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Infinitely Many Fast Homoclinic Solutions for Nonlinear Damped Systems Involving the p-Laplacian under Local Conditions

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Abstract This paper is concerned with proving that an infinite number of fast homoclinic solutions exist for a specific type of damped vibration system with nonlinearities governed by the p-Laplacian operator. Unlike most existing works, which typically require coercivity, periodicity, or global growth conditions on the potential, we establish our results under weaker, localized assumptions. In particular, the damping and stiffness terms are allowed to be non-coercive, and the potential function satisfies local conditions near the origin. Our approach relies on variational methods and the symmetric mountain pass theorem. Two main existence results are obtained, illustrating the effectiveness of this method in treating strongly nonlinear systems with nonstandard growth and damping terms.

1 Introduction

Second-order differential systems involving the p-Laplacian operator arise naturally in various scientific fields, including non-Newtonian fluid mechanics, nonlinear filtration theory, and nonlinear elasticity [1]. In this context, we study the following nonlinear damped vibration system with a p-Laplacian operator:

$$\begin{aligned}&\frac{d}{dt}\left(|\dot{u}(t)|^{p-2}\dot{u}(t)\right)+q(t)|\dot{u}(t)|^{p-2}\dot{u}(t)\\&-a(t)|u(t)|^{p-2}u(t)+\nabla W(t,u(t))=0,\quad t\in\mathbb{R}.\end{aligned}\tag{1}$$

where $p\geq 2$, $q,a\in C(\mathbb{R},\mathbb{R})$, and the potential $W:\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}$ is continuous and differentiable with respect to its second variable.

This system generalizes the classical damped vibration equation, which corresponds to the case $p=2$. When $p=2$ and $q\neq 0$, the equation reduces to a linear second-order differential system with variable coefficients, for which the existence and multiplicity of homoclinic orbits have been extensively studied via variational methods and critical point theory. Many works have focused on establishing fast homoclinic solutions under superquadratic or asymptotically quadratic conditions on the potential, particularly in periodic or coercive settings [2–15].

On the other hand, when $p\neq 2$ and $q=0$, the system becomes conservative and retains the p-Laplacian structure. In this setting, several researchers have also investigated the existence and multiplicity of homoclinic solutions using variational approaches [16–20].

However, in the general case where $p>1$ and $q\neq 0$, results remain scarce due to the technical challenges introduced by the nonlinear damping term, which breaks the self-adjoint structure and complicates the variational analysis. Existing works in this general framework often impose strong assumptions on the nonlinearity or the potential. For instance, in [21], the authors considered a potential composed of two parts satisfying super- and sub-quadratic Ambrosetti-Rabinowitz-type conditions, and applied both the classical and symmetric mountain pass theorems to obtain fast homoclinic orbits. More recently, [22] used Jeanjean's monotonicity trick along with concentration-compactness principles to establish the existence of infinitely many fast homoclinic solutions without relying on the classical Ambrosetti-Rabinowitz condition. However, such results generally assume that $W(t,x)$ is periodic in time and globally superquadratic in the spatial variable.

In contrast to these works, the present paper investigates the existence of infinitely many fast homoclinic solutions under more relaxed assumptions. Specifically, we allow the coefficient $a(t)$ to be non-coercive, and consider potentials $W(t,x)$ that satisfy only local conditions with respect to both time t and the spatial variable x. Notably, we do not require

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periodicity in t or global growth conditions at infinity for the potential. Instead, the potential may exhibit superquadratic growth locally near the origin, which significantly broadens the class of admissible nonlinearities and allows us to treat more general and realistic systems.

Our approach relies on variational methods and, in particular, on the symmetric mountain pass theorem due to Kajikiya [23]. To handle the lack of compactness, we construct suitable truncations of the potential and apply refined estimates to show that the associated energy functional satisfies the Palais-Smale condition. We then prove the existence of an infinite sequence of nontrivial critical points converging to zero in the energy space, which correspond to fast homoclinic solutions of the original system.

We now introduce the precise assumptions:

- (Q_ν) There exists a constant $\nu > 1$ such that

$$Q(t) = \int_0^t q(s)\,ds \to +\infty \quad \text{as } |t| \to \infty,$$
$$\int_{|t|\geq 1} \frac{e^{Q(t)}}{|t|\,\ln^\nu |t|}\,dt < \infty.$$

- (A) There exists $a_0 > 0$ such that

$$a(t) \geq a_0, \quad \forall t \in \mathbb{R},$$

- (A_ν) $\operatorname{meas}_Q\Big(\{t \in \mathbb{R} : a(t)/(|t|\ln^\nu |t|) < b\}\Big) < \infty, \quad \forall b > 0.$

- (W_1) $W(t,0) = 0$ and $\exists r > 0$ such that $W(t,-x) = W(t,x)$ for all (t,x) with $|x| \leq r$.

- (W_2) $\exists a < b$ and $\alpha > p$ such that

$$\lim_{|x|\to 0} \frac{W(t,x)}{|x|^\alpha} = +\infty \quad \text{uniformly for } t \in [a,b].$$

- (W_3) $\exists R_0 > 0$ such that

$$\lim_{|x|\to 0} \frac{|\nabla W(t,x)|}{a(t)|x|^{p-1}} = 0 \quad \text{uniformly for } |t| \geq R_0.$$

- (W_4) $\exists c > 0$, $\sigma \in (\frac{p}{2}, p)$, $R_1 > 0$, $\rho \in (0,r)$ with

$$|\nabla W(t,x)| \leq c\,|x|^{\sigma-1}, \qquad \forall |t| \geq R_1,\ |x| \leq \rho.$$

We now state our main results. The first covers the case where the potential satisfies a local symmetry condition near zero and has locally uniform superquadratic growth over a bounded time interval.

Theorem 1.1. Suppose assumptions (A) and (W_1) through (W_3) are satisfied. Then the system $(\mathscr{DV})$ has infinitely many nontrivial fast homoclinic solutions.

Example 1.1. Let $a(t) = |t\cos t| + 1$, and let $p < \gamma < \alpha$ be constants. Define

$$W(t,x) = a(t)\,|x|^\gamma,\ |x| \leq 1. \tag{2}$$

One can verify that a satisfies the condition (A), and $W(t,x)$ satisfies the conditions $(W_1) - (W_3)$.

The second result allows for non-coercive coefficients $a(t)$ under a weighted measure condition, and assumes certain local estimates on the gradient of the potential.

Theorem 1.2. Suppose assumptions (A), (A_ν), (W_1), (W_2), and (W_4) are satisfied. Then the system $(\mathscr{DV})$ has infinitely many nontrivial fast homoclinic solutions.

Example 1.2. Let $\sigma \in]\frac{p}{2}, p[$, $\alpha > \gamma > p+1$ be constants. Let $[a,b] = [\frac{\pi}{6}, \frac{\pi}{3}]$. Define

$$W(t,x) = |x|^\sigma \cos t + |x|^\gamma \sin t. \tag{3}$$

One can verify that $W(t,x)$ satisfies the conditions (W_1), (W_2), and (W_4).

These results represent a significant extension of the current theory, allowing for more general p-Laplacian systems under minimal local conditions. To the best of our knowledge, this is the first result establishing the existence of infinitely many fast homoclinic solutions under such minimal local assumptions in the presence of nonlinear damping and non-coercive stiffness.

2 Preliminaries

Our analysis of fast homoclinic solutions for $(\mathscr{DV})$ begins with a review of key properties of the weighted Sobolev space E. This space serves as the domain for a variational functional J such that its critical points are exactly the homoclinic solutions of $(\mathscr{DV})$. For $1 \leq s < \infty$, we define $L^s_Q(\mathbb{R})$ as the Banach space of measurable functions from $\mathbb{R}$ to $\mathbb{R}^N$ with the norm:

$$\|u\|_{L^s_Q} = \Big(\int_{\mathbb{R}} e^{Q(t)}\,|u(t)|^s\,dt\Big)^{\frac{1}{s}}. \tag{4}$$

Additionally, we define $L_Q^\infty(\mathbb{R})$ as the Banach space of functions from $\mathbb{R}$ to $\mathbb{R}^N$ with the norm:

$$\|u\|_{L_Q^\infty} = ess\sup\left\{e^{\frac{Q(t)}{2}}\,|u(t)|\,/t\in\mathbb{R}\right\}. \tag{5}$$

Next, we define the Sobolev space $W_Q^{1,p}(\mathbb{R})$ as:

$$W_Q^{1,p}(\mathbb{R}) = \left\{u\in L_Q^p(\mathbb{R})/\dot{u}\in L_Q^p(\mathbb{R})\right\}, \tag{6}$$

equipped with the usual norm:

$$\|u\|_{W_Q^{1,p}} = \Big(\int_{\mathbb{R}} e^{Q(t)}\Big[|\dot{u}(t)|^p+|u(t)|^p\Big]dt\Big)^{\frac{1}{p}}. \tag{7}$$

Under the condition (A), we introduce the Banach space:

$$E = \Big\{u\in W_Q^{1,p}(\mathbb{R}):\int_{\mathbb{R}} e^{Q(t)}\left(|\dot{u}(t)|^p+a(t)|u(t)|^p\right)dt<\infty\Big\}. \tag{8}$$

with the norm:

$$\|u\| = \Big(\int_{\mathbb{R}} e^{Q(t)}\Big[|\dot{u}(t)|^p+a(t)\,|u(t)|^p\Big]dt\Big)^{\frac{1}{p}}. \tag{9}$$

It is well known that E is continuously embedded in $L_Q^s(\mathbb{R})$ for all $p\le s\le\infty$.

Definition 2.1. We define a solution u of $(\mathscr{DV})$ to be a fast homoclinic solution provided that u belongs to the space E.

Lemma 2.1. [16] For $u\in W_Q^{1,p}(\mathbb{R})$, the following inequalities hold:

$$\|u\|_{L^\infty}^p \le \left(\frac{p-1}{2^{p'}}\right)\frac{1}{p'}\int_{\mathbb{R}}\left(|\dot{u}(s)|^p+|u(s)|^p\right)ds. \tag{10}$$

and for $u\in E$,

$$\|u\|_{L^\infty}^p \le \left(\frac{p-1}{2^q a_0}\right)^{\frac{1}{q}}\|u\|^p. \tag{11}$$

$$|u(t)|^p \le (p-1)^{\frac{1}{p'}}\int_t^\infty \times\left[a(s)\right]^{-\frac{1}{p'}}\left(|\dot{u}(s)|^p+a(s)|u(s)|^p\right)ds,\quad t\in\mathbb{R}. \tag{12}$$

$$|u(t)|^p \le (p-1)^{\frac{1}{p'}}\int_{-\infty}^t \times\left[a(s)\right]^{-\frac{1}{p'}}\left(|\dot{u}(s)|^p+a(s)|u(s)|^p\right)ds,\quad t\in\mathbb{R}. \tag{13}$$

where p' is the exponent conjugate of p: $\frac{1}{p'}+\frac{1}{p'}=1$.

Lemma 2.2. Suppose that $(Q_\nu),(A_\nu)$ are satisfied. Then E is compactly embedded in $L_Q^s(\mathbb{R})$ for any $s\in[\frac{p}{2},\infty[$. Moreover, for all $s\in[\frac{p}{2},\infty]$, there exists a constant $\eta_s>0$ such that

$$\|u\|_{L_Q^s} \le \eta_s\,\|u\|,\ \forall u\in E. \tag{14}$$

Proof: Let $\varepsilon>0$ be arbitrary. Assumption (A_ν) guarantees the existence of a radius $r_\varepsilon\ge e$ for which the measure of the set B_ε is less than or equal to ε, where

$$B_\varepsilon = \left\{t\in\mathbb{R}\setminus[-r_\varepsilon,r_\varepsilon]\;\middle|\;\frac{a(t)}{|t|\,\ln^\nu|t|}<\frac{1}{\varepsilon}\right\}. \tag{15}$$

and

$$\int_{|t|\ge r_\varepsilon} e^{Q(t)}\frac{1}{|t|\,\ln^\nu|t|}\,dt<\varepsilon. \tag{16}$$

Let

$$D_\varepsilon = \mathbb{R}\setminus\left(B_\varepsilon\cup]-r_\varepsilon,r_\varepsilon[\right), \tag{17}$$

and

$$a_\varepsilon = \inf_{t\in D_\varepsilon}\frac{a(t)}{|t|\,\ln^\nu|t|}. \tag{18}$$

Then $\frac{1}{a_\varepsilon}\le\varepsilon$. Consider a sequence (u_k) converging weakly to u in the space E. An application of the Banach-Steinhaus theorem yields

$$M = \sup_{k\in\mathbb{N}}\|u_k-u\|<\infty. \tag{19}$$

The continuous embeddings $E\hookrightarrow W_Q^{1,p}(\mathbb{R})\hookrightarrow L_Q^s(\mathbb{R})$ hold for every $s\in[p,\infty]$. Consequently, we can find a constant $M_s>0$ satisfying

$$\|u_k-u\|_{L_Q^s}\le M_s,\ \forall k\in\mathbb{N}. \tag{20}$$

Since $a(t) \geq a_0$ in $I_\varepsilon =]-r_\varepsilon, r_\varepsilon[$, the operator $E \to W_Q^{1,p}(I_\varepsilon)$, $u \mapsto u_{|I_\varepsilon}$ defines a linear operator that is continuous. Here, the symbol $W_Q^{1,p}(I_\varepsilon)$ refers to the weighted Sobolev space over the interval I_ε

$$W_Q^{1,p}(I_\varepsilon) = \left\{ u : I_\varepsilon \to \mathbb{R} / \int_{I_\varepsilon} e^{Q(t)} [|\dot{u}(t)|^p + |u(t)|^p] dt < +\infty \right\}. \tag{21}$$

An application of the Sobolev embedding theorem shows that

$$u_k \to u \; uniformly \; in \; \bar{I}_\varepsilon. \tag{22}$$

Step 1. We first demonstrate the compactness of the embedding of E into $L_Q^p(\mathbb{R})$. To see this, observe that

$$\begin{aligned}
&\int_{|t| \geq r_\varepsilon} e^{Q(t)} |u_k(t) - u(t)|^p \, dt = \\
&\quad \int_{B_\varepsilon} e^{Q(t)} |u_k(t) - u(t)|^p \, dt \\
&\quad + \int_{D_\varepsilon} e^{Q(t)} |u_k(t) - u(t)|^p \, dt \\
&\leq \text{meas}_Q(B_\varepsilon) \, \|u_k - u\|_{L^\infty}^p \\
&\quad + \int_{D_\varepsilon} e^{Q(t)} |t| \ln^\nu |t| \, |u_k(t) - u(t)|^p \, dt \\
&\leq \varepsilon \left(\frac{M_\infty}{\sqrt{m_0}} \right)^p + \frac{1}{a_\varepsilon} \int_{\mathbb{R}} e^{Q(t)} a(t) \, |u_k(t) - u(t)|^p \, dt \\
&\leq \varepsilon \left[\left(\frac{M_\infty}{\sqrt{m_0}} \right)^p + M^p \right].
\end{aligned} \tag{23}$$

where $m_0 = \inf_{t \in \mathbb{R}} e^{Q(t)}$. Combining (2.10) with (2.11) yields $\|u_k - u\|_{L_Q^p} \to 0$ as $k \to \infty$.

Step 2. $s \in]p, \infty[$. Next, we assert that the embedding of E into $L_Q^s(\mathbb{R})$ is also compact. This can be shown by noting

$$\begin{aligned}
\|u_k - u\|_{L_Q^s}^s &= \int_{\mathbb{R}} e^{Q(t)} \, |u_k - u|^s \, dt \\
&\leq \left(\frac{M_\infty}{\sqrt{m_0}} \right)^{s-p} \|u_k - u\|_{L_Q^p}^p .
\end{aligned} \tag{24}$$

Building on the result from Step 1, we conclude that $u_k \to u$ in $L_Q^s(\mathbb{R})$.

Step 3. $s \in [\frac{p}{2}, p[$. We claim that $u_k \to u$ in $L_Q^s(\mathbb{R})$. Let $\tau = \frac{\nu}{p-s}$. Then $s > \frac{p}{1+\nu}$ and $\tau s > 1$. For $v \in L_Q^s(\mathbb{R})$, Hölder's inequality implies

$$\begin{aligned}
&\int_{|t| \geq r_\varepsilon} e^{Q(t)} |v(t)|^s \, dt = \int_{B_\varepsilon} e^{Q(t)} |v(t)|^s \, dt \\
&\quad + \int_{D_\varepsilon} e^{Q(t)} |v(t)|^s \, dt \\
&\quad \leq \left(\int_{B_\varepsilon} e^{Q(t)} dt \right)^{\frac{p-1}{p}} \left(\int_{B_\varepsilon} e^{Q(t)} |v(t)|^{ps} dt \right)^{\frac{1}{p}} \\
&\quad + \int_{\substack{t \in D_\varepsilon \\ |t|^{-1/s} \ln^{-\tau} |t| \, |v(t)| \leq 1}} e^{Q(t)} |v(t)|^s \, dt \\
&\quad + \int_{\substack{t \in D_\varepsilon \\ |t|^{-1/s} \ln^{-\tau} |t| \, |v(t)| \geq 1}} e^{Q(t)} |v(t)|^s \, dt \\
&\leq \left(\text{meas}_Q(B_\varepsilon) \right)^{\frac{p-1}{p}} \|v\|_{L_Q^{ps}}^s + \int_{|t| \geq r_\varepsilon} \frac{e^{Q(t)}}{|t| \ln^{\tau s} |t|} \, dt \\
&\quad + \int_{\substack{t \in D_\varepsilon \\ |t|^{-1/s} \ln^{-\tau} |t| \, |v(t)| \geq 1}} \frac{e^{Q(t)}}{|t| \ln^{\tau s} |t|} \\
&\qquad \times \left(|t|^{1/s} \ln^\tau |t| \, |v(t)| \right)^s dt \\
&\leq \varepsilon^{\frac{p-1}{p}} \|v\|_{L_Q^{ps}}^s + \int_{|t| \geq r_\varepsilon} \frac{e^{Q(t)}}{|t| \ln^\nu |t|} \, dt \\
&\quad + \int_{\substack{t \in D_\varepsilon \\ |t|^{-1/s} \ln^{-\tau} |t| \, |v(t)| \geq 1}} \left(|t|^{1/s} \ln^\tau |t| \, |v(t)| \right)^p dt \\
&\leq \varepsilon^{\frac{p-1}{p}} \|v\|_{L^{ps}}^s + \varepsilon + \int_{|t| \geq r_\varepsilon} |t|^{\frac{p}{s}-1} \ln^{\tau(p-s)} |t| \, |v(t)|^p dt \\
&\leq \varepsilon^{\frac{p-1}{p}} \|v\|_{L^{ps}}^s + \varepsilon + \int_{|t| \geq r_\varepsilon} |t| \ln^\nu |t| \, |v(t)|^p \, dt \\
&\leq \varepsilon^{\frac{p-1}{p}} \|v\|_{L^{ps}}^s + \varepsilon + \frac{1}{a_\varepsilon} \int_{|t| \geq r_\varepsilon} a(t) \, |v(t)|^p \, dt \\
&\leq \varepsilon^{\frac{p-1}{p}} \|v\|_{L^{ps}}^s + \varepsilon + \varepsilon \, \|v\|^p \\
&\leq \varepsilon^{\frac{p-1}{p}} \left(\|v\|_{L_Q^{ps}}^s + 1 + \|v\|^p \right).
\end{aligned} \tag{25}$$

Hence, we have

$$\begin{aligned}
&\int_{|t| \geq r_\varepsilon} e^{Q(t)} \, |u_k(t) - u(t)|^s \, dt \\
&\qquad \leq \varepsilon^{\frac{p-1}{p}} \left[\|u_k - u\|_{L_Q^{ps}}^s + 1 + \|u_k - u\|^p \right].
\end{aligned} \tag{26}$$

Since $ps \geq p$, we deduce that

$$\int_{|t| \geq r_\varepsilon} e^{Q(t)} \, |u_k(t) - u(t)|^s \, dt \leq \varepsilon^{\frac{p-1}{p}} \left[M_{ps}^s + 1 + M^p \right]. \tag{27}$$

As above $\int_{I_\varepsilon} e^{Q(t)} |u_k(t) - u(t)|^s \, dt \to 0$ as $k \to \infty$. Hence $u_k \to u$ in $L^s_Q(\mathbb{R})$.

The proof of Lemma 2.2 is completed. ■

Our proof applies critical point theory, specifically the symmetric mountain pass theorem due to Kajikiya [5], to establish the main result. Before proceeding, we recall the concept of genus.

Let X be a Banach space and let A be a subset of X. The set A is said to be symmetric if for every element $u \in A$, the element $-u$ is also in A. Let A be a closed symmetric set not containing the origin. Its genus, denoted by $\gamma(A)$, is the smallest integer k for which one can construct an odd continuous mapping from $\mathbb{R}$ to $\mathbb{R}^k \setminus \{0\}$. If no such k exist, we define $\gamma(A) = +\infty$. Additionally, we define $\gamma(\emptyset) = 0$, where $\emptyset$ denotes the empty set. We now introduce the set Γ_k, defined as

$$\Gamma_k = \Big\{ A \subset X \;\Big|\; A \text{ is a closed symmetric subset, } 0 \notin A,\ \gamma(A) \geq k \Big\}. \tag{28}$$

Below, we summarize the essential properties of the genus that will be utilized in our argument.

Lemma 2.3. [5, Proposition 7.5.] Let A and B be closed symmetric subsets of a Banach space X that do not contain the origin. The following properties hold:

(i) If $A \subset B$, then $\gamma(A) \leq \gamma(B)$.

(ii) The $N-$dimensional sphere S^N has a genus of $N+1$ by the Borsuk-Ulam theorem.

Lemma 2.4. [5, Theorem 1] Let E be an infinite-dimensional Banach space and $\Phi \in C^1(X, \mathbb{R})$ be an even functional with $\Phi(0) = 0$. Suppose that Φ satisfies the following conditions:

(1) Φ is bounded from below and satisfies the $(PS)-$condition;

(2) For each $k \in \mathbb{N}$, there exists $A_k \in \Gamma_k$ such that

$$\sup_{u \in A_k} \Phi(u) < 0. \tag{29}$$

The theorem guarantees two types of sequences: (a) a sequence of critical points (u_k) with negative energy converging to zero, and (b) two distinct sequences: one of non-zero critical points with zero energy that tend to zero, and another of critical points with negative energy tending to zero, which itself converges to a non-zero limit.

3 Proof of Theorem 1.1.

Our strategy is to apply critical point theory to a modified version of the potential $W(t,x)$, denoted $\widetilde{W}(t,x)$, which coincides with $W(t,x)$ near the origin but is altered for large $|x|$. This modification is defined as follows:
Choose a constant $\varepsilon_0 \in]0,1[$. By (W_3), there exists a constant $\delta_0 \in]0,r]$ such that

$$|\nabla W(t,x)| \leq \varepsilon_0 a(t) \, |x|^{p-1}, \ \forall |t| \geq R_0 \text{ and } |x| \leq \delta_0. \tag{30}$$

Define a nonincreasing cut-off function $g \in C^1(\mathbb{R}^+, \mathbb{R}^+)$ such that $g(s) = 1$ for $0 \leq s \leq \frac{\delta_0^p}{2^p}$, $g(s) = 0$ for $s \geq \delta_0^p$ and $0 \leq g(s) \leq 1$ for all $s \in \mathbb{R}^+$. Let

$$\widetilde{W}(t,x) = g(|x|^p) \, W(t,x), \qquad \forall (t,x) \in \mathbb{R} \times \mathbb{R}^N. \tag{31}$$

This modified potential leads us to consider the following associated system $\widetilde{\mathscr{DV}}$:

$$\frac{d}{dt}\Big(|\dot{u}(t)|^{p-2} \dot{u}(t) \Big) + q(t)\dot{u}(t) - a(t)u(t) + \nabla \widetilde{W}(t, u(t)) = 0, \qquad t \in \mathbb{R}. \tag{32}$$

and we can now define the action functional J for the modified system $(\widetilde{\mathscr{DV}})$ by

$$\begin{aligned} J(u) &= \frac{1}{p} \int_{\mathbb{R}} e^{Q(t)} \left(|\dot{u}(t)|^p + a(t)|u(t)|^p \right) dt \\ &\quad - \int_{\mathbb{R}} e^{Q(t)} \widetilde{W}(t, u(t)) \, dt \\ &= \frac{1}{p} \, \|u\|^p - \int_{\mathbb{R}} e^{Q(t)} \widetilde{W}(t, u(t)) \, dt. \end{aligned} \tag{33}$$

It is well-known that $J \in C^1(E, \mathbb{R})$ and for all $u, v \in E$

$$\begin{aligned} J'(u)v &= \int_{\mathbb{R}} e^{Q(t)} \Big(|\dot{u}(t)|^{p-2} \dot{u}(t) \cdot \dot{v}(t) \\ &\qquad + a(t) \, |u(t)|^{p-2} u(t) \cdot v(t) \Big) \, dt \\ &\quad - \int_{\mathbb{R}} e^{Q(t)} \nabla \widetilde{W}(t, u(t)) \cdot v(t) \, dt. \end{aligned} \tag{34}$$

Moreover, critical points of J are fast homoclinic solutions of $(\widetilde{\mathscr{DV}})$.

Lemma 3.1. Assume that (A), (W_1), and (W_3) hold. Then $J(0) = 0$ and J is even and bounded from below.

Proof By (W_1), it is clear that J is even and $J(0)=0$. From Eq. (30) and the fact that $W(t,0)=0$, we obtain

$$W(t,x) \le \frac{\varepsilon_0}{p}\, a(t)\,|x|^p, \qquad \forall\,|t| \ge R_0 \text{ and } |x| \le \delta_0. \tag{35}$$

By combining Eq. (35) and the properties of g, we get

$$\begin{aligned} J(u) &= \frac{1}{p}\,\|u\|^p - \int_{\mathbb{R}} e^{Q(t)}\, g(|u(t)|^p)\, W(t,u(t))\,dt \\ &= \frac{1}{p}\,\|u\|^p - \int_{\{|t|\ge R_0\}} e^{Q(t)}\, g(|u(t)|^p)\, W(t,u(t))\,dt \\ &\quad - \int_{\{|t|\le R_0\}} e^{Q(t)}\, g(|u(t)|^p)\, W(t,u(t))\,dt \\ &= \frac{1}{p}\,\|u\|^p - \int_{\{|t|\ge R_0,\ |u(t)|\le\delta_0\}} e^{Q(t)} \\ &\qquad\qquad \times g(|u(t)|^p)\, W(t,u(t))\,dt \\ &\quad - \int_{\{|t|\le R_0,\ |u(t)|\le\delta_0\}} e^{Q(t)}\, g(|u(t)|^p)\, W(t,u(t))\,dt \\ &\ge \frac{1}{p}\,\|u\|^p - \frac{\varepsilon_0}{p}\int_{\mathbb{R}} e^{Q(t)}\, a(t)\,|u(t)|^p\,dt - c_1 \\ &\ge \frac{1-\varepsilon_0}{p}\,\|u\|^p - c_1. \end{aligned} \tag{36}$$

where $c_1 = \sup_{|t|\le R_0, |x|\le\delta_0} e^{Q(t)} g(|x|^p) W(t,x)$. Hence, J is bounded from below and coercive in E.

Lemma 3.2. Under assumptions (A) and (W_3), the functional J satis es the Palais-Smale condition.

Proof We will prove that J satisfies the (PS)-condition. Let (u_n) be a (PS)-sequence, that is, there exists a positive constant M_0 such that

$$|J(u_n)| \le M_0, \quad \forall n \in \mathbb{N}, \quad \text{and} \quad J'(u_n) \longrightarrow 0 \text{ as } n \to \infty. \tag{37}$$

Combining estimates (3.3) and (3.4), we deduce the boundedness of the sequence (u_n) in the space E, that is,

$$\|u_n\| \le c_2, \qquad \forall n \in \mathbb{N}. \tag{38}$$

The reflexivity of E allows us to find a weakly convergent subsequence, which we continue to denote by (u_n) for convenience:

$$u_n \rightharpoonup u_0 \text{ weakly in } E \text{ as } n \to \infty. \tag{39}$$

By (W_3), for any $0<\varepsilon<\varepsilon_0$, there exists a constant $0<\delta<\delta_0$ such that

$$|\nabla W(t,x)| \le \varepsilon\, a(t)\,|x|^{p-1}, \quad \forall\,|t| \ge R_0,\ |x| \le \delta, \tag{40}$$

which, with the condition $W(t,0)=0$, yields

$$|W(t,x)| \le \frac{\varepsilon}{p}\, a(t)\,|x|^p, \quad \forall\,|t|\ge R_0,\ |x|\le\delta. \tag{41}$$

By (Q_v), there exists a constant $T_0 \ge R_0$ such that

$$Q(t) \ge \ln\left(\frac{1}{\delta^p}(p-1)^{\frac{1}{p'}}\, c_2^p\right). \tag{42}$$

Combining Lemma 2.1, (38), and (42), we have

$$\begin{aligned} |u_n(t)|^p &\le (p-1)^{\frac{1}{p'}} \\ &\quad \times \int_t^{\infty} e^{-Q(s)} e^{Q(s)} \left[|\dot u_n(s)|^p + a(s)\,|u_n(s)|^p\right] ds \\ &\le (p-1)^{\frac{1}{p'}} \frac{\delta^p}{(p-1)^{\frac{1}{p'}}\, c_2^{p'}}\, \|u_n\|^{p'} \le \delta^p, \\ &\qquad \forall t \ge T_0,\ n \in \mathbb{N}. \end{aligned} \tag{43}$$

Similarly, we get

$$Q(t) \ge \ln\left(\frac{1}{\delta^p}(p-1)^{\frac{1}{p'}}\, c_2^p\right). \tag{44}$$

Inequalities (38) and (39) yield that $\|u_0\| \le \liminf_{n\to\infty}\|u_n\| \le c_2$, thus by similar steps one can get

$$|u_0(t)|^p \le \delta^p, \quad \forall\,|t| \ge T_0. \tag{45}$$

It follows from (3.7), (3.10)-(3.12), and the properties of g that

$$\begin{aligned} &\Bigg|\int_{|t|\ge T_0} e^{Q(t)} g\big(|u_n(t)|^p\big) \\ &\qquad \nabla W(t,u_n(t))\cdot(u_n(t)-u_0(t))\,dt \\ &\qquad - \int_{|t|\ge T_0} e^{Q(t)} g\big(|u_0(t)|^p\big) \\ &\qquad \times \nabla W(t,u_0(t))\cdot(u_n(t)-u_0(t))\,dt\Bigg| \\ &\le \int_{|t|\ge T_0} e^{Q(t)} \big(|\nabla W(t,u_n(t))| \\ &\qquad + |\nabla W(t,u_0(t))|\big)\big(|u_n(t)|+|u_0(t)|\big)\,dt \\ &\le 2\varepsilon \int_{|t|\ge T_0} e^{Q(t)} a(t)\big(|u_n(t)|^{p-1}+|u_0(t)|^{p-1}\big) \\ &\qquad \times \big(|u_n(t)|+|u_0(t)|\big)\,dt \\ &\le 4\varepsilon \int_{|t|\ge T_0} e^{Q(t)} a(t)\big(|u_n(t)|^p+|u_0(t)|^p\big)\,dt \\ &\le 4\varepsilon\big(\|u_n\|^p+\|u_0\|^p\big) \le 8\varepsilon\, c_2^p. \end{aligned} \tag{46}$$

Conditions (41), (43)-(45), and the properties of g yield

$$\begin{aligned}
&\Big| p \int_{|t|\geq T_0} e^{Q(t)} g'\big(|u_n(t)|^p\big)\, |u_n(t)|^{p-2} u_n(t) \\
&\quad \times W(t,u_n(t))\cdot\big(u_n(t)-u_0(t)\big)\,dt \\
&\quad - p\int_{|t|\geq T_0} e^{Q(t)} g'\big(|u_0(t)|^p\big)\, |u_0(t)|^{p-2} u_0(t) \\
&\quad \times W(t,u_0(t))\cdot\big(u_n(t)-u_0(t)\big)\,dt \Big| \\
&\leq 2p\,\delta_0^p \int_{|t|\geq T_0} e^{Q(t)} \big|g'\big(|u_n(t)|^p\big)\big|\, |W(t,u_n(t))|\,dt \\
&\quad + 2p\,\delta_0^p \int_{|t|\geq T_0} e^{Q(t)} \big|g'\big(|u_0(t)|^p\big)\big|\, |W(t,u_0(t))|\,dt \\
&\leq 2p\,\delta_0^p c_3 \int_{\mathbb{R}} e^{Q(t)} \frac{\varepsilon}{p} a(t)\Big(|u_n(t)|^p+|u_0(t)|^p\Big)\,dt \\
&\leq 4\,\delta_0^p\, c_3\, c_2^p\, \varepsilon.
\end{aligned} \tag{47}$$

where $c_3 = \max_{t\in[\frac{\delta_0^p}{2^p},\delta_0^p]} |g'(t)|$. From Lemma 2.2 and (3.5), we obtain

$$|u_n(t)| \leq \|u_n\|_{L^\infty} \leq \eta_\infty \|u_n\| \leq \eta_\infty c_2 = c_4, \quad \forall t\in\mathbb{R},\ \forall n\in\mathbb{N}. \tag{48}$$

and

$$|u_0(t)| \leq \|u_0\|_{L^\infty} \leq \eta_\infty \|u_0\| \leq \eta_\infty c_2 = c_4, \quad \forall t\in\mathbb{R}. \tag{49}$$

Combining (48) and (49), we get for a positive constant c_5

$$\begin{aligned}
&|\nabla W(t,u_n(t))| \leq c_5, \\
&|\nabla W(t,u_0(t))| \leq c_5, \qquad \forall t\in[-T_0,T_0],\ \forall n\in\mathbb{N}.
\end{aligned} \tag{50}$$

and

$$\begin{aligned}
&|W(t,u_n(t))| \leq c_5, \\
&|W(t,u_0(t))| \leq c_5, \qquad \forall t\in[-T_0,T_0],\ \forall n\in\mathbb{N}.
\end{aligned} \tag{51}$$

Since E is continuously embedded in $W_Q^{1,p}([-T_0,T_0],\mathbb{R}^N)$, Sobolev's embedding theorem implies that

$$u_n \to u_0 \quad \text{uniformly on } [-T_0,T_0]. \tag{52}$$

Then, by (50), (52), and the properties of g, we have

$$\begin{aligned}
&\Big| \int_{|t|\leq T_0} e^{Q(t)} g\big(|u_n(t)|^p\big) \\
&\quad \times \nabla W(t,u_n(t))\cdot\big(u_n(t)-u_0(t)\big)\,dt \\
&\quad - \int_{|t|\leq T_0} e^{Q(t)} g\big(|u_0(t)|^p\big) \\
&\quad \times \nabla W(t,u_0(t))\cdot\big(u_n(t)-u_0(t)\big)\,dt \Big| \\
&\leq \int_{|t|\leq T_0} e^{Q(t)} \Big(|\nabla W(t,u_n(t))| \\
&\quad + |\nabla W(t,u_0(t))|\Big)\, |u_n(t)-u_0(t)|\,dt \\
&\leq 2c_5 \int_{|t|\leq T_0} e^{Q(t)}\, |u_n(t)-u_0(t)|\,dt \longrightarrow 0 \\
&\qquad \text{as } n\to\infty.
\end{aligned} \tag{53}$$

so there exists $N_1\in\mathbb{N}$ such that

$$\begin{aligned}
&\Big| \int_{|t|\leq T_0} e^{Q(t)} g\big(|u_n(t)|^p\big) \\
&\quad \times \nabla W(t,u_n(t))\cdot\big(u_n(t)-u_0(t)\big)\,dt \\
&\quad - \int_{|t|\leq T_0} e^{Q(t)} g\big(|u_0(t)|^p\big) \\
&\quad \times \nabla W(t,u_0(t))\cdot\big(u_n(t)-u_0(t)\big)\,dt \Big| \leq \varepsilon, \quad \forall n\geq N_1.
\end{aligned} \tag{54}$$

In addition, from (51), (52), and the properties of g, one gets

$$\begin{aligned}
&\Big| p\int_{|t|\leq T_0} e^{Q(t)} g'\big(|u_n(t)|^p\big)\, |u_n(t)|^{p-2} u_n(t) \\
&\quad \times W(t,u_n(t))\cdot\big(u_n(t)-u_0(t)\big)\,dt \\
&\quad - p\int_{|t|\leq T_0} e^{Q(t)} g'\big(|u_0(t)|^p\big)\, |u_0(t)|^{p-2} u_0(t) \\
&\quad \times W(t,u_0(t))\cdot\big(u_n(t)-u_0(t)\big)\,dt \Big| \leq 2p\,c_5\,\delta_0^{p-1} \\
&\quad \times \sup_{s\in[\delta_0^p/2^p,\delta_0^p]} |g'(s)| \int_{|t|\leq T_0} e^{Q(t)}\, |u_n(t)-u_0(t)|\,dt \\
&\quad \longrightarrow 0 \qquad \text{as } n\to\infty.
\end{aligned} \tag{55}$$

Hence, there exists $N_2\in\mathbb{N}$ such that

$$\begin{aligned}
&\Big| p\int_{|t|\leq T_0} e^{Q(t)} g'\big(|u_n(t)|^p\big)\, |u_n(t)|^{p-2} u_n(t) \\
&\quad \times W(t,u_n(t))\cdot\big(u_n(t)-u_0(t)\big)\,dt \\
&\quad - p\int_{|t|\leq T_0} e^{Q(t)} g'\big(|u_0(t)|^p\big)\, |u_0(t)|^{p-2} u_0(t) \\
&\quad \times W(t,u_0(t))\cdot\big(u_n(t)-u_0(t)\big)\,dt \Big| \\
&\quad \leq \varepsilon, \qquad \forall n\geq N_2.
\end{aligned} \tag{56}$$

Combining (46), (47), (54), and (56), we obtain

$$\begin{aligned}
&\|u_n-u_0\|^p-\left(J'(u_n)-J'(u_0)\right)(u_n-u_0)\\
&=\int_{\mathbb{R}} e^{Q(t)}\, g\left(|u_n(t)|^p\right)\\
&\quad\times\nabla W(t,u_n(t))\cdot\left(u_n(t)-u_0(t)\right)dt\\
&\quad-\int_{\mathbb{R}} e^{Q(t)}\, g\left(|u_0(t)|^p\right)\\
&\quad\times\nabla W(t,u_0(t))\cdot\left(u_n(t)-u_0(t)\right)dt\\
&\quad+p\int_{\mathbb{R}} e^{Q(t)}\, g'\left(|u_n(t)|^p\right)|u_n(t)|^{p-2}u_n(t)\\
&\quad\times W(t,u_n(t))\cdot\left(u_n(t)-u_0(t)\right)dt\\
&\quad-p\int_{\mathbb{R}} e^{Q(t)}\, g'\left(|u_0(t)|^p\right)|u_0(t)|^{p-2}u_0(t)\\
&\quad\times W(t,u_0(t))\cdot\left(u_n(t)-u_0(t)\right)dt\\
&\le\left(8c_2^p+4p\,\delta_0^p c_3 c_2^p+2\right)\varepsilon,\qquad\forall n\ge\max(N_1,N_2).
\end{aligned}\tag{57}$$

This with (3.5) and (3.6) shows that $u_n\to u_0$ in E. Hence J satisfies the (PS)-condition.

Lemma 3.3. Assume that (A) and (W_2) hold, then for all $k\in\mathbb{N}$, there exists an $A_k\in\Gamma_k$ such that $\sup_{u\in A_k}J(u)<0$.

Proof For any fixed $k\in\mathbb{N}$, we divide the interval $I=[a,b]$ into k equal closed subintervals, denoted by I_i for $i=1,\dots,k$. For each i, let $K_i\subset I_i$ be a subset such that K_i has the same center as I_i and has a length of $\frac{b-a}{2k}$. Now, choose k functions $\varphi_i\in C^1(\mathbb{R},\mathbb{R}^N)$ for $i=1,\dots,k$ such that

$$\begin{aligned}
&\operatorname{supp}(\varphi_i)\subset I_i,\quad \operatorname{supp}(\varphi_i)\cap\operatorname{supp}(\varphi_j)=\emptyset,\ \forall i\ne j,\\
&|\varphi_i(t)|\le 1,\qquad |\varphi_i'(t)|\le M,\ \forall t\in\mathbb{R},\\
&|\varphi_i(t)|=1,\qquad \forall t\in K_i.
\end{aligned}\tag{58}$$

where M is a constant independent of i. Let

$$\begin{aligned}
V_k&=\left\{(t_1,\dots,t_k)\in\mathbb{R}^k\ \middle|\ \max_{1\le i\le k}|t_i|=1\right\},\\
X_k&=\left\{\sum_{i=1}^k t_i\varphi_i\ \middle|\ (t_1,\dots,t_k)\in V_k\right\}.
\end{aligned}\tag{59}$$

It is evident that $supp(u)\subset I$ for any $u=\sum_{i=1}^k t_i\varphi_i$. The paper [8] implies that X_k is a closed subset of E with $\gamma(X_k)=k$. By the properties of φ_i, for fixed $k\in\mathbb{N}$, there exists a constant $\beta_k>0$ such that

$$\|u\|\le\beta_k,\qquad\forall u\in X_k.\tag{60}$$

From (W_2), there exist two positive constants η and R such that

$$W(t,x)\ge R|x|^\alpha,\qquad\forall t\in[a,b],\ |x|\le\eta.\tag{61}$$

For every $u=\sum_{i=1}^k t_i\varphi_i\in X_k$, let $s_k\in\left]0,\min\left(\eta,\frac{\delta_0}{2\eta_\infty\beta_k}\right)\right[$. Then, by (3.22), we have

$$\begin{aligned}
|s_k u(t)|^p&\le s_k^p\,\|u\|_{L^\infty}^p\le s_k^p\,\eta_\infty^p\,\|u\|^p\\
&\le\frac{\delta_0^p}{2^p\eta_\infty^p\beta_k^p}\,\eta_\infty^p\beta_k^p=\frac{\delta_0^p}{2^p}.
\end{aligned}\tag{62}$$

which implies that $g(|s_k u(t)|^p)=1$. Moreover, by the properties of φ_i, we obtain

$$\begin{aligned}
|s_k t_i\varphi_i(t)|&=s_k\,|t_i|\,|\varphi_i(t)|\\
&\le s_k\le\eta,\qquad\forall i=1,\dots,k.
\end{aligned}\tag{63}$$

The definitions of V_k and φ_i imply that there exists a $j\in\{1,\dots,k\}$ such that $|t_j|=1$ and $|\psi_j(t)|=1$ for any $t\in K_j$. Now, from $(3.22)-(3.24)$, we get

$$\begin{aligned}
J(s_k u)&=\frac{1}{p}\,\|s_k u\|^p-\int_{\mathbb{R}}e^{Q(t)}W(t,s_k u(t))dt\\
&=\frac{s_k^p}{p}\|u\|^p-\int_a^b e^{Q(t)}W(t,s_k u(t))dt\\
&\le\frac{s_k^p}{p}\beta_k^p-\sum_{i=1}^k\int_{I_i}e^{Q(t)}W(t,s_k t_i\varphi_i(t))dt\\
&\le\frac{s_k^p}{p}\beta_k^p-\int_{K_j}e^{Q(t)}W(t,s_k t_j\psi_j(t))dt\\
&\le\frac{s_k^p}{p}\beta_k^p-R\int_{K_j}e^{Q(t)}\left|s_k t_j\psi_j(t)\right|^\alpha dt\\
&\le\frac{s_k^p}{p}\beta_k^p-m_0Rs_k^\alpha meas(K_j)\\
&=\frac{s_k^p}{p}\beta_k^p-\frac{b-a}{2k}m_0Rs_k^\alpha,\qquad\forall u\in X_k.
\end{aligned}\tag{64}$$

Note that R can be any sufficiently large number, so choose $R\ge\frac{2k\beta_k^p}{m_0(b-a)}s_k^{p-\alpha}$. From (3.25), we have

$$J(s_k u_k) \leq \frac{s_k^p}{p}\beta_k^p - s_k^p\beta_k^p = -\frac{1}{q}s_k^p\beta_k^p < 0, \quad \forall u \in X_k. \tag{65}$$

For each $k \in \mathbb{N}$, let $A_k = s_k X_k$. Then, we have $\gamma(A_k) = \gamma(s_k X_k) = k$, so $A_k \in \Gamma_k$ and $\sup_{u\in A_k} J(u) < 0$. The proof is complete.

Lemmas 3.1 − 3.3 establish that J satisfies all the hypotheses of Lemma 2.4, implying the existence of a sequence of nontrivial critical points $(u_n) \subset E$ of the functional J, where $u_n \to 0$ in E as $n \to \infty$. Therefore, (u_n) forms a sequence of fast homoclinic solutions for the system $(\widetilde{\mathscr{DV}})$. From Lemma 2.2, it follows that $\sup_{t\in\mathbb{R}} |u_n(t)| \to 0$ as $n \to \infty$. Consequently, there exists a positive constant n_0 such that for all $n \geq n_0$, we have $\sup_{t\in\mathbb{R}} |u_n(t)| \leq \delta_0$, where δ_0 is as defined previously. This implies that for all $n \geq n_0$, u_n is a fast homoclinic solution to the system $(\mathscr{DV})$. The proof of Theorem 1.1 is thus complete.

4 Proof of Theorem 1.2.

Let the cut-off function $h \in C^1(\mathbb{R}^+, \mathbb{R}^+)$ be defined such that $h(t) = 1$ for $t \in [0, \frac{\rho}{2}]$, $h(t) = 0$ for $t \geq \rho$, and $-\rho \leq h'(t) < 0$ for $\frac{\rho}{2} < t < \rho$. We define the modified function

$$\begin{aligned}\widehat{W}(t,x) = h(|x|)\,W(t,x) + c\,(1-h(|x|))\,|x|^\sigma,\\ \forall (t,x) \in \mathbb{R}\times\mathbb{R}^N.\end{aligned} \tag{66}$$

where c is defined in (W_4). Next, we introduce the following modified system $(\widehat{\mathscr{DV}})$

$$\begin{aligned}\frac{d}{dt}\Big(|\dot u(t)|^{p-2}\dot u(t)\Big) + q(t)\,\dot u(t) - L(t)\,u(t)\\ + \nabla\widehat{W}(t,u(t)) = 0, \qquad t \in \mathbb{R}.\end{aligned} \tag{67}$$

We then define the variational functional $\widehat{J}$ associated with the modified system as

$$\begin{aligned}\widehat{J}(u) &= \frac{1}{p}\int_{\mathbb{R}} e^{Q(t)}\Big(|\dot u(t)|^p + a(t)\,|u(t)|^p\Big)\,dt\\ &\quad - \int_{\mathbb{R}} e^{Q(t)}\,\widehat{W}(t,u(t))\,dt\\ &= \frac{1}{p}\,\|u\|^p - \int_{\mathbb{R}} e^{Q(t)}\,\widehat{W}(t,u(t))\,dt.\end{aligned} \tag{68}$$

It is well established that $\widehat{J} \in C^1(E,\mathbb{R})$, and for any $u, v \in E$, the derivative of $\widehat{J}$ is given by

$$\begin{aligned}\widehat{J}'(u)v = \int_{\mathbb{R}} e^{Q(t)}\Big(&|\dot u(t)|^{p-2}\dot u(t)\cdot\dot v(t)\\ &+ a(t)\,|u(t)|^{p-2}u(t)\cdot v(t)\Big)\,dt\\ - \int_{\mathbb{R}} &e^{Q(t)}\,\nabla\widehat{W}(t,u(t))\cdot v(t)\,dt.\end{aligned} \tag{69}$$

Furthermore, the critical points of $\widehat{J}$ correspond to fast homoclinic solutions of $(\widehat{\mathscr{DV}})$.

Lemma 4.1. Assume that the conditions (A), (A_ν), (W_1), and (W_4) are satisfied. Then, the functional $\widehat{J}$ is bounded from below and fulfills the (PS)-condition.

Proof From (W_4) and the fact that $W(t,0) = 0$, it follows that

$$|W(t,x)| \leq \frac{c}{\sigma}\,|x|^\sigma \leq c\,|x|^\sigma, \quad \forall |t| \geq R_1 \text{ and } |x| \leq \rho. \tag{70}$$

which, together with the properties of h, gives

$$|\widehat{W}(t,x)| \leq c\,|x|^\sigma, \qquad \forall |t| \geq R_1 \text{ and } |x| \leq \rho. \tag{71}$$

For $\frac{\rho}{2} \leq |x| < \rho$, we have

$$\begin{aligned}\big|\nabla\widehat{W}(t,x)\big| = \bigg|\,&h(|x|)\,\nabla W(t,x) + h'(|x|)\,\frac{x}{|x|}\,W(t,x)\\ &+ c\sigma\big(1-h(|x|)\big)\,|x|^{\sigma-2}x\\ &- c\,h'(|x|)\,|x|^{\sigma-1}x\,\bigg|\\ \leq{}& d\,|x|^{\sigma-1}.\end{aligned} \tag{72}$$

where $d = c\left(1 + \frac{\rho^2}{\sigma} + \sigma + \rho^2\right)$. For $|x| > \rho$, we get

$$|\nabla\widehat{W}(t,x)| = c\,\sigma\,|x|^{\sigma-1}. \tag{73}$$

For $|x| < \frac{\rho}{2}$, it follows that

$$|\nabla\widehat{W}(t,x)| = |\nabla W(t,x)| \leq c\,|x|^{\sigma-1}. \tag{74}$$

Thus, we conclude that

$$|\nabla\widehat{W}(t,x)| \leq d\,|x|^{\sigma-1}, \qquad \forall |t| \geq R_1,\; x \in \mathbb{R}^N. \tag{75}$$

Hence, we can write the functional $\widehat{J}$ as

$$
\begin{aligned}
\widehat{J}(u) &= \frac{1}{p}\|u\|^p - \int_{\mathbb{R}} e^{Q(t)}\,\widehat{W}(t,u(t))\,dt \\
&= \frac{1}{p}\|u\|^p - \int_{\{|t|\geq R_1\}} e^{Q(t)}\,\widehat{W}(t,u(t))\,dt \\
&\quad - \int_{\{|t|\leq R_1\}} e^{Q(t)}\,\widehat{W}(t,u(t))\,dt \\
&= \frac{1}{p}\|u\|^p - \int_{\{|t|\geq R_1,\,|u(t)|\leq\rho\}} e^{Q(t)}\,\widehat{W}(t,u(t))\,dt \\
&\quad - \int_{\{|t|\leq R_1,\,|u(t)|\leq\rho\}} e^{Q(t)}\,\widehat{W}(t,u(t))\,dt \\
&\geq \frac{1}{p}\|u\|^p - c\int_{\{|t|\geq R_1\}} e^{Q(t)}\,|u(t)|^\sigma\,dt - c_6 \\
&\geq \frac{1}{p}\|u\|^p - c\int_{\mathbb{R}} e^{Q(t)}\,|u(t)|^\sigma\,dt - c_6 \\
&= \frac{1}{p}\|u\|^p - c\,\|u\|^\sigma_{L^\sigma_Q} - c_6 \\
&\geq \frac{1}{p}\|u\|^p - c\,\eta^\sigma_\sigma\,\|u\|^\sigma - c_6.
\end{aligned} \tag{76}
$$

where $c_6 = 2R_1\sup_{\{|t|\leq R_1,|x|\leq\rho\}} e^{Q(t)}\left|\nabla\widehat{W}(t,x)\right|$. Since $\sigma < p$, we conclude that $\widehat{J}$ is bounded from below and coercive in E. Next, let $(u_n)\subset E$ be a (PS)-sequence of $\widehat{J}$, i.e., there exists a positive constant c_7 such that

$$
|\widehat{J}(u_n)| \leq c_7, \qquad \widehat{J}'(u_n)\to 0 \text{ as } n\to\infty. \tag{77}
$$

From (76) and (77), we deduce that there exists a positive constant c_8 such that

$$
\|u_n\| \leq c_8, \qquad \forall n\in\mathbb{N}. \tag{78}
$$

Since E is reflexive, (78) implies that (u_n) has a subsequence (denoted by (u_n)) such that

$$
u_n \rightharpoonup u_n \text{ weakly in } E \text{ as } n\to\infty. \tag{79}
$$

Using similar arguments as in Lemma 3.2, we can show that there exists a constant $T_1 \geq R_1$ such that

$$
|u_n(t)| \leq \rho \quad \text{and} \quad |u_0(t)| \leq \rho, \quad \forall |t| \geq T_1,\ \forall n\in\mathbb{N}. \tag{80}
$$

Moreover, we have

$$
u_n \to u_0 \quad \text{uniformly on } [-T_1,T_1]. \tag{81}
$$

$$
\begin{aligned}
&\|u_n-u_0\|^p - \left(\widehat{J}'(u_n)-\widehat{J}'(u_0)\right)(u_n-u_0) \\
&= \int_{\mathbb{R}} e^{Q(t)}\left[\nabla\widehat{W}(t,u_n(t)) - \nabla\widehat{W}(t,u_0(t))\right] \\
&\qquad \times (u_n(t)-u_0(t))\,dt \\
&= \int_{\{|t|\geq T_1\}} e^{Q(t)}\left[\nabla\widehat{W}(t,u_n(t)) - \nabla\widehat{W}(t,u_0(t))\right] \\
&\qquad \times (u_n(t)-u_0(t))\,dt \\
&\quad + \int_{\{|t|\leq T_1\}} e^{Q(t)}\left[\nabla\widehat{W}(t,u_n(t)) - \nabla\widehat{W}(t,u_0(t))\right] \\
&\qquad \times (u_n(t)-u_0(t))\,dt \\
&\leq d\int_{\{|t|\geq T_1\}} e^{Q(t)}\left(|u_n(t)|^{\sigma-1}+|u_0(t)|^{\sigma-1}\right) \\
&\qquad \times |u_n(t)-u_0(t)|\,dt \\
&\quad + c_9\int_{\{|t|\leq T_1\}} e^{Q(t)}\,|u_n(t)-u_0(t)|\,dt \\
&\leq d\left[\left(\int_{\{|t|\geq T_1\}} e^{Q(t)}\,|u_n(t)|^{\frac{(\sigma-1)p}{p-2}}\,dt\right)^{\frac{p-2}{p}}\right. \\
&\quad \left. + \left(\int_{\{|t|\geq T_1\}} e^{Q(t)}\,|u_0(t)|^{\frac{(\sigma-1)p}{p-2}}\,dt\right)^{\frac{p-2}{p}}\right] \\
&\qquad \times \left(\int_{\{|t|\geq T_1\}} e^{Q(t)}\,|u_n(t)-u_0(t)|^{\frac{p}{2}}\,dt\right)^{\frac{2}{p}} \\
&\quad + c_9\int_{\{|t|\leq T_1\}} e^{Q(t)}\,|u_n(t)-u_0(t)|\,dt \\
&\leq d\left(\|u_n\|^{\sigma-1}_{L_Q^{\frac{(\sigma-1)p}{p-2}}} + \|u_0\|^{\sigma-1}_{L_Q^{\frac{(\sigma-1)p}{p-2}}}\right)\|u_n-u_0\|_{L_Q^{\frac{p}{2}}} \\
&\quad + c_9\int_{\{|t|\leq T_1\}} e^{Q(t)}\,|u_n(t)-u_0(t)|\,dt \\
&\leq 2d\,\eta^{\sigma-1}_{\frac{(\sigma-1)p}{p-2}}\,c_8^{\sigma-1}\,\|u_n-u_0\|_{L_Q^{\frac{p}{2}}} \\
&\quad + c_9\int_{\{|t|\leq T_1\}} e^{Q(t)}\,|u_n(t)-u_0(t)|\,dt \\
&\quad \longrightarrow 0 \qquad \text{as } n\to\infty.
\end{aligned} \tag{82}
$$

Hence, $u_n \to u_0$ in E as $n\to\infty$, that is $\widehat{J}$ satisfies the (PS)-condition.

By (W_1), the fact that $W(t,0)=0$ and the properties of h, it follows that $\widehat{J}$ is even and $\widehat{J}(0)=0$. Combining this with Lemma 3.2 and Lemma 4.1, we conclude that $\widehat{J}$ satisfies all the conditions of Lemma 2.4. Therefore, $\widehat{J}$ has a sequence (u_n) of nontrivial critical points that converges to 0 as $n\to\infty$ in E. As a result, (u_n) is a sequence of fast homoclinic solutions for $(\widehat{\mathscr{DV}})$. By Lemma 2.2, we deduce that $\sup_{t\in\mathbb{R}} |u_n(t)| \to 0$ as $n \to \infty$. Thus, there exists a positive constant n_0 such that for all $n \geq n_0$, $\sup_{t\in\mathbb{R}} |u_n(t)| \leq \rho$, where ρ is as defined earlier. Hence, for all $n \geq n_0$, u_n is a fast homoclinic solution of $(\mathscr{D}\mathscr{V})$. This completes the proof of Theorem 1.2.

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