Fock space expansion. We analyze their mathematical structure, normalization and completeness, and we examine the corresponding photon number distribution and statistical properties. The results show that the hypergeometric coherent states exhibit both classical-like and non-classical features, such as sub-Poissonian statistics and quadrature squeezing. This framework provides a unifying approach to a broad class of coherent states and may open pathways for new applications in quantum optics and related areas of mathematical physics.

Furthermore, we focus on generalized hypergeometric coherent states (GH-CSs) for a continuous spectrum. We establish their basic mathematical properties (normalization, completeness, overlaps) and investigate their statistical characteristics (photon distribution, sub-Poissonian behavior). We also point out possible applications in quantum mechanics, as well as in quantum optics and related domains, where these states may serve as useful models for non-classical light. The formalism of such coherent states naturally leads to a mathematical "feedback": one is led to obtain and solve integrals involving ν-functions, thereby enlarging the range of applications of these functions.

1 Introduction

In this work, we focus on a new family of coherent states based on generalized hypergeometric functions. Hypergeometric functions appear naturally in many areas of mathematical physics, including solutions of differential equations, special function theory, and representations of orthogonal polynomials. By employing hypergeometric functions in the definition of coherent states, we aim to unify and extend several previously studied classes of states within a single framework. The purpose of this paper is to construct and analyze these hypergeometric coherent states, to establish their mathematical properties such as normalization and completeness, and to investigate their statistical characteristics, including photon number distribution, sub- Poissonian behavior, and quadrature squeezing. We also highlight possible applications in quantum optics and related domains, where these states may serve as useful models for non-classical light. The purpose of this paper is to construct and analyse these generalized hypergeometric coherent states (GHCSs) for continuous spectrum, to establish some mathematical properties (normalization, completeness, overlap), and to investigate their statistical characteristics (photon distribution, sub-Poissonian behavior). We also highlight some possible applications in quantum mechanics, as well as in quantum optics and related domains, where these states may serve as useful models for non-classical light. Also, the formalism of this type of CSs implicitly leads to a mathematical feedback: obtaining and solving integrals involving the ν-functions, which implicitly means an expansion of the applications that use these functions.

As is well known, for some function $f(E)$, the Laplace transform is defined by the integral

$$\mathcal{L}\{f\}(s) = \int_0^\infty dE\, e^{-sE} f(E), \tag{1}$$

where s is a complex number.

On the other hand, the reciprocal gamma function can be written as

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$$f(E+1) = \frac{1}{\Gamma(E+1)}. \tag{2}$$

By combining Eqs. (1) and (2), we obtain

$$\mathcal{L}\{f\}(e^{-s}) = \int_0^\infty dE\, \frac{e^{-sE}}{\Gamma(E+1)}. \tag{3}$$

If we set

$$e^{-s} = z, \qquad z = |z|e^{i\varphi}, \qquad 0 \le |z| \le R_c \le \infty, \tag{4}$$

where R_c is the radius of convergence of the power series in the variable $|z|$, then Eq. (3) becomes

$$\nu(z) \equiv \mathcal{L}\{f\}(s) = \int_0^\infty dE\, \frac{z^E}{\Gamma(E+1)}. \tag{5}$$

A new function $\nu(z)$ thus appears on the right-hand side. This is the so-called ν-function, which may be regarded as a generalization of the Laplace transform of the reciprocal gamma function. The ν-function was introduced by Volterra in 1916 [1]. Apart from a few classical monographs [1, 2], there are relatively few references in the literature dealing explicitly with $\nu(z)$.

A further generalization is the function $\nu(z,\alpha)$, defined by [1]

$$\nu(z,\alpha) = \int_0^\infty dE\, \frac{z^{E+\alpha}}{\Gamma(E+1+\alpha)}, \tag{6}$$

which reduces to $\nu(z)$ for $\alpha = 0$. The relation between these two functions can be expressed in terms of derivatives with respect to z as

$$\frac{d^n}{dz^n}\nu(z) = \nu(z,-n), \tag{7}$$

so that higher-order derivatives of $\nu(z)$ generate the family $\nu(z,\alpha)$.

On the other hand, one may consider the most general form of coherent states (CSs) in quantum mechanics, expanded in the Fock basis $\{|n\rangle, n = 0,1,2,\ldots,n_{\max} \le \infty\}$. These are the generalized hypergeometric coherent states (GH-CSs), defined by [3]

$$|z\rangle = \frac{1}{\sqrt{F_{p,q}(\{a_i\}_1^p, \{b_j\}_1^q; |z|^2)}} \sum_{n=0}^{n_{\max}\le\infty} \frac{z^n}{\sqrt{\rho_{p,q}(n)}}\,|n\rangle, \tag{8}$$

where p, q are non-negative integers and $\{a_i\}$, $\{b_j\}$ are parameter sets entering the generalized hypergeometric functions. We adopt the notation $\{x_i\}_{i=1}^m \equiv (x_1, x_2, \ldots, x_m)$.

The Pochhammer symbols $(a_i)_n$ appear in the structure constants. The name "generalized hypergeometric coherent states" comes from the fact that their normalizing function ${}_pF_q\big(\{a_i\}_{i=1}^p; \{b_j\}_{j=1}^q; |z|^2\big)$ is a generalized hypergeometric function. Moreover, the positive constants $\rho_{p,q}(n)$ are assumed to arise as the moments of a probability distribution [4], and for GH-CSs they are defined as follows [5]:

$$\rho_{p,q}(n) \equiv n!\, \frac{\prod_{j=1}^{q}(b_j)_n}{\prod_{i=1}^{p}(a_i)_n}, \tag{9}$$

so that the normalization function can be written as [6]

$${}_pF_q\big(\{a_i\}_{i=1}^p; \{b_j\}_{j=1}^q; |z|^2\big) = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{p}(a_i)_n}{\prod_{j=1}^{q}(b_j)_n} \frac{|z|^{2n}}{n!} = \sum_{n=0}^{\infty} \frac{|z|^{2n}}{\rho_{p,q}(n)}. \tag{10}$$

Any family of coherent states must satisfy the well-known Klauder prescriptions [7]:

- Normalization and non-orthogonality:

$$\langle z \mid z'\rangle = \begin{cases} 1, & \text{if } z = z', \\ \neq 0, & \text{if } z \neq z'. \end{cases} \tag{11}$$

- Continuity in the label z:

$$\lim_{z'\to z} \|\, z - z' \| = 0. \tag{12}$$

- Resolution of the identity:

$$\int d\mu(z)\,|z\rangle\langle z| = \hat{I} = \sum_{n=0}^{\infty} |n\rangle\langle n|, \tag{13}$$

where $d\mu(z) = \frac{d^2 z}{\pi} = \frac{d\varphi}{2\pi}\, d(|z|^2) h(|z|^2)$ is a measure with a positive weight function $h(|z|^2)$, to be determined for each family of coherent states.

Although at first glance there seems to be no connection between the two concepts introduced above, namely the ν-function $\nu(z)$ and the generalized coherent states (GH-CSs) $|z\rangle$, it was shown in Ref. [8] that such a connection does exist. This may be regarded as the first physical application of the ν-function. In the present paper we aim to deepen this connection, leading to a series of new properties of the ν-function $\nu(z)$ that do not appear in the specialized literature.

In Ref. [8] we also examined the transition from a discrete spectrum (d) to a continuous spectrum (c) for a given quantum system. We found that if a certain limiting procedure—referred to as the discrete–continuous limit $d \to c$—is applied, any quantity associated with a system possessing a discrete spectrum is mapped to the corresponding quantity associated with the continuous spectrum.

There exist quantum systems whose spectra include both discrete and continuous parts. A well-known example is the diatomic molecule, whose internuclear potential is accurately represented by the Morse potential. This potential supports a finite number of discrete energy levels corresponding to bound states. For sufficiently large values of the vibrational quantum number n, i.e. for energies exceeding the dissociation energy D_e, the molecule dissociates under external influences such as increasing temperature. In this regime the two nuclei (together with their electrons) behave as free particles, and the spectrum becomes continuous.

Let us emphasize that, for a given quantum system, two types of coherent states can be defined: (a) Barut–Girardello (BG) coherent states, introduced as the eigenstates of an annihilation operator [9]; and (b) Klauder–Perelomov (KP) coherent states, obtained by applying a displacement operator to the vacuum state [10]. Although their expansions in the Fock basis differ, the two families are dual. This duality manifests itself in the fact that the indices p and q and the corresponding parameter sets $\{a_i\}_{i=1}^{p}$ and $\{b_j\}_{j=1}^{q}$ can be interchanged [11]. However, when the discrete–continuous limit $d \to c$ is applied, both families converge to the same mathematical structure, identical to that of the coherent states of the one-dimensional harmonic oscillator (HO–1D). For this reason, and in light of the aims of the present work, we shall restrict our attention to the Barut–Girardello type coherent states as our starting point [11].

2 The main results from Ref. [8]—in short

To begin with, let us consider a dimensionless Hamiltonian $\mathcal{H}$ with a non-degenerate continuous spectrum, together with a corresponding set of dimensionless eigenstates $|E\rangle$ (where $\hbar\omega = 1$ is assumed). The energies lie in the interval $0 \le E < \infty$ and the eigenstates are normalized in the sense of Dirac delta functions:

$$\mathcal{H}\,|E\rangle = E\,|E\rangle, \qquad \langle E \mid E'\rangle = \delta(E - E'). \tag{14}$$

The closure, or completeness relation, for a continuous spectrum takes the form

$$\int_0^{\infty} dE\,|E\rangle\langle E| = \hat{I}, \tag{15}$$

and therefore

$$\int_0^{\infty} dE\,\langle E'|E\rangle\langle E|E''\rangle = \delta(E' - E''). \tag{16}$$

The coherent states corresponding to the continuous spectrum can be obtained by applying a suitable limiting procedure, which we refer to as the discrete–continuous limit $d \to c$ (for brevity). This limit has been introduced and discussed in detail in Ref. [13]; see also Ref. [8].

$$X_c(E) = \lim_{\substack{n\to E \\ n_{\max}\to\infty \\ \sum_0^\infty \to \int_0^\infty \\ p=q \\ \{a_i\}=\{b_j\}}} X_d(n, n_{\max}) \equiv \lim_{d\to c} X_d(n, n_{\max}), \tag{17}$$

and

$$\sum_{n=0}^{n_{\max}} X_d(n, n_{\max}) \;\longrightarrow\; \int_0^{\infty} dE\, X_c(E). \tag{18}$$

Thus, establishing the correspondence between the observables (or quantities) of the discrete spectrum, X_d, and those of the continuous spectrum, X_c, requires the following operations: (a) the discrete quantum number n must be replaced by the dimensionless energy E; (b) the maximal number of bound states must tend to infinity,

$n_{\max} \to \infty$; (c) simultaneously, the summation over n must be replaced by an integral over E; (d) the indices p and q of the generalized hypergeometric functions, as well as the parameter sets $\{a_i\}_{i=1}^{p}$ and $\{b_j\}_{j=1}^{q}$, must be taken equal.

Consequently, we obtain the following limits:

$$\lim_{d\to c} \rho_{p,q}(n) = \lim_{d\to c} n! \, \frac{\prod_{j=1}^{q} (b_j)_n}{\prod_{i=1}^{p} (a_i)_n} = \Gamma(E+1)\,, \tag{19}$$

$$\begin{aligned} &\lim_{d\to c} {}_pF_q\big(\{a_i\}_{i=1}^{p}; \{b_j\}_{j=1}^{q}; |z|^2\big) \\ &\quad = \lim_{d\to c} \sum_{n=0}^{\infty} \frac{|z|^{2n}}{\rho_{p,q}(n)} \\ &\quad = \int_0^{\infty} \frac{|z|^{2E}}{\Gamma(E+1)}\, dE = \nu(|z|^2). \end{aligned} \tag{20}$$

This leads to the following limit for GH-CSs:

$$|z\rangle = \lim_{d\to c} |z\rangle = \frac{1}{\sqrt{\nu(|z|^2)}} \int_0^{\infty} \frac{z^E}{\sqrt{\Gamma(E+1)}}\, |E\rangle \, dE. \tag{21}$$

The expression of coherent states (CSs) for a continuous spectrum was first obtained—by different methods and considerations—for the Gazeau–Klauder coherent states in Ref. [4], and later reconsidered in Refs. [13] and [14].

The overlap of two coherent states follows immediately and is given by

$$\langle z \mid z' \rangle = \lim_{d\to c} \langle z \mid z' \rangle = \frac{\nu(z^* z')}{\sqrt{\nu(|z|^2)\, \nu(|z'|^2)}}. \tag{22}$$

Observation: To avoid overloading the notation, we keep the same complex variable z for both the discrete and the continuous spectra; in other words, $\lim_{d\to c} z = z$. Whenever necessary, and only in order to avoid ambiguity, we will explicitly indicate the indices d or c.

Using the Mellin transform of the Meijer G-function [1],

$$\begin{aligned} &\int_0^{\infty} x^{s-1}\, G_{p,q}^{\,m,n}\left(\omega x \,\middle|\, \begin{matrix} \{a_i\}_{i=1}^{n}\,;\ \{a_i\}_{i=n+1}^{p} \\ \{b_j\}_{j=1}^{m}\,;\ \{b_j\}_{j=m+1}^{q} \end{matrix}\right) dx \\ &\quad = \omega^{-s} \frac{\prod_{j=1}^{m} \Gamma(b_j+s) \prod_{i=1}^{n} \Gamma(1-a_i-s)}{\prod_{j=m+1}^{q} \Gamma(1-b_j-s) \prod_{i=n+1}^{p} \Gamma(a_i+s)}. \end{aligned} \tag{23}$$

The integration measure of a GH-CS, as defined in Eq. (6), for a discontinuous spectrum was obtained in Ref. [15], and it is given by

$$\begin{aligned} d\mu_{p,q}^{(d)}(z) &= \frac{d\varphi\, d(|z|^2)}{2\pi} \frac{\prod_{i=1}^{p} \Gamma(a_i)}{\prod_{j=1}^{q} \Gamma(b_j)} \\ &\quad \times F_{p,q}\big(\{a_i\}_{i=1}^{p};\ \{b_j\}_{j=1}^{q};\ |z|^2\big) \\ &\quad \times G_{p,\,q+1}^{\,q+1,\,0}\left(|z|^2 \,\middle|\, \begin{matrix} /; & \{a_i-1\}_{i=1}^{p} \\ 0,\ \{b_j-1\}_{j=1}^{q}; & / \end{matrix}\right). \end{aligned} \tag{24}$$

so that its limit is

$$\lim_{d\to c} d\mu_{p,q}^{d}(z) = \frac{d\varphi\, d(|z|^2)}{2\pi}\, e^{-|z|^2}\, \nu(|z|^2) \ \equiv\ d\mu^{c}(z). \tag{25}$$

where we have used the specialized value of the Meijer G-function, namely $G_{0,1}^{\,1,0}\left(|z|^2 \,\middle|\, 0\right) = e^{-|z|^2}$, see Ref. [6].

In this context, the following relationship is also valid

$$\begin{aligned} &\lim_{d\to c} \int d\mu_{p,q}^{d}(z)\, |z\rangle\langle z| \\ &\quad = \int d\mu^{c}(z)\, |z\rangle\langle z| = \int_0^{\infty} dE\, |E\rangle\langle E| = \hat{I}. \end{aligned} \tag{26}$$

In performing the demonstration through the corresponding substitutions, after the angular integration, we used a fundamental integral:

$$\int_0^\infty d(|z|^2)\,(|z|^2)^E \times G_{p,\,q+1}^{q+1,\,0}\left(|z|^2 \middle| \begin{matrix} /; & \{a_i-1\}_{i=1}^p \\ 0,\ \{b_j-1\}_{j=1}^q; & / \end{matrix}\right) = \Gamma(E+1)\frac{\prod_{j=1}^q \Gamma(b_j+E)}{\prod_{i=1}^p \Gamma(a_i+E)} = \frac{\prod_{j=1}^p \Gamma(b_j)}{\prod_{i=1}^q \Gamma(a_i)}\rho_{p,q}(E), \tag{27}$$

which will be useful in the following.

As we showed previously in Ref. [5], the GH-CSs are generated by a pair of creation and annihilation operators, $\mathcal{A}_+$ and $\mathcal{A}_-$. A new operational ordering procedure, called DOOT (the diagonal ordering operation technique), can be applied to these operators. This approach leads to several new results, both for the discontinuous spectrum [15] and for the continuous one [8].

The pair operators are $(\mathcal{A})^+ = \mathcal{A}_+$ and satisfy the following equations:

$$\begin{aligned} \mathcal{A}^-\,|n\rangle &= \sqrt{n}\,|n-1\rangle, \\ \mathcal{A}^+\,|n\rangle &= \sqrt{n+1}\,|n+1\rangle, \\ \mathcal{A}^+\mathcal{A}^-\,|n\rangle &= |n\rangle. \end{aligned} \tag{28}$$

Their action on the vectors $|E\rangle$ follows from applying the discrete–continuous limit $d \to c$ to their discrete counterparts:

$$\begin{aligned} \mathcal{A}_-\,|E\rangle &= E\,|E-1\rangle, \\ \mathcal{A}_+\,|E\rangle &= (E+1)\,|E+1\rangle, \\ \mathcal{A}_+\mathcal{A}_-\,|E\rangle &= E\,|E\rangle. \end{aligned} \tag{29}$$

In Ref. [8], for the continuous spectrum we introduced a real, dimensionless energy parameter $\varepsilon > 0$, which is not a quantum number but may be interpreted as a suitable "unit jump" in the energy scale of continuous spectra. By choosing $\varepsilon = 1$, the system energy can be written simply as $E = m$. If we apply the raising operator $\mathcal{A}_+$ successively m times to the ground (vacuum) state $|0\rangle$, then for the continuous spectrum we obtain

$$\begin{aligned} |E\rangle &= \frac{1}{\Gamma(E+1)}\,(\mathcal{A}_+)^E\,|0\rangle, \\ \langle E| &= \frac{1}{\Gamma(E+1)}\,\langle 0|\,(\mathcal{A}_-)^E. \end{aligned} \tag{30}$$

Starting from Eqs (14)-(15) and applying the DOOT rules (for details, see Refs. [12],[15]), we obtain the expression of the vacuum-state projector for the continuous spectrum, $|0\rangle\langle 0|$:

$$\begin{aligned} \int_0^\infty dE\,|E\rangle\langle E| &= |0\rangle\langle 0|\,\int_0^\infty \frac{\#\,(\mathcal{A}_+\mathcal{A}_-)^E\,\#}{\Gamma(E+1)}\,dE \\ &= |0\rangle\langle 0|\,\#\,\nu(\mathcal{A}_+\mathcal{A}_-)\,\# = 1, \end{aligned} \tag{31}$$

$$|0\rangle\langle 0| = \frac{1}{\#\nu(\mathcal{A}_+\mathcal{A}_-)\#}\,. \tag{32}$$

The symbol $\#\cdot\#$ denotes the normal (diagonal) ordering of operators within the DOOT formalism.

Consequently, the projector in the energy eigenvector basis, $|E\rangle\langle E|$, is given by

$$|E\rangle\langle E| = \#\,\frac{(\mathcal{A}_+\mathcal{A}_-)^E}{\Gamma(E+1)}\,\frac{1}{\nu(\mathcal{A}_+\mathcal{A}_-)}\,\#. \tag{33}$$

With the above relations and the DOOT rules, the CSs for the continuous spectrum becomes:

$$\begin{aligned} |z\rangle &= \frac{1}{\sqrt{\nu(|z|^2)}}\int_0^\infty dE\,\frac{(z\,\mathcal{A}_+)^E}{\Gamma(E+1)}\,|0\rangle \\ &= \frac{1}{\sqrt{\nu(|z|^2)}}\;\nu(z\,\mathcal{A}_+)\,|0\rangle. \end{aligned} \tag{34}$$

and similarly for their dual counterparts, so that the projector onto the coherent state $|z\rangle$ is given by

$$|z\rangle\langle z| = \frac{1}{\nu(|z|^2)}\#\frac{\nu(z\,\mathcal{A}_+)\;\nu(z^*\,\mathcal{A}_-)}{\nu'(\mathcal{A}_+\mathcal{A}_-)}\,\#\,. \tag{35}$$

The correctness of this expression can be verified by using the completeness relation for coherent states, Eq. (25).

$$\int d\mu^c(z)\,|z\rangle\langle z| = \#\,\frac{1}{\nu(\mathcal{A}_+\mathcal{A}_-)}\,\#\int_0^\infty d(|z|^2)\,e^{-|z|^2} \int_0^{2\pi}\frac{d\varphi}{2\pi}\,\#\nu(z\,\mathcal{A}_+)\,\nu(z^*\,\mathcal{A}_-)\,\# = 1. \tag{36}$$

because the angular integral is

$$\int_0^{2\pi}\frac{d\varphi}{2\pi}\,\#\nu(z\,\mathcal{A}_+)\;\nu(z^*\,\mathcal{A}_-)\,\# = \int_0^\infty dE\,\frac{\#\,(\mathcal{A}_+\mathcal{A}_-)^E\,\#}{\left[\Gamma(E+1)\right]^2}\,|z|^{2E}. \tag{37}$$

On the other hand, the probability density for the transition from the energy eigenstate $|E\rangle$ to the coherent state $|z\rangle$ is given by

$$P(|z|^2)\;\equiv\;|\langle z\,|\,E\rangle|^2 = \frac{1}{\nu(|z|^2)}\,\frac{|z|^{2E}}{\Gamma(E+1)}. \tag{38}$$

If we apply the inverse limit, that is, the continuous–discrete limit $c\to d$, we recover precisely the Poisson probability density function corresponding to the discrete case.

$$\lim_{c\to d}\nu(|z|^2) = \lim_{\substack{E\to n\\ \int\to\sum}}\int_0^\infty dE\frac{|z|^{2n}}{\Gamma(E+1)} = \sum_{n=0}^\infty\frac{|z|^{2n}}{\Gamma(n+1)} = e^{|z|^2}, \tag{39}$$

$$\lim_{c\to d}P(|z|^2) = \lim_{\substack{E\to n\\ \int\to\sum}}P(|z|^2) = e^{-|z|^2}\,\frac{|z|^{2n}}{n!}\;\equiv\;P_n^{\text{Poiss}}\big(|z|^2\big)\,. \tag{40}$$

This result shows that the coherent states $|z\rangle$ obey sub-Poissonian statistics, i.e. they exhibit non-classical features.

From the relations above, it is clear that the ν-function, $\nu(|z|^2)$, naturally appears in the description of the continuous spectrum of a quantum system. To the best of our knowledge, this represents the first application of the ν-function $\nu(|z|^2)$ in a non-mathematical scientific context, namely in quantum mechanics and, more specifically, in quantum optics.

3 The generalized discrete–continuous limit $d\to c$

Let us now generalize the main results obtained in Ref. [8] and examine the practical consequences of this generalization. Compared with the limit used in this paper, Eqs (17)-(18), we will now adopt a less restrictive limit, namely:

$$X_c(E) = \lim_{\substack{n\to E\\ n_{\max}\to\infty\\ \sum_{n=0}^\infty\to\int_0^\infty dE}} X_d(n,n_{\max}) \equiv \lim_{d\to c}X_d(n,n_{\max}) \qquad \sum_{n=0}^{n_{\max}}X_d(n,n_{\max})\;\to\;\int_0^\infty X_c(E)\,dE. \tag{41}$$

This means that all observables X_c that characterize a system with a continuous spectrum are obtained as limiting cases of the corresponding observables X_d of the discrete spectrum, through the following three operations: (a) replacing the discrete quantum number n by the dimensionless energy E; (b) extending the upper bound $n_{\max}\to\infty$; (c) simultaneously replacing the sum over n by an integral over E.

For this reason, we shall refer to this transformation as the *generalized discrete–continuous limit* $d\to c$, or simply the *Gd–cL*. In what follows, we also introduce the notation *d-GH-CSs* for the generalized hypergeometric coherent states associated with the discrete spectrum, and *c-GH-CSs* for those associated with the continuous spectrum.

Let us now apply the Gd–cL to the objects that characterize the discontinuous spectrum.

$$\lim_{d\to c}\rho_{p,q}(n) = \lim_{d\to c}\,n!\,\frac{\prod_{j=1}^{q}(b_j)_n}{\prod_{i=1}^{p}(a_i)_n} = \Gamma(E+1)\frac{\prod_{j=1}^{p}(b_j)_E}{\prod_{i=1}^{p}(a_i)_E}\;\equiv\;\rho_{p,q}(E). \tag{42}$$

where the Pochhammer symbols are defined by $(x)_E = \dfrac{\Gamma(x+E)}{\Gamma(x)}$.

$$\lim_{d\to c} \mathcal{F}\big(\{a\}_p;\ \{b\}_q;\ |z|^2\big) = \lim_{d\to c} \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{p}(a_i)_n}{\prod_{j=1}^{q}(b_j)_n} \frac{|z|^{2n}}{n!}$$

$$= \int_0^\infty dE\, \frac{\prod_{i=1}^{p}(a_i)_n}{\prod_{j=1}^{q}(b_j)_n} \frac{|z|^{2E}}{\Gamma(E+1)} \tag{43}$$

$$= \int_0^\infty dE\, \frac{|z|^{2E}}{\rho_{p,q}(E)}$$

$$\equiv\ {}_p\mathcal{F}_q\big(\{a\}_1^p;\ \{b\}_1^q;\ |z|^2\big) = \nu_{p,q}\big(|z|^2\big)\,.$$

It is observed that a function similar to the generalized hypergeometric function is obtained, but defined not by a sum, but by an integral. We will call this new object the *integral generalized hypergeometric function* (int-GHF), and we denote it by ${}_p\mathcal{F}_q\big(\{a\}_1^p;\ \{b\}_1^q;\ |z|^2\big)$, which is precisely the ν-function in the continuous formulation. Let us note that the last integral has the same structure as the ordinary ν-function $\nu(|z|^2)$, but in a much more general form, in accordance with Eq. (27). For this reason we call it the *generalized* ν-function (Gν), $\nu_{p,q}(|z|^2)$, which is in fact equal to the integral generalized hypergeometric function (int-GHF) ${}_p\mathcal{F}_q\big(\{a\}_1^p;\ \{b\}_1^q;\ |z|^2\big)$.

$$\nu_{p,q}\big(|z|^2\big) \equiv \int_0^\infty \frac{|z|^{2E}}{\rho_{p,q}(E)}\, dE$$

$$= \int_0^\infty dE\, \frac{\prod_{i=1}^{p}(a_i)_E}{\prod_{j=1}^{q}(b_j)_E} \frac{|z|^{2E}}{\Gamma(E+1)} \tag{44}$$

$$\equiv {}_p\mathcal{F}_q\big(\{a\}_1^p;\ \{b\}_1^q;\ |z|^2\big)\,.$$

In the special case $p = q = 0$, we have

$$\nu_{0,0}\big(|z|^2\big) \equiv \nu\big(|z|^2\big)$$

$$= \int_0^\infty \frac{|z|^{2E}}{\Gamma(E+1)}\, dE \tag{45}$$

$$\equiv {}_0\mathcal{F}_0\big(\,;\,;\,|z|^2\big) = \exp(|z|^2).$$

In this context, we may state that the function $\nu(|z|^2)$ is an *integral exponential function*, namely the integral representation of $\exp(|z|^2)$.

Note: This integral exponential function $\exp(|z|^2)$ must not be confused with the *exponential integral* $\mathrm{Ei}(x)$, for real x, which is a special function defined on the complex plane by

$$\mathrm{Ei}(x) = -\int_{-x}^{\infty} \frac{e^{-t}}{t}\, dt = \int_{-\infty}^{x} \frac{e^{t}}{t}\, dt. \tag{46}$$

Consequently, the Gd–cL of the GH-CSs is

$$\lim_{d\to c} |z\rangle \equiv |z\rangle$$

$$= \frac{1}{\sqrt{\nu_{p,q}(|z|^2)}} \int_0^\infty \frac{z^E}{\sqrt{\rho_{p,q}(E)}}\, |E\rangle\, dE. \tag{47}$$

Since the construction involves the same complex variable z, we keep the same notation as before for the GH-CSs in the continuous spectrum, namely $|z\rangle$.

The overlap of two GH-CSs for the continuous spectrum is, consequently,

$$\langle z \mid z'\rangle = \frac{\nu_{p,q}(z^* z')}{\sqrt{\nu_{p,q}(|z|^2)\,\sqrt{\nu_{p,q}(|z'|^2)}}}. \tag{48}$$

Using the action of the generalized creation and annihilation operators, together with the DOOT rules, the projector onto the coherent state $|z\rangle$ is

$$|z\rangle\langle z| = \frac{1}{\nu_{p,q}(|z|^2)} \times \int_0^\infty dE\, \frac{z^E |E\rangle}{\sqrt{\rho_{p,q}(E)}} \int_0^\infty dE'\, \frac{(z^*)^{E'} \langle E'|}{\sqrt{\rho_{p,q}(E')}}\,. \tag{49}$$

Following the same calculation procedure as before, it will be obtained that the measure of integration in the continuous space will be

$$d\mu^c(z) = \frac{d\varphi}{2\pi}\, d(|z|^2)\, \frac{\prod_{i=1}^{p}\Gamma(a_i)}{\prod_{j=1}^{q}\Gamma(b_j)}\, \nu_{p,q}\, |z|^2 \times G^{\,q+1,\,0}_{\,p,\,q+1}\left(|z|^2 \,\middle|\, \begin{matrix} /; & \{a_i-1\}_{i=1}^p \\ 0,\ \{b_j-1\}_{j=1}^q; & / \end{matrix}\right), \tag{50}$$

where we took into account that the angular integral has the value

$$\int_0^{2\pi} \frac{d\varphi}{2\pi}\, z^E\, (z^*)^{E'} = |z|^{2E}\, \delta(E - E'), \tag{51}$$

and we have also used the fundamental integral, Eq. (27), that is, the general formula for reducing a classical integral to a single Meijer G–function [6], together with the completeness relation for the energy eigenvectors $|E\rangle$:

$$\int_0^{\infty} |E\rangle\langle E|\, dE = 1. \tag{52}$$

In this context, it is confirmed that the c-GH-CSs associated with a continuous spectrum admit an expansion in the energy basis $\{|E\rangle\}$, of the form

$$|z\rangle = \frac{1}{\sqrt{\nu_{p,q}(|z|^2)}} \int_0^{\infty} \frac{z^E}{\sqrt{\rho_{p,q}(E)}}\, |E\rangle\, dE. \tag{53}$$

Therefore, this integration measure ensures the validity of the completeness relationship of the unit operator:

$$\int d\mu^c(z)\, |z\rangle\langle z| = 1. \tag{54}$$

It is observed that the difference between the mathematical expressions of $\nu(|z|^2)$, Eq. (5), and the generalized form $\nu_{p,q}(|z|^2)$, Eq. (3.13), lies in the fact that in the denominator, instead of $\Gamma(E+1)$, one now finds $\rho_{p,q}(E)$, where $\nu(|z|^2) \equiv \nu_{0,0}(|z|^2)$.

After applying the discrete–continuous generalized limit $d \to c$, the actions of the generalized creation and annihilation operators on the vacuum state lead to the relations:

$$|E\rangle = \frac{1}{\sqrt{\rho_{p,q}(E)}} \left(\mathcal{A}_+\right)^E |0\rangle, \tag{55a}$$

$$\langle E| = \frac{1}{\sqrt{\rho_{p,q}(E)}} \langle 0| \left(\mathcal{A}_-\right)^E. \tag{55b}$$

In general, the pair of operators $\mathcal{A}_-$ and $\mathcal{A}_+$ acts on the vectors $|E\rangle$ in a manner that follows directly from the application of the discrete–continuous limit $d \to c$:

$$\mathcal{A}_-\, |E\rangle = \sqrt{\rho_{p,q}(E)}\, |E-1\rangle, \tag{56a}$$

$$\mathcal{A}_+\, |E\rangle = \sqrt{\rho_{p,q}(E+1)}\, |E+1\rangle, \tag{56b}$$

$$\mathcal{A}_+\mathcal{A}_-\, |E\rangle = \rho_{p,q}(E)\, |E\rangle, \tag{56c}$$

$$\#\left(\mathcal{A}_+\mathcal{A}_-\right)^E \#\, |E\rangle = \left[\rho_{p,q}(E)\right]^E |E\rangle. \tag{56d}$$

or, equivalently

$$\mathcal{A}_- = \int_0^{\infty} \sqrt{\rho_{p,q}(E)}\, |E-1\rangle\langle E|\, dE, \tag{57a}$$

$$\mathcal{A}_+ = \int_0^{\infty} \sqrt{\rho_{p,q}(E+1)}\, |E+1\rangle\langle E|\, dE, \tag{57b}$$

$$\mathcal{A}_+\mathcal{A}_- = \int_0^{\infty} \rho_{p,q}(E)\, |E\rangle\langle E|\, dE, \tag{57c}$$

$$\#(\mathcal{A}_+\mathcal{A}_-)^E\# = \int_0^{\infty} \left[\rho_{p,q}(E)\right]^E |E\rangle\langle E|\, dE. \tag{57d}$$

Similarly, by applying the DOOT rules, we obtain the expression for the generalized projector of the vacuum state associated with the continuous spectrum:

$$\begin{aligned} \int_0^{\infty} |E\rangle\langle E|\, dE &= |0\rangle\langle 0| \int_0^{\infty} \frac{\#\left(\mathcal{A}_+\mathcal{A}_-\right)^E \#}{\rho_{p,q}(E)}\, dE \\ &= |0\rangle\langle 0|\, \#\, \nu_{p,q}(\mathcal{A}_+\mathcal{A}_-)\, \# = 1, \end{aligned} \tag{58}$$

$$|0\rangle\langle 0| = \frac{1}{\#\, \nu_{p,q}(\mathcal{A}_+\mathcal{A}_-)\, \#}\,. \tag{59}$$

The projector onto the state $|z\rangle$ is obtained in complete analogy with the usual case, by applying the DOOT rules:

$$|z\rangle\langle z| = \frac{1}{\nu_{p,q}(|z|^2)} \# \frac{\nu_{p,q}(z\,\mathcal{A}_+)\ \nu_{p,q}(z^*\,\mathcal{A}_-)}{\nu_{p,q}(\mathcal{A}_+\mathcal{A}_-)} \#. \tag{60}$$

where, according to the DOOT rules, the vacuum projector $|0\rangle\langle 0|$ can be taken outside the symbols $\# \cdot \#$.

The completeness relation of the c-GH-CSs then leads to

$$\int d\mu_{p,q}^{(c)}(z)\,|z\rangle\langle z| = \frac{\prod_{j=1}^{q}\Gamma(b_j)}{\prod_{i=1}^{p}\Gamma(a_i)}\#\frac{1}{\nu_{p,q}(\mathcal{A}_+\mathcal{A}_-)}\#$$
$$\times\int_0^\infty d|z|^2\, G_{p,\,q+1}^{\,q+1,\,0}\left(|z|^2\,\middle|\,\begin{matrix} /; & \{a_i-1\}_1^p \\ 0,\ \{b_j-1\}_1^q; & / \end{matrix}\right)$$
$$\times\int_0^{2\pi}\frac{d\varphi}{2\pi}\,\#\,\nu_{p,q}(z\,\mathcal{A}_+)\;\nu_{p,q}(z^*\,\mathcal{A}_-)\,\#\;=\;1, \tag{61}$$

because the angular integral is

$$\int_0^{2\pi}\frac{d\varphi}{2\pi}\,\#\nu_{p,q}(z\,\mathcal{A}_+)\;\nu_{p,q}(z^*\mathcal{A}_-)\,\#$$
$$=\int_0^\infty dE\,\frac{\#\,(\mathcal{A}_+\mathcal{A}_-)^{E/2}\,\#}{\left[\rho_{p,q}(E)\right]^2}\,|z|^{\,2E}. \tag{62}$$

In addition, from the completeness relation, the following integral in complex space results, which refers to the ν-function with operator argument:

$$\int d^2z\; G_{p,\,q+1}^{\,q+1,\,0}\left(|z|^2\,\middle|\,\begin{matrix} /; & \{a_i-1\}_{i=1}^p \\ 0,\ \{b_j-1\}_{j=1}^q; & / \end{matrix}\right)$$
$$\times\,\#\,\nu_{p,q}(z\,\mathcal{A}_+)\;\nu_{p,q}(z^*\mathcal{A}_-)\,\#$$
$$=\frac{\prod_{j=1}^{q}\Gamma(b_j)}{\prod_{i=1}^{p}\Gamma(a_i)}\,\#\,\nu_{p,q}(\mathcal{A}_+\mathcal{A}_-)\,\#\,. \tag{63}$$

On the other hand, the probability density for the transition from the state $|E\rangle$ to the coherent state $|z\rangle$ is given by

$$P_{E;\,p,q}\big(|z|^2\big)\;\equiv\;|\langle z|E\rangle|^2=\frac{|z|^{\,2E}}{\nu_{p,q}(|z|^2)\;\rho_{p,q}(E)}\,, \tag{64}$$

whose generalized inverse limit $c\to d$ also leads to a generalized Poisson distribution. In fact, if we apply the inverse limit, that is, the continuous–discrete transition $c\to d$, we recover precisely the usual Poisson probability density function corresponding to the discrete case.

4 Other interesting properties of the generalized ν-function

To verify the expressions obtained for the continuous spectrum, we now apply the reciprocal limit, that is, the continuous–discrete transition $c\to d$. As a result, we must recover the corresponding expressions for the discrete (discontinuous) spectrum. For example:

$$\lim_{c\to d}\,\nu_{p,q}\big(|z|^2\big)=\lim_{c\to d}\int_0^\infty\frac{|z|^{2E}}{\rho_{p,q}(E)}\,dE$$
$$=\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}(a_i)_n}{\prod_{j=1}^{q}(b_j)_n}\,\frac{|z|^{2n}}{n!}$$
$$\equiv\;{}_p\mathcal{F}_q\big(\{a\}_1^p;\,\{b\}_1^q;\,|z|^2\big)\,. \tag{65}$$

In particular, for the case $p=0$ and $q=0$, we have ${}_0\mathcal{F}_0\big(\,;\,;\,|z|^2\big)=e^{|z|^2}$, and therefore we obtain

$$\lim_{c\to d}\,\nu_{0,0}\big(|z|^2\big)\equiv\lim_{c\to d}\,\nu\big(|z|^2\big)$$
$$=\lim_{c\to d}\int_0^\infty\frac{|z|^{2E}}{\Gamma(E+1)}\,dE$$
$$=\sum_{n=0}^{\infty}\frac{|z|^{2n}}{n!}={}_0\mathcal{F}_0\big(\,;\,;\,|z|^2\big)=e^{\,|z|^2}. \tag{66}$$

We can also define a ν operator (that is, a ν function that has an operator as argument):

$$\nu_{p,q}(\mathcal{A}_-)\equiv\int_0^\infty\frac{(\mathcal{A}_-)^E}{\rho_{p,q}(E)}\,dE, \tag{67a}$$

$$\nu_{p,q}(\mathcal{A}_+)\equiv\int_0^\infty\frac{(\mathcal{A}_+)^E}{\rho_{p,q}(E)}\,dE. \tag{67b}$$

Their action on c-GH-CSs is easily obtained if we take into account the definition of coherent states in the Barut-Girardello manner [9]:

$$\mathcal{A}_-\,|z\rangle=z\,|z\rangle, \tag{68a}$$

$$\langle z|\,\mathcal{A}_+=z^*\,\langle z|, \tag{68b}$$

$$\mathcal{A}_+\mathcal{A}_-\,|z\rangle=|z|^2\,|z\rangle. \tag{68c}$$

and we obtain

$$\nu_{p,q}(\mathcal{A}_-)\,|z\rangle = \int_0^\infty \frac{1}{\rho_{p,q}(E)}\,(\mathcal{A}_-)^E\,|z\rangle\,dE = \int_0^\infty \frac{z^E}{\rho_{p,q}(E)}\,|z\rangle\,dE = \nu_{p,q}(z)\,|z\rangle. \tag{69}$$

$$\langle z|\,\nu_{p,q}(\mathcal{A}_+) = \nu_{p,q}(z^*)\,\langle z|. \tag{70}$$

Therefore, the function $\nu_{p,q}(z)$ in the complex variable z is the eigenvalue of the operator-valued function $\nu_{p,q}(\mathcal{A}_-)$ in the coherent state $|z\rangle$.

The mean (expected) value of a DOOT-ordered product in the coherent state $|z\rangle$ is then

$$\langle z|\,\#\,\nu_{p,q}(\mathcal{A}_+)\,\nu_{p,q}(\mathcal{A}_-)\,\#\,|z\rangle = \nu_{p,q}(z^*)\,\nu_{p,q}(z)\,. \tag{71}$$

It is easy to verify that the action of the ν–operator (that is, a ν–function whose argument is an operator) on the c–GH–CSs is

$$\nu_{p,q}(z'^*\,\mathcal{A}_-)\,|z'\rangle = \nu_{p,q}\big(|z'|^2\big)\,|z'\rangle, \tag{72a}$$

$$\langle z|\,\nu_{p,q}(z\,\mathcal{A}_+) = \nu_{p,q}\big(|z|^2\big)\,\langle z|. \tag{72b}$$

The mean (expected) value in the coherent–state representation $|z\rangle$ is

$$\langle z|\,\#\,\nu_{p,q}(z\,\mathcal{A}_+)\,\nu_{p,q}(z^*\mathcal{A}_-)\,\#\,|z'\rangle = \sqrt{\nu_{p,q}(|z|^2)}\,\sqrt{\nu_{p,q}(|z'|^2)}\,\nu_{p,q}(z^*z')\,. \tag{73}$$

Let us now continue by calculating several matrix elements in the c–GH–CS representation of the function $\nu_{p,q}(\,\cdot\,)$ when it is evaluated on different operator arguments.

$$\langle z|\,\nu_{p,q}(z'^*\mathcal{A}_-)\,|z'\rangle = \nu_{p,q}\big(|z'|^2\big)\,\langle z|z'\rangle, \tag{74a}$$

$$\langle z|\,\nu_{p,q}(z\,\mathcal{A}_+)\,|z'\rangle = \nu_{p,q}\big(|z|^2\big)\,\langle z|z'\rangle. \tag{74b}$$

$$\langle z|\,\nu_{p,q}\Big(e^{-\,z'^*\mathcal{A}_-}\Big)\,|z'\rangle = \nu_{p,q}\Big(e^{-|z'|^2}\Big)\,\langle z|z'\rangle, \tag{75a}$$

$$\langle z|\,\nu_{p,q}(e^{\,z\mathcal{A}_+})\,|z'\rangle = \nu_{p,q}\Big(e^{\,|z|^2}\Big)\,\langle z|z'\rangle. \tag{75b}$$

$$\langle z|\,\#\,\nu_{p,q}(e^{\,z\mathcal{A}_+})\,\nu_{p,q}\Big(e^{-\,z'^*\mathcal{A}_-}\Big)\,\#\,|z'\rangle = \nu_{p,q}\Big(e^{\,|z|^2}\Big)\,\nu_{p,q}\Big(e^{-|z'|^2}\Big)\,\langle z|z'\rangle, \tag{76a}$$

$$\langle z|\,\#\,\nu_{p,q}(e^{\,z\mathcal{A}_+})\,\nu_{p,q}\Big(e^{-\,z^*\mathcal{A}_-}\Big)\,\#\,|z\rangle = \nu_{p,q}\Big(e^{\,|z|^2}\Big)\,\nu_{p,q}\Big(e^{-|z|^2}\Big)\,. \tag{76b}$$

Considering that, within the GH–CS formalism combined with the DOOT rules, the creation and annihilation operators $\mathcal{A}_+$ and $\mathcal{A}_-$ commute, i.e. $[\mathcal{A}_+, \mathcal{A}_-] = 0$, we may apply the Baker–Campbell–Hausdorff formula

$$\exp(\mathcal{X})\,\exp(\mathcal{Y}) = \exp(\mathcal{Z}), \tag{77a}$$

$$\mathcal{Z} = \mathcal{X} + \mathcal{Y} + \frac{1}{2}[\mathcal{X},\mathcal{Y}] + \frac{1}{12}[\mathcal{X},[\mathcal{X},\mathcal{Y}]] + \frac{1}{12}[\mathcal{Y},[\mathcal{Y},\mathcal{X}]] + \cdots\,. \tag{77b}$$

Let us calculate the diagonal elements of the function $\nu_{p,q}(\,\cdot\,)$ when its argument is the displacement operator:

$$\mathcal{D}(z) \;\equiv\; \#\,e^{\,z\,\mathcal{A}_+\,-\,z^*\mathcal{A}_-}\,\# \;=\; \#\,e^{\,z\,\mathcal{A}_+}\,e^{-\,z^*\mathcal{A}_-}\,\#. \tag{78}$$

The final result is (see the deduction in the Appendix)

$$\langle z|\,\nu_{p,q}(\mathcal{D}(z))\,|z\rangle = \langle z|\,\nu_{p,q}\Big(\#\,e^{\,z\mathcal{A}_+-z^*\mathcal{A}_-}\,\#\Big)\,|z\rangle = \int_0^\infty \frac{1}{\Gamma(E+1)}\,dE = \nu_{p,q}(1)\,. \tag{79}$$

One of the DOOT rules states that, inside the symbols $\#\cdot\#$, the creation and annihilation operators $\mathcal{A}_+$ and $\mathcal{A}_-$ are treated as ordinary c–numbers. Consequently, they may be removed from under the integral sign [15]. Therefore, in such expressions we may replace these operators by scalar quantities, $\mathcal{A}_+ \to x$ and $\mathcal{A}_- \to y$.

Let us now consider several integrals involving the function $\nu_{p,q}(\,\cdot\,)$.

Starting from the closure (completeness) relation of the unit operator and performing the above substitutions, we are led to two integrals of fundamental importance in the coherent–state approach.

Integral over the complex plane (from Eq. (63)):

$$\int \frac{d^2z}{\pi}\, G^{\,q+1,\,0}_{\,p,\,q+1}\left(|z|^2 \,\middle|\, \begin{matrix} /; & \{a_i-1\}_{i=1}^{p} \\ 0,\ \{b_j-1\}_{j=1}^{q}; & / \end{matrix}\right) \times \nu_{p,q}(xz)\,\nu_{p,q}(yz^*) = \left(\prod_{j=1}^{q}\Gamma(b_j)\right)\left(\prod_{i=1}^{p}\Gamma(a_i)\right)^{-1}\nu_{p,q}(xy). \tag{80}$$

Integral in real space, from Eqs. (49), (50), and (54):

$$\int_0^\infty d(|z|^2)\,|z|^{2E} \times G^{\,q+1,\,0}_{\,p,\,q+1}\left(|z|^2 \,\middle|\, \begin{matrix} /; & \{a_i-1\}_{i=1}^{p} \\ 0,\ \{b_j-1\}_{j=1}^{q}; & / \end{matrix}\right) = \left(\prod_{j=1}^{q}\Gamma(b_j)\right)\left(\prod_{i=1}^{p}\Gamma(a_i)\right)^{-1}\rho_{p,q}(E). \tag{81}$$

By specifying the indices p and q, the sets of parameters $\{a_i\}_{i=1}^{p}$ and $\{b_j\}_{j=1}^{q}$, as well as by performing a suitable change of integration variable, the above integrals allow us to evaluate a variety of other integrals involving the function $\nu_{p,q}\big(x\,|z|^2\big)$, where x is a real or complex number.

As a first example, we obtain

$$\int_0^\infty d\big(|z|^2\big)\ \nu_{p,q}\big(\frac{1}{x}\,|z|^2\big) \times G^{\,q+1,\,0}_{\,p,\,q+1}\left(|z|^2 \,\middle|\, \begin{matrix} /; & \{a_i-1\}_{i=1}^{p} \\ 0,\ \{b_j-1\}_{j=1}^{q}; & / \end{matrix}\right) = \left(\prod_{j=1}^{q}\Gamma(b_j)\right)\left(\prod_{i=1}^{p}\Gamma(a_i)\right)^{-1}\frac{1}{\ln x}\,. \tag{82}$$

Particularly, if we take $x = s$, make the substitution $|z|^2 = st$, and specify $p = 0$, $q = 0$ (so that $\nu_{0,0}(\,\cdot\,) = \nu(\,\cdot\,)$), we obtain (see Appendix):

$$\int_0^\infty e^{-st}\,\nu(t)\,dt = \frac{1}{s\,\ln s}, \qquad s>0,\ s\neq 1\,, \tag{83}$$

which is, in fact, the Laplace transform of the function $\nu(t)$, a formula that appears in Erdélyi's book [1], p. 222. On the other hand, if we take $p = 1$, $a_1 = 1$, $q = 1$ and $b_1 = b+1$, then in this situation we have $\rho_{1,1}(E) = \Gamma(b+1+E)$, and

$$G^{\,2,\,0}_{\,1,\,2}\left(|z|^2 \,\middle|\, \begin{matrix} /; & 0 \\ 0,\ b; & / \end{matrix}\right) = G^{\,0,\,1}_{\,1,\,0}\big(|z|\,\big|\,b\big) = e^{-|z|^2}\,(|z|)^{2b}. \tag{84}$$

Then, the integral becomes

$$\begin{aligned} &\int_0^\infty d\big(|z|^2\big)\ \nu_{1,1}\left(\frac{1}{x}\,|z|^2\right)\,e^{-|z|^2}\,\big(|z|^2\big)^b \\ &= \int_0^\infty dE\,\frac{x^{-E}}{\rho_{1,1}(E)}\int_0^\infty d\big(|z|^2\big)\ e^{-|z|^2}\,\big(|z|^2\big)^{b+E} \\ &= \int_0^\infty dE\,\frac{x^{-E}}{\Gamma(b+1+E)}\,\Gamma(b+1)\Gamma(b+1+E) \\ &= \Gamma(b+1)\int_0^\infty x^{-E}\,dE = \frac{\Gamma(b+1)}{\ln x}\,. \end{aligned} \tag{85}$$

Generally, if we choose the function $\nu_{p,q}\Big(e^{-s|z|^2}\Big)$, then, after expanding it into its power–series representation, we must appeal to the Laplace transform of the corresponding Meijer G–function (see Appendix):

$$\int_0^\infty d\big(|z|^2\big)\ \nu_{0,0}\,e^{-s|z|^2} \times G^{\,q+1,\,0}_{\,p,\,q+1}\left(|z|^2 \,\middle|\, \begin{matrix} /; & \{a_i-1\}_{i=1}^{p} \\ 0,\ \{b_j-1\}_{j=1}^{q}; & / \end{matrix}\right) = \sum_{l=0}^{\infty}\rho_{p,q}(l)\,\frac{(-1)^l}{l!}\,s^l\left(\frac{\partial}{\partial s}\right)^l\nu\big(e^{-s}\big)\,. \tag{86}$$

A similar integral can be derived for the two–variable function $\nu(z,\alpha)$ (see Appendix):

$$\int_0^\infty d\big(|z|^2\big)\ \nu\left(\frac{1}{C}|z|^2,\alpha\right) \times G^{\,q+1,\,0}_{\,p,\,q+1}\left(|z|^2 \,\middle|\, \begin{matrix} /; & \{a_i-1\}_{i=1}^{p} \\ 0,\ \{b_j-1\}_{j=1}^{q}; & / \end{matrix}\right) = C^{-\alpha}\,\frac{\prod_{j=1}^{q}\Gamma(b_j+\alpha)}{\prod_{i=1}^{p}\Gamma(a_i+\alpha)}\,\nu_{p,\,q+1}(C^{-1}). \tag{87}$$

Particularly, in Eq. (80), for the case $p=0$ and $q=0$, we have $G^{1,0}_{0,1}(|z|^2\,|\,0)=e^{-|z|^2}$ and we obtain

$$\frac{1}{\pi}\int d^2z\, e^{-|z|^2}\,\nu(xz)\,\nu(yz^*)=\nu(xy). \tag{88}$$

At the end of this section, it is necessary to make the following observation: when the generalized function $\nu(x)$ appears under an integral sign, it may also be defined through other structure functions, characterized by different indices and different parameter sets. For example:

$$\begin{aligned}\nu_{r,s}(X) \;&\equiv\; \int_0^\infty \frac{X^E}{\rho_{r,s}(E)}\,dE,\\ &\equiv\; {}_r\mathcal{F}_s\big(\{c_i\}_{i=1}^r\;;\;\{d_j\}_{j=1}^s\;;\;X\big)\,,\end{aligned} \tag{89a}$$

$$\rho_{r,s}(E)=\Gamma(E+1)\frac{\prod\limits_{j=1}^{s}(d_j)_E}{\prod\limits_{i=1}^{r}(c_i)_E}. \tag{89b}$$

The only reason why, throughout this paper, we employed the structure function $\rho_{r,s}(E)$ was to avoid overcomplicating the formulas.

5 Concluding remarks

In this paper we have sought to extend the properties and applications of the ν–function $\nu(\,\cdot\,)$, a subject that has been very little (or almost not at all) addressed in the specialized literature. Our analysis started from a first application presented in our previous work [8], where we examined the connection between the coherent–state formalism for continuous spectra and the ν–function $\nu(\,\cdot\,)$. There we showed that the ν–function plays the role of the normalization function of the coherent states $|z\rangle$ associated with the continuous spectrum of a quantum system. This role becomes apparent through the formulation and use of the discrete–continuous limit $d\to c$, by which any quantity or observable $X_d(n,n_{\max})$ characteristic of the discrete spectrum has a well-defined counterpart $X_c(E)$ in the continuous spectrum. In the present paper we have generalized the definition of the function $\nu(z)$, in the sense that in the denominator of its integral definition, instead of the simple gamma function $\Gamma(E+1)$, where E denotes the continuous spectrum of the Hamiltonian, a more general structure function $\rho_{p,q}(E)$ appears, containing products and ratios of gamma functions. This structure function, in its discrete (discontinuous) form, is directly related to the most general coherent states, namely the generalized hypergeometric coherent states (GH–CSs). In this way, we introduced a new function—the *generalized ν–function* $\nu_{p,q}(z)$—for which the usual ν–function becomes a particular case: $\nu(z)=\nu_{0,0}(z)$. We have examined a number of structural and operational properties of this generalized function.

In addition, we also dealt with the generalized functions $\nu_{p,q}(\,\cdot\,)$ whose argument depends on the creation or annihilation operators that define the GH–CSs. The results involving the functions $\nu_{p,q}(\,\cdot\,)$ are fully consistent with the well–known relations of coherent–state theory:

$$f(\mathcal{A}_-)\,|z\rangle = f(z)\,|z\rangle, \tag{90a}$$

$$\langle z|\,f(\mathcal{A}_+) = \langle z|\,f(z^*). \tag{90b}$$

These relations involving operators are important because, due to the rules of the diagonal operational ordering technique (DOOT), the operators are treated as simple c-numbers, which can be removed from under the DOOT symbol $\#\,\cdot\,\#$ [15]. Consequently, the operators can be replaced by scalar quantities, leading to new classical mathematical identities. As applications, we have derived several integrals involving the generalized functions $\nu_{p,q}(\cdot)$ and examined explicit examples. In this way, we opened the possibility of obtaining many further integrals and relations involving the functions $\nu_{p,q}(\cdot)$, by particularizing the indices p and q, the parameter sets $\{a_i\}_{i=1}^p$ and $\{b_j\}_{j=1}^q$, and by applying suitable changes of the integration variables.

Moreover, let us point out that all calculations in this paper were performed using coherent states defined in the Barut–Girardello sense [9]. In principle, the previous results also remain valid for the dual coherent states defined in the sense of Klauder–Perelomov, denoted by $|z\rangle_{\rm KP}$ [10].

$$\begin{aligned}|z\rangle_{\rm KP} &= \frac{e^{\,z\mathcal{A}_+-z^*\mathcal{A}_-}\;|0\rangle}{{}_q\mathcal{F}_p\big(\{b_j\}_{j=1}^q;\,\{a_i\}_{i=1}^p;\,|z|^2\big)}\\ &=\frac{1}{{}_q\mathcal{F}_p\big(\{b_j\}_{j=1}^q;\,\{a_i\}_{i=1}^p;\,|z|^2\big)}\\ &\quad\times\sum_{n=0}^\infty\frac{z^n}{\rho_{q,p}(n)}\,|n\rangle,\end{aligned} \tag{91a}$$

$$\rho_{q,p}(n)=\Gamma(n+1)\,\frac{\prod\limits_{i=1}^{p}(a_i)_n}{\prod\limits_{j=1}^{q}(b_j)_n}. \tag{91b}$$

By applying the discrete–continuous limit $d \to c$, the same results are obtained, taking into account the duality between the two types of coherent states, BG-CSs versus KP-CSs [11]. The role of the CSs normalization function for the continuous spectrum represents, to our knowledge, the first application of the ν–function in quantum optics. Conversely, the coherent-state formalism leads to several mathematical applications involving the ν–function, in the sense that it allows one to evaluate integrals containing this function. This establishes a genuine feedback mechanism between quantum mechanics (through the coherent-state formalism) and higher mathematics (through the theory of special functions).

As a general conclusion, we may state that coherent states—both those associated with discrete spectra and those corresponding to continuous energy spectra—constitute a fundamental bridge between quantum and classical (phase-space) representations. The coherent-state formalism has found applications in many diverse fields, including quantum physics (quantum optics, squeezing and photon-added states, measurement theory, quantum Hall effect), quantum information theory and practice (entanglement phenomena, cryptography), solid-state physics (ferromagnetism, superconductivity, collective phenomena), and mathematical physics (theory and applications of special functions).

All these areas of application fully justify the growing interest, in recent years, in the coherent states formalism, both for systems with discrete spectra and for those with continuous energy spectra.

Appendix

1. Derivation of Equation (86)

$$\int_0^\infty d(|z|^2)\, \nu_{0,0}\, e^{-s|z|^2} \times G^{q+1,\,0}_{p,\,q+1}\left(|z|^2 \middle| \begin{array}{ll} /; & \{a_i-1\}_{i=1}^p \\ 0,\ \{b_j-1\}_{j=1}^q; & / \end{array} \right)$$
$$= \int_0^\infty dE\, \frac{(e^{-s})^E}{\Gamma(E+1)} \sum_{l=0}^\infty \frac{(-sE)^{\,l}}{l!} \times \int_0^\infty d(|z|^2)\, (|z|^2)^l \times G^{q+1,\,0}_{p,\,q+1}\left(|z|^2 \middle| \begin{array}{ll} /; & \{a_i-1\}_{i=1}^p \\ 0,\ \{b_j-1\}_{j=1}^q; & / \end{array} \right)$$

$$\begin{aligned}
&= \left(\prod_{j=1}^q \Gamma(b_j)\right)\left(\prod_{i=1}^p \Gamma(a_i)\right)^{-1} \\
&\quad \times \sum_{l=0}^\infty \frac{(-s)^l}{l!} \left(\prod_{j=1}^q \Gamma(b_j)_l\right)\left(\prod_{i=1}^p \Gamma(a_i)l\right)^{-1} \\
&\quad \times \int_0^\infty dE\, \frac{(e^{-s})^E}{\Gamma(E+1)}\, (-E)^l \\
&= \sum_{l=0}^\infty \frac{(-1)^l}{l!} \left(\prod_{j=1}^q \Gamma(b_j+l)\right)\left(\prod_{i=1}^p \Gamma(a_i+l)\right)^{-1} \\
&\quad \times (-1)^l \left(\frac{\partial}{\partial s}\right)^l \int_0^\infty dE\, \frac{(e^{-s})^E}{\Gamma(E+1)}\, (-E)^l \\
&= \sum_{l=0}^\infty \rho_{p,q}(l)\, \frac{(-1)^l}{l!} \left(\frac{\partial}{\partial s}\right)^l \nu\left(e^{-s}\right).
\end{aligned} \tag{92}$$

2. Derivation of Equation (79)

$$\langle z|\nu_{p,q}(\mathcal{D}(z))|z\rangle = \langle z|\nu_{p,q}\left(\# e^{\,z\mathcal{A}_+ - z^*\mathcal{A}_-}\#\right)|z\rangle$$
$$\begin{aligned}
&= \langle z| \int_0^\infty dE\, \# \frac{(e^{\,z\mathcal{A}_+ - z^*\mathcal{A}_-})^E}{\rho_{p,q}(E)} \#|z\rangle \\
&= \int_0^\infty dE\, \frac{1}{\rho_{p,q}(E)}\, \langle z|\#(e^{\,z\mathcal{A}_+ - z^*\mathcal{A}_-})^E\#|z\rangle \\
&= \int_0^\infty dE\, \frac{1}{\rho_{p,q}(E)}\, \langle z|\#\,(e^{\,zE\mathcal{A}_+})(e^{-z^*E\mathcal{A}_-})\#|z\rangle \\
&= \int_0^\infty dE\, \frac{1}{\rho_{p,q}(E)}\, \#\left[\left(\sum_{j=0}^\infty \frac{(zE)^j}{j!}\langle z|(\mathcal{A}_+)^j\right) \times \left(\sum_{l=0}^\infty \frac{(z^*E)^l}{l!}(\mathcal{A}_-)^{\,l}|z\rangle\right)\right]\# \\
&= \int_0^\infty dE\, \frac{1}{\rho_{p,q}(E)} \left[\left(\sum_{j=0}^\infty \frac{(zE)^j}{j!}\, \langle z|(z^*)^j\right) \times \left(\sum_{l=0}^\infty \frac{(z^*E)^l}{l!}\, z^l|z\rangle\right)\right] \\
&= \int_0^\infty dE\, \frac{1}{\rho_{p,q}(E)} \left(\sum_{j=0}^\infty \frac{(|z|^2E)^j}{j!}\right) \times \left(\sum_{l=0}^\infty \frac{(|z|^2E)^l}{l!}\right)
\end{aligned}$$

$$= \int_0^\infty dE\, \frac{1}{\rho_{p,q}(E)}\, \exp(|z|^2)\exp(|z|^2)$$
$$= \int_0^\infty dE\, \frac{1}{\rho_{p,q}(E)} = \nu_{p,q}(1). \tag{93}$$

3. Derivation of Equation (82)

$$\begin{aligned}
\mathrm{Int} \equiv & \int_0^\infty d(|z|^2)\, \nu\left(\frac{1}{C}|z|^2, \alpha\right) \\
& \times G^{q+1,0}_{p,q+1}\left(|z|^2 \middle| \begin{matrix} /; & \{a_i-1\}_{i=1}^p \\ 0,\ \{b_j-1\}_{j=1}^q; & / \end{matrix}\right) \\
& = \int_0^\infty dE\, \frac{C^{-(\alpha+E)}}{\Gamma(\alpha+E+1)} \frac{\prod_{j=1}^q \Gamma(b_j+\alpha+E)}{\prod_{i=1}^p \Gamma(a_i+\alpha+E)} \\
& = \int_0^\infty dE\, \frac{C^{-(\alpha+E)}}{\Gamma(\alpha+E+1)}\, \rho_{p,q}(\alpha+E) \\
& \times \int_0^\infty d(|z|^2)\, \nu_{p,q}(\frac{1}{x}|z|^2) \\
& \times G^{q+1,0}_{p,q+1}\left(|z|^2 \middle| \begin{matrix} /; & \{a_i-1\}_{i=1}^p \\ 0,\ \{b_j-1\}_{j=1}^q; & / \end{matrix}\right) \\
& = \int_0^\infty dE \frac{x^{-E}}{\rho_{p,q}(E)} \int_0^\infty d(|z|^2)\, (|z|^2)^E \\
& \times G^{q+1,0}_{p,q+1}\left(|z|^2 \middle| \begin{matrix} /; & \{a_i-1\}_{i=1}^p \\ 0,\ \{b_j-1\}_{j=1}^q; & / \end{matrix}\right) \\
& = \left(\prod_{j=1}^q \Gamma(b_j)\right)\left(\prod_{i=1}^p \Gamma(a_i)\right)^{-1} \int_0^\infty dE\, x^{-E} \\
& = \left(\prod_{j=1}^q \Gamma(b_j)\right)\left(\prod_{i=1}^p \Gamma(a_i)\right)^{-1} (-1)\frac{1}{\ln x}\, x^{-E}\Big|_0^\infty \\
& = \left(\prod_{j=1}^q \Gamma(b_j)\right)\left(\prod_{i=1}^p \Gamma(a_i)\right)^{-1} \frac{1}{\ln x}\,.
\end{aligned} \tag{94}$$

4. Derivation of Equation (87)

$$\begin{aligned}
\mathrm{Int} \equiv & \int_0^\infty d(|z|^2)\, \nu\left(\frac{1}{C}|z|^2, \alpha\right) \\
& \times G^{q+1,0}_{p,q+1}\left(|z|^2 \middle| \begin{matrix} /; & \{a_i-1\}_{i=1}^p \\ 0,\ \{b_j-1\}_{j=1}^q; & / \end{matrix}\right) \\
& = \int_0^\infty dE\, C^{-(\alpha+E)} \frac{\prod_{j=1}^q \Gamma(b_j+\alpha+E)}{\prod_{i=1}^p \Gamma(a_i+\alpha+E)} \\
& = \int_0^\infty dE\, \frac{C^{-(\alpha+E)}}{\Gamma(\alpha+E+1)}\, \rho_{p,q}(\alpha+E)\,,
\end{aligned} \tag{95}$$

where

$$\begin{aligned}
\rho_{p,q}(\alpha+E) = & \Gamma(\alpha+E+1) \\
& \times \frac{\prod_{j=1}^q \Gamma(b_j+\alpha) \prod_{j=1}^q (b_j+\alpha)_E}{\prod_{i=1}^p \Gamma(a_i+\alpha) \prod_{i=1}^p (a_i+\alpha)_E}\,,
\end{aligned} \tag{96}$$

$$\begin{aligned}
\mathrm{Int} \equiv & C^{-\alpha}\left(\prod_{j=1}^q \Gamma(b_j+\alpha)\right)\left(\prod_{i=1}^p \Gamma(a_i+\alpha)\right)^{-1} \\
& \times \int_0^\infty dE\, (l)_E \frac{\prod_{j=1}^q (b_j+\alpha)_E}{\prod_{i=1}^p (a_i+\alpha)_E} \frac{C^{-E}}{\Gamma(E+1)} \\
& = C^{-\alpha} \frac{\prod_{j=1}^q (b_j+\alpha)}{\prod_{i=1}^p (a_i+\alpha)}\, \nu_{p,q+1}\left(C^{-1}\right).
\end{aligned} \tag{97}$$

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