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The existence of solutions for two types of nonlinear equations on locally finite graphs

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Abstract In this paper, we focus on connected locally finite graphs $G=(V,E)$. First we assume that there are two constants μ_0 and ω_0, which make the measure function and symmetric weight function satisfy $\mu(x)\geq\mu_0$ $\forall x\in V$, $\omega_{xy}\geq\omega_0$, $\forall xy\in E$. Based on this assumption, we obtain two interesting embedding theorems on finite graphs: $W_0^{1,2}(B_k)\hookrightarrow L^p(B_k)$, $W^{1,2}(B_k)\hookrightarrow L^p(B_k)$. Although their inclusion relations are obvious on finite graphs, here we mainly give the control relations under the same control coefficient. Secondly, Δ is the Laplace operator on a general graph. Due to Lin and Yang (2022), using calculus of variations from local to global, we establish the existence of solutions to the exponential power type nonlinear Schrodinger equation, says $-\Delta u+hu=fue^{u^2}+g$, $x\in V$, and the existence of solutions for fractional nonlinear mean field equations, says $-\Delta u+hu=\frac{ge^u}{\int_V ge^u d\mu}+\frac{f}{u+m}$, $x\in V$. When f, g and h satisfy some conditions, we prove the existence of non explicit solutions for the above two kinds of equations in a specific space.

1 Introduction

The nonlinear Schrödinger equation and the mean field equation are two crucial and widely studied models in mathematical physics. The classical nonlinear Schrödinger equation describes a series of profound phenomena in continuous Euclidean space, from the transmission of light waves in nonlinear optical fibers to the dynamics of macroscopic quantum wave functions in Bose Einstein condensates[1, 2].Its mathematical theory and physical applications have been developed to a very mature level. At the same time, the mean field equation, such as the Liouville equation derived from mean field theory in statistical physics or geometric analysis[3], plays a central role in understanding the collective behavior of multi-body systems, vortex point distribution, and problems in conformal geometry.

In recent years, significant progress has been made in the study of partial differential equations on graphs.We can refer to article[4–16] and the references in it for details. As a discrete extension of Euclidean spaces and Riemannian manifolds, the Laplacian operator and its related equations on the graphs have attracted widespread attention. Grigor'yan, Lin, and Yang[17–19] successfully solved the existence problem of solutions for several types of elliptical equations on graphs by using variational methods, including classic problems such as the Kazdan-Warner equation, Yamabe equation, and Schrödinger equation. Subsequently, as for certain nonlinear Schrödinger equation on locally finite graphs, Zhang and Zhao[20] have obtained non trivial solutions. Fabio Punzo and Marcello Svagna[21] explored the uniqueness problem of solutions to the Schrödinger equation on an infinite graph,with a focus on the case where potential energy tends to zero at infinity. Yang and Zhao[22] first applied the quality constrained variational method system to NLS problems on graphs. In terms of research methods, Sun and Wang[23] innovatively applied the Brouwer degree theory to prove the existence of solutions to the Kazdan- Warner equation on connected finite graphs from another perspective. Liu[24] conducted similar research on the mean field equation. In addition,Huang Wang and Yang[25] studied the mean field equation on finite graphs and its relationship with the relativistic Abel Chern Simons model.

The graph-based NLSE finds applications across multiple physical domains:

In the field quantum networks, Modeling Bose-Einstein condensates in optical lattice potentials, where the discrete nonpolynomial NLSE describes self-attractive BECs in combined trap geometries. On-site collapse phenomena

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and soliton stability follow Vakhitov-Kolokolov criteria[26]; In the field of plasma physics, Network-structured plasma systems support novel wave patterns describable by graph NLSE, with applications to fusion device modeling[27]; For 2D Turbulence, Neri's mean field equation describes vortex statistics under stochastic circulation assumptions, with mass quantization phenomena for blow-up sequences[28]; As for social/epidemiological dynamics, Network-structured population models use mean field approaches for phase transition analysis[29, 30].

In the latest research progress, Lin and Yang[12] developed a variational method from local to global on local finite graphs, Qiu and Liu[31] studied the Schrödinger equation with fe^u as the nonlinear term. Based on their research ideas, the current research focus has shifted to exploring the exponential power type fue^{u^2} nonlinear Schrödinger equation and fractional term $\frac{f}{u+m}$ nonlinear mean field equation. Simultaneously we provide two embedding theorems on finite graphs. These works laid an important foundation for the theory of partial differential equations in discrete spaces.

We first declare some annotations and concepts about graphs. Let V is a set of vertices, $E = \{xy|x, y \in V\ x \sim y\}$ where $x \sim y$ represents the connection between x and y. Take $G = (V, E)$ is a graph, in this paper, we discuss connected locally finite graphs with symmetric weights and positive finite measures, we always assume that G satisfies the following conditions $(a) - (d)$.

(a) (Locally finite) For any $x \in V$, there exist only finite vertices $y \in V$ such that $xy \in E$.

(b) (Connected) For any $x, y \in V$, there exist finite edges connecting x and y.

(c) (Symmetric weight) For any $x, y \in V$, let $\omega : V \times V \to \mathbb{R}$ be a positive symmetric weight, i.e. $\omega_{xy} > 0$ and $\omega_{xy} = \omega_{yx}$.

(d) (Positive finite measure) $\mu : V \to \mathbb{R}^+$ with $x \mapsto \mu(x)$ is a measure function.

Take $C(V)$ as the space composed of all real valued functions on the graph. Regarding any function $u \in C(V)$ on the graph, its Laplacian operator is defined as follows

$$\Delta u(x) = \frac{1}{\mu(x)} \sum_{y \sim x} \omega_{xy}(u(y) - u(x)). \tag{1}$$

In addition, we immediately provide the definition of the gradient modulus of u

$$|\nabla u|(x) = \left(\frac{1}{2\mu(x)} \sum_{y \sim x} \omega_{xy}(u(y) - u(x))^2 \right)^{\frac{1}{2}}. \tag{2}$$

We define the integral of function $u \in C(V)$ as

$$\int_V f d\mu = \sum_{x \in V} \mu(x) f(x). \tag{3}$$

For $\forall p \in [1, +\infty)$, the Lebesgue space $L^p(V) \triangleq \{u| \int_V |u|^p d\mu < +\infty\}$, and the norm on it is

$$\|u\|_{L^p(V)} = (\sum_{x \in V} |u(x)|^p \mu(x))^{\frac{1}{p}}, \tag{4}$$

when $p = +\infty$,

$$\|u\|_\infty = \sup_{x \in V} |u(x)|. \tag{5}$$

For any $x, y \in V$, since graph G is connected, there exists a shortest path γ connecting x and y. The distance between x and y is defined by $\rho(x,\ y)$, which means the number of edges belonging to the shortest path γ. That is, if $xy \in E$, then $\rho(x,\ y) = 1$, if $xy \in E$, without loss of generality, we may choose a shortest path $\gamma = \{x_1,\ x_2,\ \ldots,\ x_{k+1}\}$ connecting x and y, then $\rho(x,\ y) = k$. Take a certain point $O \in V$, for O, establish a distance function as follows

$$\rho(x) = \rho(x,\ O). \tag{6}$$

The opening ball with O as the center and radius k is denoted by

$$B_k = \{x \in V :\ \rho(x) < k\}, \tag{7}$$

and the boundary of B_k is written as

$$\partial B_k = \{x \in V :\ \rho(x) = k\}. \tag{8}$$

For any fixed k, Grigor'yan et al. [18] defined the Sobolev space $W_0^{1,2}(B_k)$ and its norm by

$$W_0^{1,2}(B_k) = \left\{ u:\ B_k \cup \partial B_k \to \mathbb{R} \ \middle|\ u|_{\partial B_k} = 0, \int_{B_k} |\nabla u|^2 d\mu < +\infty \right\}, \tag{9}$$

and

$$\|u\|_{W_0^{1,2}(B_k)} = \left(\int_{B_k} |\nabla u|^2 d\mu \right)^{\frac{1}{2}}. \tag{10}$$

Next we provide another important Sobolev space $W^{1,2}(V)$ and its norm, which are defined by

$$W^{1,2}(V) = \left\{ u:\ V \to \mathbb{R} \ \middle|\ \int_V (|\nabla u|^2 + u^2) d\mu < +\infty \right\} \tag{11}$$

and

$$\|u\|_{W^{1,2}(V)} = \left(\int_V (|\nabla u|^2 + u^2) d\mu \right)^{\frac{1}{2}}. \tag{12}$$

The above two spaces are both Hilbert spaces.

Let $h(x) \geq h_0 > 0$ for all $x \in V$, we define a space of functions

$$\mathscr{H} = \left\{ u \in W_0^{1,2}(V) :\ \int_V (|\nabla u|^2 + hu^2) d\mu < \infty \right\}, \tag{13}$$

with a norm

$$\|u\|_{\mathscr{H}} = \left(\int_V (|\nabla u|^2 + hu^2) d\mu \right)^{\frac{1}{2}}. \tag{14}$$

It is clear that $\mathscr{H}$ is a Hilbert space with the inner product

$$\langle u,\ v \rangle_{\mathscr{H}} = \int_V (\nabla u \cdot \nabla v + huv) d\mu, \quad \forall\ u,\ v \in \mathscr{H}. \tag{15}$$

where $\nabla u \cdot \nabla v$ is defined as

$$\nabla u \cdot \nabla v = \frac{1}{2\mu(x)} \sum_{y \sim x} \omega\, (u(y) - u(x))(v(y) - v(x)). \tag{16}$$

When $\omega_{xy} > \omega_0\ \forall xy \in E$, Lin and Yang[32] proposed a more general embedding theorem.

In this paper, we will consider the following exponential power type the following nonlinear Schrödinger equation on locally finite graph, says

$$\begin{cases} -\Delta u + hu = fue^{u^2} + g, \text{ in } V, \\ u \in \mathscr{H}, \end{cases} \tag{17}$$

and the following fractional term nonliear mean field equation on locally finite graph, says

$$\begin{cases} -\Delta u + hu = \frac{ge^u}{\int_V ge^u d\mu} + \frac{f}{u+m}, \text{ in } V, \\ u \in \mathscr{H} \bigcap L^\infty(V), \end{cases} \tag{18}$$

where Δ is the Laplacian operator given as in (1), and $\mathscr{H}$ is defined as in (14).

2 Notations and main results

Theorem 1 *(Embedding theorem I) Let G=(V, E) be a graph satisfying conditions (a)–(d). B_k is the opening ball with O as the center and a radius of k, Among them,$\forall O \in V$, $\forall k \in \mathbb{Z}^+$, $\rho(x)$ is the distance function with respect to O on B_k. At the same time, it satisfies the following properties:*
(1) $\omega_{xy} \geq \omega_0 > 0$; (2) $h(x) \geq h_0 > 0$; (3) $p > 0$ is a constant Then, for $\forall u \in W_0^{1,2}(B_k)$, we have $\|u\|_{L^p(B_k)} \leq C\|u\|_{W_0^{1,2}(B_k)}$.

Theorem 2 *(Embedding theorem II) Let G=(V, E) be a graph satisfying conditions (a)–(d). B_k is the opening ball with o as the center and a radius of k, Among them,$\forall O \in V$, $\forall k \in \mathbb{Z}^+$, If it meets the two conditions of its subordinates:*
(1) $\mu(x) \geq \mu_0 > 0$ established for any $x \in V$; (2) $0 < q \leq \infty$
Then for $\forall u \in W^{1,2}(B_k)$, we have $\|u\|_{L^q(B_k)} \leq C\|u\|_{W^{1,2}(B_k)}$. Among them, C is related to B_k, μ_0, q.

Theorem 3 *(Conclusion on the Existence of Solutions)*

$$\begin{cases} -\Delta u + hu = fue^{u^2} + g, \ in\ V, \\ u \in \mathscr{H}. \end{cases} \tag{19}$$

If the equation satisfies the following conditions:

(1) $h(x) \geq h_0 > 0$;

(2) $-h_0 < -M < f < 0$;

(3) $f \in L^1(V)$; (4) $g \in L^2(V)$.

We have that the equation (17) has a solution in $\mathscr{H}$.

we present an embedding theorem 4 that will be used , which has already been proven in [31].

Theorem 4 *(Embedding theorem [31]) Let G=(V, E) be a graph satisfying conditions (a)–(d). For any* $u \in W_0^{1,2}(B_k)$ *and any* $1 \le q \le \infty$*, there exists a positive constant C depending only on q,* h_0 *and* B_k *such that*

$$\|u\|_{L^q(V)} \le C\|u\|_{W_0^{1,2}(B_k)}. \tag{20}$$

Theorem 5 *(Conclusion on the Existence of Solutions) If the equation (18) satisfies the following six conditions*

(1). *G is a connected locally finite graph*

(2). $\mu(x) \ge \mu_0 > 0$

(3). $h(x) \ge a_0 > 0$

(4). $g \ge 0$, $g \in L^1(V)$ *and* $\forall O \in V$, $\forall l > 1, g \not\equiv 0$ *in* B_l

(5). $f \in L^q(V)$, $q \in [1,2]$, *and* $f \ge 1$

(6). $m \in \mathbb{R}$, $0 <| m |< 1$

then the equation (18) has a solution.

3 Proof of Theorem 1-3

Let's first prove Theorem 1

Proof For $\forall O \in V$, We establish a distance function for point O, denoted as $\rho(x)$. $\forall k \in \mathbb{Z}^+$, We take the opening ball with O as the center and k as the radius, it is recorded as $B_k = \{x \in V, \rho(x,O) < k\}$. We will discuss on the arbitrary opening ball B_k. $\forall x \in B_k$, $\rho(x,O)$ represents the distance from a point in B_k to a fixed point O, The shortest path is labeled according to the direction from that point to the fixed point, and we have $\gamma = \{x_1, x_2, x_3, \ldots, x_{m+1}\}$, Among them, $x = x_1$, $O = x_{m+1}$, x_i is the point adjacent to x_{i+1}, and $\rho(x,O) = m$. According to the definition of integral on finite graph, we can naturally know that $\rho(x) \in L^p(B_k)$. In the following proof, we will show that $\|\rho\|_{L^p(B_k)}$ constitutes the control coefficient.

In [31],Qiu and Liu have proven that $\|u\|_{W_0^{1,2}(B_k)} = (\int_{B_k}(| \nabla u |^2 + hu^2)d\mu)^{\frac{1}{2}}$ is the equivalent norm of the original norm (10). We will use the newly defined norm to discuss the following proof.

$\forall u \in W_0^{1,2}(B_k)$, due to the property of opening the ball with O as the center and radius k, the node of the shortest path is still taken in B_k, we take the node interpolation on the shortest path for processing. $| u(x) |=| u(x_1) |=| u(x_1)-u(x_2)+u(x_2)-\ldots-u(x_{m+1})+u(x_{m+1}) |\le| u(x_1)-u(x_2) | + | u(x_2) - u(x_3) | \ldots + | u(x_m) - u(x_{m+1}) | + | u(O) |$.

Firstly, let's establish the relationship between $| u(x) |$ and $\|u\|_{W_0^{1,2}(B_k)}$ below.

$$\begin{aligned}
\|u\|^2_{W_0^{1,2}(B_k)} &= \int_{B_k}(| \nabla u |^2 + hu^2)d\mu \\
&= \int_{B_k} | \nabla u |^2 \, d\mu + \int_{B_k} hu^2 d\mu \\
&= \sum_{x\in B_k} \mu(x) \cdot \frac{1}{2\mu(x)} \sum_{y\sim x} \omega_{xy}(u(y) \\
&\quad - u(x))^2 + \sum_{x\in B_k} \mu(x)h(x)u^2(x) \\
&\ge \sum_{x\in B_k} \mu(x)h(x)u^2(x) \\
&\ge h_0 \sum_{x\in B_k} \mu(x)u^2(x) \\
&\ge h_0\mu(O)u^2(O).
\end{aligned} \tag{21}$$

According to the expansion above, we have $| u(O) |\le \frac{1}{\sqrt{h_0\mu(O)}}\|u\|_{W_0^{1,2}(B_k)}$. Regarding

$$\sum_{i=1}^{m} | u(x_i) - u(x_{i+1}) |\le m \max_{1\le i\le m} | u(x_i) - u(x_{i+1}) | . \tag{22}$$

Since $\omega_{xy} \ge \omega_0 > 0$ holds true for $\forall xy \in E$, so we naturally have that $\frac{\omega_{xy}}{\omega_0} \ge 1$, $\sqrt{\frac{\omega_{xy}}{\omega_0}} \ge 1$, therefore, which leads to $| u(x_i) - u(x_{i+1}) |\le \sqrt{\frac{\omega_{x_ix_{i+1}}}{\omega_0}} | u(x_i) - u(x_{i+1}) |$, And we can obtain that

$$\begin{aligned}
&\max_{1\le i\le m} |u(x_i) - u(x_{i+1}) | \\
&\quad\le \max_{1\le i\le m} \sqrt{\frac{\omega_{x_ix_{i+1}}}{\omega_0}} | u(x_i) - u(x_{i+1}) | \\
&\quad= \frac{1}{\sqrt{\omega_0}} \max_{1\le i\le m} \sqrt{\omega_{x_ix_{i+1}}} | u(x_i) - u(x_{i+1}) |,
\end{aligned} \tag{23}$$

Inserting (23) into (22), we have that the original formula $\le \frac{m}{\sqrt{\omega_0}} \max_{1\le i\le m} \sqrt{\omega_{x_ix_{i+1}}} \; | u(x_i) - u(x_{i+1}) |$, where $m = \rho(x)$.

Next, we will establish the relationship between $\max\limits_{1\le i\le m}\sqrt{\omega_{x_ix_{i+1}}}\mid u(x_i)-u(x_{i+1})\mid$ and $W_0^{1,2}(B_k)$ norms to discover the connection between them.

Let's assume that $\max\limits_{1\le i\le m}\sqrt{\omega_{x_ix_{i+1}}}\mid u(x_i)-u(x_{i+1})\mid = \sqrt{\omega_{x_mx_{m+1}}}\mid u(x_m)-u(x_{m+1})\mid$, due to

$$\|u\|_{W_0^{1,2}(B_k)}=\left(\int_{B_k}\left(|\nabla u|^2+h\,u^2\right)\,d\mu\right)^{1/2}. \tag{24}$$

we have

$$\begin{aligned}\|u\|^2_{W_0^{1,2}(B_k)}&=\int_{B_k}(\mid\nabla u\mid^2+hu^2)d\mu\\&\ge\int_{B_k}\mid\nabla u\mid^2\,d\mu\\&=\sum_{x\in B_k}\mu(x)\cdot\frac{1}{2\mu(x)}\\&\quad\times\sum_{y\sim x}\omega_{xy}(u(y)-u(x))^2\\&=\frac{1}{2}\sum_{x\in B_k,y\sim x}\omega_{xy}(u(y)-u(x))^2\\&\ge\frac{1}{2}\omega_{x_mx_{m+1}}(u(x_m)-u(x_{m+1}))^2\end{aligned}\tag{25}$$

Then, we get $\|u\|^2_{W_0^{1,2}(B_k)}\ge\frac{1}{2}\omega_{x_mx_{m+1}}(u(x_m)-u(x_{m+1}))^2$, $\sqrt{2}\|u\|_{W_0^{1,2}(B_k)}\ge\sqrt{\omega_{x_mx_{m+1}}}\mid u(x_m)-u(x_{m+1})\mid$. Which leads to $\sum\limits_{i=1}^{m}\mid u(x_i)-u(x_{i+1})\mid\le\frac{m\sqrt{2}}{\sqrt{\omega_0}}\|u\|_{W_0^{1,2}(B_k)}$. So far, we have

$$\begin{aligned}\mid u(x)\mid&\le\frac{m\sqrt{2}}{\sqrt{\omega_0}}\|u\|_{W_0^{1,2}(B_k)}+\frac{1}{\sqrt{h_0\mu(O)}}\|u\|_{W_0^{1,2}(B_k)}\\&=(\frac{m\sqrt{2}}{\sqrt{\omega_0}}+\frac{1}{\sqrt{h_0\mu(O)}})\|u\|_{W_0^{1,2}(B_k)},\end{aligned}\tag{26}$$

which is established for $\forall x\in B_k$.

Finally, let's establish the relationship between $W_0^{1,2}(B_k)$ norm and $L^p(B_k)$ norm below.

We have known that $\rho(x,O)\in L^p(B_k)$. From the connectivity, we have obviously obtained that $\forall y\in B_k, x\ne y$ we have $\rho(x,y)\ge 1$. Below we investigate $\|u\|_{L^p(B_k)}$.

$$\begin{aligned}\|u\|_{L^p(B_k)}=&(\int_{B_k}\mid u(x)\mid^p\,d\mu)^{\frac{1}{p}}\le\\&(\int_{B_k}(\frac{\rho(x)\sqrt{2}}{\sqrt{\omega_0}}\\&\quad+\frac{1}{\sqrt{h_0\mu(O)}})^p\|u\|^p_{W_0^{1,2}(B_k)}d\mu)^{\frac{1}{p}}\\&=\|u\|_{W_0^{1,2}(B_k)}(\int_{B_k}(\frac{\rho(x)\sqrt{2}}{\sqrt{\omega_0}}\\&\quad+\frac{1}{\sqrt{h_0\mu(O)}})^pd\mu)^{\frac{1}{p}}\\&\le\|u\|_{W_0^{1,2}(B_k)}((\int_{B_k}(\frac{\rho(x)\sqrt{2}}{\sqrt{\omega_0}})^pd\mu)^{\frac{1}{p}}\\&\quad+(\int_{B_k}(\frac{1}{\sqrt{h_0\mu(O)}})^pd\mu)^{\frac{1}{p}})\\&=\|u\|_{W_0^{1,2}(B_k)}(\frac{\sqrt{2}}{\sqrt{\omega_0}}\|\rho\|_{L^p(B_k)}\\&\quad+\frac{1}{\sqrt{h_0\mu(O)}}(\int_{B_k}1d\mu)^{\frac{1}{p}}).\end{aligned}\tag{27}$$

Where $\|1\|^p_{L^p(B_k)}=\sum\limits_{x\in B_k}\mu(x)$. In view of $\rho(x)\ge 1$, one has $\mid\rho(x)\mid^p\ge 1$, $\mu(x)\mid\rho(x)\mid^p\ge\mu(x)$, $\sum\limits_{x\in B_k\setminus\{O\}}\mu(x)\mid\rho(x)\mid^p\ge\sum\limits_{x\in B_k\setminus\{O\}}\mu(x)$. And thus, we obtain the following comparison expression, which says

$$\begin{aligned}\|1\|_{L^p(B_k)}&\le(\sum_{x\in B_k\setminus\{O\}}\mu(x)\mid\rho(x)\mid^p+\mu(O))^{\frac{1}{p}}\\&\le 2^{\frac{1}{p}}\cdot max\{(\sum_{x\in B_k}\mu(x)\mid p\mid^p)^{\frac{1}{p}},\ \mu(O)^{\frac{1}{p}}\}\\&\le 2^{\frac{1}{p}}(\|\rho\|_{L^p(B_k)}+\mu(O)^{\frac{1}{p}}).\end{aligned}\tag{28}$$

Based on the above discussion, one has

$$\begin{aligned}\|u\|_{L^p(B_k)}\le&\|u\|_{W_0^{1,2}(B_k)}(\frac{\sqrt{2}}{\sqrt{\omega_0}}\|\rho\|_{L^p(B_k)}\\&\quad+\frac{2^{\frac{1}{p}}}{\sqrt{h_0\mu(O)}}(\|\rho\|_{L^p(B_k)}+\mu(O)^{\frac{1}{p}}))\\&=\|u\|_{W_0^{1,2}(B_k)}((\frac{\sqrt{2}}{\sqrt{\omega_0}}\\&\quad+\frac{2^{\frac{1}{p}}}{\sqrt{h_0\mu(O)}})\|\rho\|_{L^p(B_k)}\\&\quad+\frac{2^{\frac{1}{p}}}{\sqrt{h_0\mu(O)}}\mu(O)^{\frac{1}{p}}),\end{aligned}\tag{29}$$

where the coefficient only depends on ω_0, p, $\mu(O)$, h_0, B_k. By now, we have completed the proof of Theorem 1.

In the following narrate, we will provide the proof of Theorem 2.

Proof The $W^{1,2}(B_k)$ norm of u is denoted as (12), and we investigate $\|u\|^2_{W^{1,2}(B_k)}$.
When $o < p < \infty$,

$$\begin{aligned}\|u\|^2_{W^{1,2}(B_k)} &= \int_{B_k} \mid \nabla u \mid^2 + u^2 d\mu \\ &= \int_{B_k} \mid \nabla u \mid^2 d\mu + \int_{B_k} u^2 d\mu \\ &= \sum_{x\in B_k} \mu(x) \cdot \frac{1}{2\mu(x)} \sum_{y\sim x} \omega_{xy}(u(y) \\ &\quad - u(x))^2 + \sum_{x\in B_k} \mu(x) u^2(x) \\ &\geq \sum_{x\in B_k} \mu(x) u^2(x) \\ &\geq \mu_0 \sum_{x\in B_k} u^2(x) \\ &\geq \mu_0 u^2(x).\end{aligned} \tag{30}$$

The above unequal relationship holds for $\forall x \in B_k$. Which will immediately lead to $\mid u(x) \mid \leq \sqrt{\frac{1}{\mu_0}} \|u\|_{W^{1,2}(B_k)}$. When $p = \infty$, $sup\{\mid u \mid, x \in B_k\} \leq \sqrt{\frac{1}{\mu_0}} \|u\|_{W^{1,2}(B_k)}$, Thus, we obtain that $\|u\|_{L^\infty(B_k)} \leq \sqrt{\frac{1}{\mu_0}} \|u\|_{W^{1,2}(B_k)}$. Next, we will investigate L^p norm of u.

$$\begin{aligned}\|u(x)\|^p_{L^p(B_k)} &= \sum_{x\in B_k} \mu(x) \mid u(x) \mid^p \\ &\leq \sum_{x\in B_k} \mu(x) \frac{1}{\sqrt{\mu_0}^p} \|u\|^p_{W^{1,2}(B_k)} \\ &= \frac{1}{\sqrt{\mu_0}^p} \|u\|^p_{W^{1,2}(B_k)} \sum_{x\in B_k} \mu(x) \\ &= \frac{1}{\sqrt{\mu_0}^p} \|u\|^p_{W^{1,2}(B_k)} Vol(B_k).\end{aligned} \tag{31}$$

According to the derivation above, we get $\|u(x)\|_{L^p(B_k)} \leq Vol(B_k)^{\frac{1}{p}} \frac{1}{\sqrt{\mu_0}} \|u\|_{W^{1,2}(B_k)}$. And then, we have completed the proof of Theorem 2.

Finally, we will prove the existence conditions of the solution and provide a proof of Theorem 3.

Proof We fix $O \in V$,taking the distance function $\rho(x)$.We take the opening ball $B_k = \{x \in V : \rho(x) < k\}$ with O as the center and radius k. Discussing equations at the opening ball.

$$-\Delta u + hu - fue^{u^2} - g = 0. \tag{32}$$

Its variational energy functional is

$$\begin{aligned}J_k(u) = &\frac{1}{2} \int_{B_k} (\mid \nabla u \mid^2 + hu^2) d\mu \\ &- \frac{1}{2} \int_{B_k} fe^{u^2} d\mu - \int_{B_k} gu d\mu.\end{aligned} \tag{33}$$

Now let's find a lower bound for it. In view of $e^{u^2} \geq u^2 + 1$ and $-h_0 < -M < f < 0$, we have $fe^{u^2} \leq fu^2 + f$,

$$\mid fe^{u^2} \mid \leq \mid fu^2 + f \mid \leq \mid fu^2 \mid + \mid f \mid < Mu^2 + \mid f \mid . \tag{34}$$

Combining the above equation (34), we naturally obtain that

$$\begin{aligned}\frac{1}{2} \int_{B_k} fe^{u^2} d\mu &\leq \frac{1}{2} \int_{B_k} \mid fe^{u^2} \mid d\mu \\ &< \frac{1}{2} \int_{B_k} Mu^2 + \mid f \mid d\mu \\ &\leq \frac{M}{2} \int_{B_k} u^2 d\mu + \frac{1}{2} \int_V \mid f \mid d\mu,\end{aligned} \tag{35}$$

wherein, $f \in L^1(V)$. Due to $h(x) \geq h_0 > 0$, one has $\frac{h(x)}{h_0} \geq 1$ and $\frac{h(x)}{h_0} u^2(x) \geq u^2(x)$. In addition $\frac{|\nabla u|^2}{h_0} > 0$, so we have

$$\begin{aligned}&\frac{h}{h_0} u^2 + \frac{\mid \nabla u \mid^2}{h_0} \geq u^2, \\ &\int_{B_k} u^2 d\mu \leq \frac{1}{h_0} \int_{B_k} hu^2 + \mid \nabla u \mid^2 d\mu.\end{aligned} \tag{36}$$

It following from aforementioned (35) (36), one immediately has

$$\begin{aligned}\int_{B_k} f\, e^{u^2} d\mu < &\frac{M}{2h_0} \int_{B_k} \left(h\, u^2 + |\nabla u|^2\right) d\mu \\ &+ \frac{1}{2} \int_V |f|\, d\mu.\end{aligned} \tag{37}$$

Next, we will handle the term $\int_{B_k} gu\, d\mu$.

$$\int_{B_k} g\,u\,d\mu \le \int_{B_k} |g\,u|\,d\mu \le \|g\|_{L^2(B_k)}\,\|u\|_{L^2(B_k)} \le \|g\|_{L^2(V)}\Big(\int_{B_k} u^2\,d\mu\Big)^{1/2}. \tag{38}$$

Because $h \ge h_0 > 0$, one has $\frac{h}{h_0} \ge 1$, $\frac{h}{h_0}u^2 \ge u^2$. Besides, $\frac{|\nabla u|^2}{h_0} > 0$, which implies $\int_{B_k} u^2 d\mu \le \frac{1}{h_0}\int_{B_k} \mid \nabla u \mid^2 + hu^2 d\mu$, and $\|u\|_{L^2(B_k)} \le \frac{1}{\sqrt{h_0}}\|u\|_{W_0^{1,2}(B_k)}$. And then, we have $\int_{B_k} gud\mu \le \|g\|_{L^2(V)} \cdot \frac{1}{\sqrt{h_0}}\|u\|_{W_0^{1,2}(B_k)}$.

Using the Young's inequality, it can be obtained that $\int_{B_k} gud\mu \le \frac{\varepsilon}{h_0}\|g\|^2_{L^2(V)} + \frac{1}{4\varepsilon}\|u\|^2_{W_0^{1,2}(B_k)}$, taking $\varepsilon = \frac{h_0}{h_0-M}$, then we get $\frac{1}{2} - \frac{M}{2h_0} - \frac{1}{4\varepsilon} > 0$. So far, we have obtained that

$$\begin{aligned} J_k(u) &= \frac{1}{2}\int_{B_k} (\mid \nabla u \mid^2 + hu^2) d\mu \\ &\quad - \frac{1}{2}\int_{B_k} f e^{u^2} d\mu - \int_{B_k} gud\mu \\ &> \frac{1}{2}\int_{B_k} \mid \nabla u \mid^2 + hu^2 d\mu \\ &\quad - \frac{M}{2h_0}\int_{B_k} \mid \nabla u \mid^2 + hu^2 d\mu \\ &\quad - \frac{1}{2}\int_V \mid f \mid d\mu \\ &\quad - \frac{1}{h_0 - M}\|g\|^2_{L^2(V)} - \frac{h_0 - M}{4h_0}\|u\|^2_{W_0^{1,2}(B_k)}, \end{aligned} \tag{39}$$

where $\frac{1}{2} - \frac{M}{2h_0} - \frac{1}{4\varepsilon} > 0$, so we have $J_k(u) \ge -\frac{1}{2}\|f\|_{L^1(V)} - \frac{1}{h_0-M}\|g\|^2_{L^2(V)}$, which holds true for $\forall u \in W_0^{1,2}(B_k)$.

On openning balls with different radii, variational functionals have the same lower bound, and if there is a lower bound, there is an infimum. We mark $\Lambda_k = \inf_{u \in W_0^{1,2}(B_k)} J_k(u)$. Because

$$\begin{aligned} J_k(0) &= \frac{1}{2}\int_{B_k} \mid \nabla 0 \mid^2 + 0^2 d\mu \\ &\quad - \int_{B_k} f e^0 d\mu - \int_{B_k} g \cdot 0 d\mu \\ &= -\int_{B_k} f d\mu \\ &= \int_{B_k} \mid f \mid d\mu \\ &\le \int_V \mid f \mid d\mu = \|f\|_{L^1(V)}, \end{aligned} \tag{40}$$

and $0 \in W_0^{1,2}(B_k)$, we can see from the definition of the infimum that $\Lambda_k \le J_k(0) \le \|f\|_{L^1(V)}$. Thus, one has

$$-\frac{1}{h_0 - M}\|g\|^2_{L^2(V)} - \frac{1}{2}\|f\|_{L^1(V)} \le \Lambda_k \le \|f\|_{L^1(V)}. \tag{41}$$

Below we will explain that the infimum can be reached. From the above, it can be seen that (41) holds for $\forall k \in \mathbb{Z}^+$. So $\{\Lambda_k\}$ is a bounded sequence. Because of the nature of the infimum, on the opening ball with a fixed radius $k \in \mathbb{Z}^+$, for $n = 1$, $\Lambda_k + 1$ is not lower bound, $\exists\, u_1^{(k)}$, s.t $J_k(u_1^{(k)}) < \Lambda_k + 1$; $n = 2$, $\Lambda_k + \frac{1}{2}$ is not lower bound, similarly $\exists\, u_2^{(k)}$, s.t $J_k(u_2^{(k)}) < \Lambda_k + \frac{1}{2}$; ... , and so on, we get $u_n^{(k)}$, s.t $J_k(u_n^{(k)}) < \Lambda_k + \frac{1}{n}$. Thus, we have $\Lambda_k \le J_k(u_n^{(k)}) < \Lambda_k + \frac{1}{n}$. Let $n \to \infty$, so we get $\lim_{n\to\infty} J_k(u_n^{(k)}) = \Lambda_k$. Given $\varepsilon_0 > 0$, $\exists\, N$, when $n > N$, we have $\mid J_k(u_n^{(k)}) - \Lambda_k \mid < \varepsilon_0$, i.e. $J_k(u_n^{(k)}) < \varepsilon_0 + \Lambda_k$. Because $\Lambda_k \le \|f\|_{L^1(V)}$, we have $J_k(u_n^{(k)}) \le \|f\|_{L^1(V)} + \varepsilon_0$. In addition,

$$\begin{aligned} &\left(\frac{1}{2} - \frac{M}{2h_0} - \frac{h_0 - M}{4h_0}\right)\|u_n^{(k)}\|^2_{W_0^{1,2}(B_k)} \\ &\quad - \frac{1}{2}\|f\|_{L^1(V)} - \frac{1}{h_0 - M}\|g\|^2_{L^2(V)} \\ &\quad \le \|f\|_{L^1(V)} + \varepsilon_0. \end{aligned} \tag{42}$$

so we get

$$\begin{aligned} &\|u_n^{(k)}\|^2_{W_0^{1,2}(B_k)} \\ &\le \left(\frac{3}{2}\|f\|_{L^1(V)} + \varepsilon_0 + \frac{1}{h_0 - M}\|g\|^2_{L^2(V)}\right) \\ &\quad \times \frac{1}{\left(\frac{1}{2} - \frac{M}{2h_0} - \frac{h_0-M}{4h_0}\right)}. \end{aligned} \tag{43}$$

which holds true for $n > N$. When $n \le N$, there is a maximum value for finite terms, we mark

$$\max\{\|u_n^{(k)}\|_{W_0^{1,2}(B_k)} : n = 1, 2, \dots, N\} \triangleq M^*. \tag{44}$$

Let

$$\begin{aligned} &M^{**} = \\ &\max\left\{M^*, \left[\left(2\|f\|_{L^1(V)} + \varepsilon_0 + \frac{1}{h_0 - M}\|g\|^2_{L^2(V)}\right)\right.\right. \\ &\quad \left.\left.\times \frac{1}{\left(\frac{1}{2} - \frac{M}{h_0} - \frac{h_0-M}{4h_0}\right)}\right]^{1/2}\right\}. \end{aligned} \tag{45}$$

thus, for $\forall\ n \in \mathbb{N}^*$, we have $0 \leq \|u_n^{(k)}\|_{W_0^{1,2}(B_k)} \leq M^{**}$ and that $\{u_n^{(k)}\}$ is a bounded point sequence in $W_0^{1,2}(B_k)$ as functions defined on B_k. In consideration of $W_0^{1,2}(B_k)$ is self compacting set, there exists a $u_k \in W_0^{1,2}(B_k)$ s.t $u_n^{(k)} \to u_k$ under the $\|\cdot\|_{W_0^{1,2}(B_k)}$, where $\{u_n^{(k)}\}$ is a subsequence of the origin sequence.

Next, we will explain that convergence according to the norm must converge point by point.
$\forall\ \varepsilon > 0$, $\exists\ N$, when $n > N$, we have $\|u_n^{(k)} - u_k\|_{W_0^{1,2}(B_k)} < \varepsilon$. Unfold it, we'll get that

$$\int_{B_k} \left(|\nabla(u_n^{(k)} - u_k)|^2 + h\,(u_n^{(k)} - u_k)^2\right) d\mu < \varepsilon. \tag{46}$$

The first item is non negative, so we have $\int_{B_k} h(u_n^{(k)} - u_k)^2 d\mu < \varepsilon$. In view of $h(x) \geq h_0$, it implies

$$h_0 \int_{B_k} (u_n^{(k)} - u_k)^2 d\mu \leq \int_{B_k} h(u_n^{(k)} - u_k)^2 d\mu < \varepsilon. \tag{47}$$

Expand the above equation (47), one has

$$\begin{aligned} & h_0 \min_{x\in B_k} \mu(x)\,(u_n^{(k)}(x) - u_k(x))^2 \\ & \quad \leq h_0 \min_{x\in B_k} \mu(x) \sum_{x\in B_k} (u_n^{(k)}(x) - u_k(x))^2 \\ & \quad \leq h_0 \sum_{x\in B_k} \mu(x)\,(u_n^{(k)}(x) - u_k(x))^2 < \varepsilon. \end{aligned} \tag{48}$$

where the rightmost item is $h_0 \int_{B_k}((u_n^{(k)} - u_k)^2 d\mu$, and the above equation is correct for $\forall\ x \in B_k$. Thus, we get $|\ u_n^{(k)}(x) - u_k(x)\ | < \varepsilon \cdot \frac{1}{h_0 \min_{x\in B_k} \mu(x)}$ and $u_n^{(k)}$ converges uniformly to u_k. And then, we obtain point by point convergence, i.e. $\lim_{n\to\infty} u_n^{(k)}(x) = u_k(x)$.
At this point,we have

$$\begin{aligned} \lim_{n\to\infty} J_k(u_n^{(k)}) &= J_k\left(\lim_{n\to\infty} u_n^{(k)}\right) \\ &= J_k(u_k) = \Lambda_k, \qquad u_k \in W_0^{1,2}(B_k). \end{aligned} \tag{49}$$

Which indicates that the infimum can be reached.
The critical point function u_k satisfies Euler-Lagrange equation: $\frac{d}{dt} J_k(u_k + t\phi)|_{t=0} = 0$, so we have

$$\begin{cases} -\Delta u_k + h u_k = f u_k e^{u_k^2} + g & in\ B_k, \\ u_k = 0 & on\ \partial B_k. \end{cases} \tag{50}$$

So far, we have obtained the solution of the equation locally.

Next, we will perform extension processing.
Due to (39) holds true for $\forall\ u \in W_0^{1,2}(B_k)$, we get

$$\begin{aligned} \|f\|_{L^1(V)} \geq \Lambda_k &= J_k(u_k) \\ &\geq \left(\frac{1}{2} - \frac{M}{2h_0} - \frac{h_0 - M}{4h_0}\right) \|u_k\|^2_{W_0^{1,2}(B_k)} \\ &\quad - \frac{1}{2}\,\|f\|_{L^1(V)} - \frac{1}{h_0 - M}\,\|g\|^2_{L^2(V)}. \end{aligned} \tag{51}$$

and then

$$\begin{aligned} & \|u_k\|^2_{W_0^{1,2}(B_k)} \\ & \leq \sqrt{\left(\frac{3}{2}\,\|f\|_{L^1(V)} + \frac{1}{h_0 - M}\,\|g\|^2_{L^2(V)}\right)\left(\frac{h_0}{h_0 - M}\right)} \\ & \nsim\ k. \end{aligned} \tag{52}$$

We say that critical point function columns have a common upper bound.Next, we will explain that the infinite norm of the critical point function column on a bounded set is still bounded.
$\forall\ K \subset V$ is a bounded set, so there exists a sufficiently large radius $k \in \mathbb{Z}^+$, s.t $B_k \supset K$. We consider the infinite norm of the critical point function u_k on B_k over K. In view of

$$\begin{aligned} \|u_k\|^2_{W_0^{1,2}(B_k)} &= \int_{B_k} \left(|\nabla u_k|^2 + h\,u_k^2\right) d\mu \\ &\geq \int_{B_k} h\,u_k^2\,d\mu \\ &\geq \int_K h\,u_k^2\,d\mu = \sum_{x\in K} h(x)\,\mu(x)\,u_k^2(x) \\ &\geq h_0 \min_{x\in K} \mu(x) \sum_{x\in K} u_k^2(x). \end{aligned} \tag{53}$$

we have $u_k^2(x) \leq \frac{1}{h_0 \min_{x\in K} \mu(x)} \|u_k\|^2_{W_0^{1,2}(B_k)}$, and

$$\begin{aligned} |\ u_k(x)\ | &\leq \sqrt{\frac{1}{h_0 \min_{x\in K} \mu(x)}}\,\|u_k\|_{W_0^{1,2}(B_k)} \\ &\leq \sqrt{\frac{1}{h_0 \min_{x\in K} \mu(x)}} \\ &\quad \times \sqrt{(\frac{3}{2}\|f\|_{L^1(V)} + \frac{1}{h_0 - M}\|g\|^2_{L^2(V)})} \\ &\quad \times \sqrt{\frac{h_0}{h_0 - M}}, \end{aligned} \tag{54}$$

which holds true for $\forall\, x \in K$. Thus, we have $\|u_k\|_{L^\infty(K)} \leq C$, in which $C \sim h_0,\ M,\ K,\ \|g\|_{L^2(V)},\ \|f\|_{L^1(V)}$. $\{u_k\}$ is defined on $B_k \cup \partial B_k$, extanding it to the entire graph:

$$u_k = \begin{cases} u_k(x) & x \in B_k, \\ 0 & x \notin B_k. \end{cases} \tag{55}$$

Especially, taking $x_1 \in V$ is a bounded set. $\exists\, k \in \mathbb{Z}^+$, s.t $x_1 \in B_k$, and then we can easily get

$$\begin{aligned} |u_k(x_1)| \ \leq\ & \sqrt{\frac{1}{h_0\, \mu(x_1)}} \\ & \times \sqrt{\left(\frac{3}{2}\, \|f\|_{L^1(V)} + \frac{1}{h_0 - M}\, \|g\|^2_{L^2(V)}\right)} \\ & \times \sqrt{\frac{h_0}{h_0 - M}}. \end{aligned} \tag{56}$$

$\forall\ K > k$, we still have $x_1 \in B_K$, $|\ u_K(x_1)\ | \leq \sqrt{\frac{1}{h_0\mu(x_1)}(\frac{3}{2}\|f\|_{L^1(V)} + \frac{1}{h_0-M}\|g\|^2_{L^2(V)})\frac{h_0}{h_0-M}}$. The first $k-1$ critical point functions have finite values at x_1, so $\{u_k(x_1)\}$ is a bounded point sequence, it has convergent subsequences. We take the opening ball where the convergent subsequence is located and discuss $x_2 \in V$, similarly, take the convergent subsequence, and so on. We ultimately obtained $\exists\ u^* \in V$ s.t u_k converges locally uniformly to u^*, i.e. $\forall\ l \in \mathbb{Z}^+$, $\lim\limits_{n\to\infty} u_k(x) = u^*(x)$, which holds true for $\forall\ x \in B_l$. Finally, we will declare that the limit of locally uniformly convergent sequences falls within $\mathscr{H}$. Obviously, the critical point function sequence after extension satisfies $\{u_k\} \subset \mathscr{H}$. Below we prove $u^* \in \mathscr{H}$, just need to explain that one of the functions in u^* and $\mathscr{H}$ is equal.

$$\begin{aligned} \|u_k\|^2_{\mathscr{H}} &= \int_V |\ \nabla u_k\ |^2 + h u_k^2 d\mu \\ &= \int_{B_k \cup \partial B_k} |\ \nabla u_k\ |^2\ d\mu + \int_{B_k} h u_k^2 d\mu \\ &= \frac{1}{2} \sum_{x \in B_k \cup \partial B_k} \sum_{y \sim x} \omega_{xy}(u_k(y) - u_k(x))^2 \\ &+ \sum_{x \in B_k} h(x)\mu(x)u_k^2(x) \\ &= \frac{1}{2} \sum_{x \in B_k} \sum_{y \sim x} \omega_{xy}(u_k(y) - u_k(x))^2 \\ &+ \frac{1}{2} \sum_{x \in \partial B_k} \sum_{y \sim x} \omega_{xy}(u_k(y) - u_k(x))^2 \\ &+ \sum_{x \in B_k} h(x)\mu(x)u_k^2(x) \\ &\leq \sum_{x \in B_k} \sum_{y \sim x} \omega_{xy}(u_k(y) - u_k(x))^2 \\ &+ 2 \sum_{x \in B_k} h(x)\mu(x)u_k^2(x) \\ &= 2\|u_k\|^2_{W_0^{1,2}(B_k)} \leq 2C. \end{aligned} \tag{57}$$

Thus, we can see that $\{u_k\}$ is a bounded sequence in $\mathscr{H}$. Becuase $\mathscr{H}$ is a Hilbert space, $\exists\ \widetilde{u} \in \mathscr{H}$ s.t $\{u_k\}$ subsequence $u_k \overset{weak}{\rightharpoonup} \widetilde{u}$. i.e. $\forall\ \Phi \in C_c(V)$, we have $\int_V u_k \Phi d\mu \to \int_V \widetilde{u} \Phi d\mu$. Especially, we take

$$\Phi(x) = \begin{cases} 1 & x = x_1 \\ 0 & x \neq x_1 \end{cases},\ \Phi(x) \in C_c(V), \tag{58}$$

then $\int_V u_k \Phi d\mu = \sum\limits_{x\in V} \mu(x)u_k(x)\Phi(x) = \mu(x_1)u_k(x_1)$, $\int_V \widetilde{u}\Phi d\mu = \sum\limits_{x\in V} \mu(x)\widetilde{u}(x)\Phi(x) = \mu(x_1)\widetilde{u}(x_1)$, and $\mu(x_1)u_k(x_1) \to \mu(x)\widetilde{u}(x_1)$, as $k \to \infty$. Due to the multiplication property of limits, we deduce that $u_k(x_1) \to \widetilde{u}(x_1)$, as $k \to \infty$. And x_1 is arbitrary, so $u_k(x) \to \widetilde{u}(x)$ holds true for $\forall\ x \in V$. We have konwn that $u_k(x) \to u^*(x)$, $\forall\ x \in V$, combining the uniqueness of the existence of limits, we get that $u^*(x) = \widetilde{u}(x) \in \mathscr{H}$. Thus, u^* is a function in $\mathscr{H}$. We have already konwn that the critical point function satisfies the distribution equation:

$$\begin{aligned} &\int_{B_k} -\Delta u_k\, \Phi\, d\mu\ +\ \int_{B_k} h\, u_k\, \Phi\, d\mu \\ &= \int_{B_k} f\, u_k\, e^{u_k}\, \Phi\, d\mu \\ &\quad + \int_{B_k} g\, \Phi\, d\mu, \qquad \forall \Phi \in C_c(B_k). \end{aligned} \tag{59}$$

For $\forall\ x_1 \in V$, $\exists\ k \in \mathbb{Z}^+$, s.t $x_1 \in B_k$. The critical point function u_k on B_k still satisfies the above equation. Taking

$$\Phi(x) = \begin{cases} 1 & x = x_1, \\ 0 & x \neq x_1. \end{cases} \tag{60}$$

and then we have

$$\begin{aligned} &\mu(x_1)\big(-\Delta u_k(x_1)\big)\ +\ \mu(x_1)\, h(x_1)\, u_k(x_1) \\ &\quad = \mu(x_1)\, f(x_1)\, u_k(x_1)\, e^{u_k(x_1)} \\ &\qquad +\ \mu(x_1)\, g(x_1). \end{aligned} \tag{61}$$

i.e.

$$\begin{aligned} &-\Delta u_k(x_1)\ +\ h(x_1)\, u_k(x_1) \\ &\quad = f(x_1)\, u_k(x_1)\, e^{u_k(x_1)} +\ g(x_1). \end{aligned} \tag{62}$$

When $K > k$, we take the corresponding characteristic function, and there is still a value of u_k at x_1 that satisfies the above equation. Let $k \to \infty$, we get

$$-\Delta u^*(x_1) + h(x_1)\, u^*(x_1) \\ = f(x_1)\, u^*(x_1)\, e^{u^*(x_1)} + g(x_1). \tag{63}$$

i.e. u^* satisfies the equation at x_1. From the arbitrariness of x_1, u^* is the solution of the equation (17), and $u^* \in \mathscr{H}$. Proof completed.

4 Proof of Theorem 5

Proof Before starting our discussion, let's do some preparation work first. Fixing one point $O \in V$ on the graph, $\forall x \in V$, there is a distance function $\rho(x) = \rho(x, O)$. Taking $B_k = \{x \in V : \ \rho(x, O) < k\}$ as the opening ball on the graph, and we might as well restrict our discussion to $k > 1$. Actually, only the situation where k is sufficiently large needs to be considered. $W_0^{1,2}(B_k)$ is a Sobolev space, which satisfies $u = 0$ on ∂B_k. We take the norm on it as $\|u\|_{W_0^{1,2}(B_k)} = (\int_{B_k} \mid \nabla u \mid^2 + hu^2 d\mu)^{\frac{1}{2}}$, where $h(x) \geq a_0 > 0$ and $\mu(x) \geq \mu_0 > 0$ hold true for $\forall x \in V$. We define variational functionals : $W_0^{1,2}(B_k) \to \mathbb{R}$ as follows

$$J_k(u) = \frac{1}{2}\int_{B_k} (\mid \nabla u \mid^2 + hu^2) d\mu \\ - \int_{B_k} fln \mid u + m \mid d\mu - log \int_{B_k} g e^u d\mu \tag{64}$$

Since $ln \mid u + m \mid < 1 + \mid u + m \mid \leq 1 + \mid u \mid + \mid m \mid$, there are $fln \mid u + m \mid < f + \mid fu \mid + \mid fm \mid$ naturally.And then, we have

$$\int_{B_k} fln \mid u + m \mid d\mu \\ < \int_{B_k} \mid f \mid d\mu + \int_{B_k} \mid fu \mid d\mu \\ + \mid m \mid \cdot \int_{B_k} \mid f \mid d\mu \\ = (1 + \mid m \mid) \int_{B_k} \mid f \mid d\mu + \int_{B_k} \mid fu \mid d\mu \tag{65}$$

Due to $f \geq 1$ and $q \in [1, 2]$, $\forall x \in V$ we can get $f(x)^q \geq f(x)$, which leads to

$$\int_{B_k} |f|\, d\mu \leq \int_{B_k} |f|^{\,q}\, d\mu \\ \leq \int_V |f|^{\,q}\, d\mu = \|f\|^q_{L^q(V)}. \tag{66}$$

Now we discuss $\int_{B_k} \mid fu \mid d\mu$. It follows from Hölder inequality that $\|fu\|_{L^1(B_k)} \leq \|f\|_{L^q(B_k)} \cdot \|u\|_{L^p(B_k)}$, wherein $\frac{1}{p} + \frac{1}{q} = 1$, $p = 1 + \frac{1}{q-1}$, $q \in [1, 2]$. Using Theorem 4, we can see that $\|u\|_{L^p(B_k)} \leq C\|u\|_{W_0^{1,2}(B_k)}$, $C \sim q,\ h_0,\ \mu_0$. And thus, one has

$$\|f\, u\|_{L^1(B_k)} \leq C\, \|f\|_{L^q(B_k)}\, \|u\|_{W_0^{1,2}(B_k)} \\ \leq C\, \|f\|_{L^q(V)}\, \|u\|_{W_0^{1,2}(B_k)}. \tag{67}$$

By using the Young's inequality to the above equation, it can be obtained that

$$\|f\, u\|_{L^1(B_k)} \ \leq\ \frac{1}{4\varepsilon}\, \|u\|^2_{W_0^{1,2}(B_k)} \ +\ \varepsilon\, C^2\, \|f\|^2_{L^q(V)}. \tag{68}$$

which holds true for $\forall \varepsilon > 0$. Taking $\varepsilon = 1$, then we have

$$-\int_{B_k} fln \mid u + m \mid d\mu \\ \geq -(1 + \mid m \mid) \cdot \|f\|^q_{L^q(V)} \\ - \frac{1}{4}\|u\|^2_{W_0^{1,2}(B_k)} - C^2\|f\|^2_{L^q(V)}. \tag{69}$$

What's more,

$$\|v\|^2_{W_0^{1,2}(B_k)} = \int_{B_k} \mid \nabla v \mid^2 + hv^2 d\mu \geq \int_{B_k} hv^2 d\mu \\ = \sum_{x \in B_k} h(x)\mu(x)v(x)^2 \\ \geq a_0\mu_0 \sum_{x \in B_k} v(x)^2 \\ \geq a_0\mu_0 v(x)^2, \tag{70}$$

so we have $\mid v(x) \mid \leq \frac{1}{\sqrt{a_0\mu_0}}\|v\|_{W_0^{1,2}(B_k)}$, and it holds for $\forall x \in B_k$. Furthermore, one has $(\frac{v(x)}{\|v\|_{W_0^{1,2}(B_k)}})^2 \leq \frac{1}{a_0\mu_0}$, which established for $\forall v(x) \in W_0^{1,2}(B_k)$. Becuase $u = \frac{u}{\|u\|_{W_0^{1,2}(B_k)}} \cdot \|u\|_{W_0^{1,2}(B_k)}$, it follows from Young's inequality that

$$u = \frac{u}{\|u\|_{W_0^{1,2}(B_k)}} \cdot \|u\|_{W_0^{1,2}(B_k)} \\ \leq \frac{u^2}{4\varepsilon\|u\|^2_{W_0^{1,2}(B_k)}} + \varepsilon\|u\|^2_{W_0^{1,2}(B_k)} \\ \leq \frac{1}{4\varepsilon\mu_0 a_0} + \varepsilon\|u\|^2_{W_0^{1,2}(B_k)}. \tag{71}$$

Thus ,we have $e^u \le e^{\frac{1}{4\varepsilon\mu_0 a_0}+\varepsilon\|u\|^2_{W_0^{1,2}(B_k)}}$. In view of $g \ge 0$ and $g \not\equiv 0$, one has $ge^u \le ge^{\frac{1}{4\varepsilon\mu_0 a_0}+\varepsilon\|u\|^2_{W_0^{1,2}(B_k)}}$. Together with $g \in L^1(V)$, we get

$$\begin{aligned}\int_{B_k} ge^u d\mu &\le e^{\frac{1}{4\varepsilon\mu_0 a_0}+\varepsilon\|u\|^2_{W_0^{1,2}(B_k)}} \cdot \int_{B_k} g d\mu \\ &\le e^{\frac{1}{4\varepsilon\mu_0 a_0}+\varepsilon\|u\|^2_{W_0^{1,2}(B_k)}} \cdot \int_V g d\mu \\ &= e^{\frac{1}{4\varepsilon\mu_0 a_0}+\varepsilon\|u\|^2_{W_0^{1,2}(B_k)}} \cdot \|g\|_{L^1(V)}.\end{aligned} \tag{72}$$

This will directly lead to

$$\begin{aligned} &log \int_{B_k} ge^u d\mu \\ &\quad \le log(e^{\frac{1}{4\varepsilon\mu_0 a_0}+\varepsilon\|u\|^2_{W_0^{1,2}(B_k)}} \cdot \|g\|_{L^1(V)}) \\ &\quad = \frac{1}{4\varepsilon\mu_0 a_0} + \varepsilon\|u\|^2_{W_0^{1,2}(B_k)} \\ &\quad\quad + log\|g\|_{L^1(V)}\end{aligned} \tag{73}$$

Combining with formula (64)(69)(73), we can obtain that

$$\begin{aligned}J_k(u) \ge& (\frac{1}{2} - \frac{1}{4} - \varepsilon)\|u\|^2_{W_0^{1,2}(B_k)} \\ &- (1+ \mid m \mid)\|f\|^q_{L^q(V)} - C^2\|f\|^2_{L^q(V)} \\ &- log\|g\|_{L^1(V)} - \frac{1}{4\varepsilon\mu_0 a_0}.\end{aligned} \tag{74}$$

Taking $\varepsilon = \frac{1}{8}$, one has

$$\begin{aligned}J_k(u) \ge& \frac{1}{8}\|u\|^2_{W_0^{1,2}(B_k)} - (1+ \mid m \mid)\|f\|^q_{L^q(V)} \\ &- C^2\|f\|^2_{L^q(V)} - log\|g\|_{L^1(V)} - \frac{2}{\mu_0 a_0}.\end{aligned} \tag{75}$$

$$\begin{aligned}J_k(u) \ge& -(1+ \mid m \mid)\|f\|^q_{L^q(V)} \\ &- C^2\|f\|^2_{L^q(V)} - log\|g\|_{L^1(V)} - \frac{2}{\mu_0 a_0},\end{aligned} \tag{76}$$

which holds for $\forall u \in W_0^{1,2}(B_k)$. Hence, $J_k(u)$ has a lower bound in $W_0^{1,2}(B_k)$, and where there is a lower bound, there must be a infimum. Then, we mark $\Lambda_k = \inf_{u \in W_0^{1,2}(B_k)} J_k(u)$ and take the minimized subsequence $(\widetilde{u_j}) \subset W_0^{1,2}(B_k)$ s.t $J_k(\widetilde{u_j}) \to \Lambda_k$ as $j \to \infty$. For $n = 1$, $\Lambda_k + 1$ is not a lower bound, then $\exists\ \widetilde{u_1} \in W_0^{1,2}(B_k)$ s.t $J_k(\widetilde{u_1}) < \Lambda_k + 1$; what's more, for $n = 2$, $\Lambda_k + \frac{1}{2}$ is also not a lower bound, similarly, $\exists\ \widetilde{u_2} \in W_0^{1,2}(B_k)$ s.t $J_k(\widetilde{u_2}) < \Lambda_k + \frac{1}{2}$; Repeat the above operation, we ultimately get $\Lambda_k \le J_k(\widetilde{u_j}) < \Lambda_k + \frac{1}{j}$. Let $j \to \infty$, one has $J_k(\widetilde{u_j}) \to \Lambda_k$. In view of (22), we have

$$J_k(0) = -\int_{B_k} fln \mid m \mid d\mu - log \int_{B_k} g d\mu \tag{77}$$

In consideration of $f \ge 1$ and $0 <\mid m \mid< 1$, so one has $ln \mid m \mid< 0$ and $f^q \ge f$. Which directly leads to

$$\int_{B_k} f d\mu \le \int f^q d\mu \le \|f\|^q_{L^q(V)} \tag{78}$$

and

$$-ln \mid m \mid \int_{B_k} f d\mu \le -ln \mid m \mid \cdot \|f\|^q_{L^q(V)} \tag{79}$$

For the second item in (77), due to $g(x) \ge 0$ and $g \not\equiv 0$, $\exists x_0 \in B_k$ such that $g(x_0) > 0$.

$$\int_{B_k} g d\mu = \sum_{x \in B_k} g(x)\mu(x) \ge g(x_0)\mu(x_0) > 0, \tag{80}$$

and then, we have

$$-log \int_{B_k} g d\mu < -log(g(x_0)\mu(x_0)). \tag{81}$$

Inserting (79), (81) into (77), one has

$$J_k(0) \le -ln \mid m \mid \cdot \|f\|^q_{L^q(V)} - log(g(x_0)\mu(x_0)), \tag{82}$$

and one thing needs to be pointed out, since we only discuss the case where the radius of the opening ball is sufficiently large, when the radius $l > k$, $x_0 \in B_k \subset B_l$ can still be taken as a non-zero point, and thus x_0 is considered independent of the radius k. Because

$$\begin{aligned}J_k(u) \ge& \frac{1}{8}\|u\|^2_{W_0^{1,2}(B_k)} - C^2\|f\|^2_{L^q(V)} \\ &- (1+ \mid m \mid)\|f\|^q_{L^q(V)} - log\|g\|_{L^1(V)} - \frac{2}{\mu_0 a_0}.\end{aligned} \tag{83}$$

holds for $\forall u \in W_0^{1,2}(B_k)$, especially, we have $J_k(\widetilde{u_j}) \ge \frac{1}{8}\|\widetilde{u_j}\|^2_{W_0^{1,2}(B_k)} - C$, where

$$C = C^2 \, \|f\|^2_{L^q(V)} \; + \; (1+|m|) \, \|f\|^q_{L^q(V)} \\ + \; \log\big(\|g\|_{L^1(V)}\big) \; + \; \frac{2}{\mu_0 \, a_0}. \tag{84}$$

Due to $J_k(\widetilde{u_j}) \to \Lambda_k$ as $j \to \infty$ and the boundedness of convergent sequences, taking $\varepsilon = M > 0$, $\exists \, J$, when $j > J$, one has $| \, J_k(\widetilde{u_j}) - \Lambda_k \, | < M$, i.e. $J_k(\widetilde{u_j}) < \Lambda_k + M$. And combined with

$$\Lambda_k \le J_k(0) \le -ln \mid m \mid \cdot \|f\|^q_{L^q(V)} - log(g(x_0)\mu(x_0)), \tag{85}$$

it implies that

$$\frac{1}{8}\|\widetilde{u_j}\|^2_{W_0^{1,2}(B_k)} - C \le J_k(\widetilde{u_j}) < \Lambda_k + M \\ \le -ln \mid m \mid \cdot \|f\|^q_{L^q(V)} - log(g(x_0)\mu(x_0)) + M. \tag{86}$$

and

$$\|\widetilde{u_j}\|^2_{W_0^{1,2}(B_k)} \le 2\sqrt{2} \\ \cdot (-ln \mid m \mid \|f\|^q_{L^q(V)} - log(g(x_0)\mu(x_0)) + M + C)^{\frac{1}{2}} \\ \triangleq C. \tag{87}$$

where $C \not\sim k$. The above indicates that all terms of the minimization sequence $(\widetilde{u_j})$ before the J term are bounded according to the norm $\| \cdot \|_{W_0^{1,2}(B_k)}$. We take the $j > J$ terms and still mark the sequence as $(\widetilde{u_j})$, its corresponding functional value sequence $(j_k(\widetilde{u_j}))$ is a subsequence of the original sequence, and $(j_k(\widetilde{u_j}))$ converges to Λ_k similarly. For the new sequence $(\widetilde{u_j})$, noting that $\|\widetilde{u_j}\|_{W_0^{1,2}(B_k)}$ are bounded and $W_0^{1,2}(B_k)$ is pre-compact, so $\exists \; u_k \in W_0^{1,2}(B_k)$, such that $\widetilde{u_j} \overset{\|\cdot\|_{W_0^{1,2}(B_k)}}{\to} u_k$, where $(\widetilde{u_j})$ is a subsequence.

In the following statement, we will deduce that sequences that converge according to the norm must converge uniformly.

We have konwn that $\forall \; \varepsilon > 0$, $\exists \; J$, when $j > J$, one has $\|\widetilde{u_j} - u_k\|_{W_0^{1,2}(B_k)} < \varepsilon$. Expand the above equation, we naturally obtain that

$$\mu_0 a_0 \sum_{x \in B_k} (\widetilde{u_j}(x) - u_k(x))^2 \\ \le \sum_{x \in B_k} h(x)\mu(x)(\widetilde{u_j}(x) - u_k(x))^2 \\ = \int_{B_k} h(\widetilde{u_j} - u_k)^2 d\mu \\ \le \int_{B_k} \mid \nabla(\widetilde{u_j} - u_k) \mid^2 + h(\widetilde{u_j} - u_k)^2 d\mu < \varepsilon^2. \tag{88}$$

which deduces that $\forall x \in B_k$, $\mu_0 a_0(\widetilde{u_j}(x) - u_k(x))^2 < \varepsilon^2$, i.e. $| \; \widetilde{u_j}(x) - u_k(x) \; | < \frac{1}{\sqrt{\mu_0 a_0}} \varepsilon$. Combining the above statements, we can easily get $\widetilde{u_j} \rightrightarrows u_k$ in B_k, hence, $(\widetilde{u_j})$ uniformly converges to u_k. Specially, one has pointwise convergence $\lim\limits_{j \to \infty} \widetilde{u_j} = u_k$. Let $j \to \infty$, we get

$$\Lambda_k = \lim_{j \to \infty} J_k(\widetilde{u_j}) = J_k(\lim_{j \to \infty} \widetilde{u_j}) = J_k(u_k). \tag{89}$$

Thus, u_k is the reachable point of the infimum. What's more, u_k satisfies the Euler-Lagrange equation.

$$\begin{cases} -\Delta u_k + h u_k = \frac{1}{\gamma_k} g e^{u_k} - \frac{f}{u_k + m}, & in \; B_k, \\ u_k \in W_0^{1,2}(B_k), \;\; \gamma_k = \int_{B_k} g e^{u} d\mu. \end{cases} \tag{90}$$

Up to now, we have obtained the local solution u_k of the equation (18).

Combining (83) and (85), we can easily obtain that

$$\|u_k\|_{W_0^{1,2}(B_k)} \le C, \;\; C \sim k. \tag{91}$$

For any bounded set $K \subset V$, there is always a sufficiently large $k \in \mathbb{Z}^+$, s.t. $K \subset B_k$. Using Th 4 and (91), we imply that

$$\|u_k\|_{L^\infty(K)} \le \frac{1}{\sqrt{a_0 \mu_0}} \|u_k\|_{W_0^{1,2}(B_k)} \le C \sim k. \tag{92}$$

Given that u_k is a function defined on $B_k \cup \partial B_k$, let's extend it to the entire V below. Let

$$u_k = \begin{cases} u_k(x) & x \in B_k, \\ 0 & x \notin B_k. \end{cases} \tag{93}$$

Next, we will take the convergent subsequence point by point. For a set composed of individual points $x_1 \in V$, $\exists \; k \in \mathbb{Z}^+$, as can be seen from the above, $|u_k(x_1)| \le C$ and for any $l > k$, which holds $|u_l(x_1)| \le C$. Due to the existence of convergent sub columns in bounded point columns, it deduece that $u_k(x_1)$ is a convergent sub column, converging to $u^*(x_1)$; For $x_2 \in V$, we consider the kick-off corresponding to $\{u_k(x_1)\}$. $\exists \; k \in \mathbb{Z}^+$ s.t. $x_2 \in B_k$ and $|u_k(x_2)| \le C$, which holds for any $l > k$. By the same token, $\{u_k(x_2)\}$ is a convergent sub column, meanwhile, maintain the astringency of $\{u_k(x_1)\}$. Repeat the above operation, we get $u_k(x) \to u^*(x)$, $\forall x \in V$. Hence, u_k locally converges uniformly to u^*. Combining (91) and Th 4, we can deduce

$$\|u_k\|_{L^\infty(B_k)} \le \frac{1}{\sqrt{\mu_0 a_0}} \|u_k\|_{W_0^{1,2}(B_k)} \le C. \tag{94}$$

And then, $|u_k(x)| \le C$ holds true for any $x \in B_k$. Considering $-C \le u_k(x) \le C$ and $g \ge 0$, thus we obtain $ge^{-c} \le ge^{u_k} \le ge^c$.Integrate its two sides on B_k, one has $e^{-c}\|g\|_{L^1(B_k)} \le \gamma_k \le e^c\|g\|_{L^1(B_k)}$, where $\gamma_k = \int_{B_k} ge^{u_k}d\mu$. It follows from

$$e^{-c} \le \frac{\gamma_k}{\|g\|_{L^1(B_k)}} \le e^c, \tag{95}$$

and $\{\frac{\gamma_k}{\|g\|_{L^1(B_k)}}\}$ is a bounded sequence, we can get a convergence subsequence, we still mark it as $\{\frac{\gamma_k}{\|g\|_{L^1(B_k)}}\}$. Because $\lim\limits_{k\to\infty} \|g\|_{L^1(B_k)} = \|g\|_{L^1(V)}$, $\{\frac{\gamma_k}{\|g\|_{L^1(B_k)}}\}$ is convergent, according to the four operations of the limit, we can know that γ_k is convergent.We mark $\lim\limits_{k\to\infty} \gamma_k = \gamma^*$. Take the limit for (95), one has

$$\lim_{k\to\infty} e^{-c} \le \lim_{k\to\infty} \frac{\gamma_k}{\|g\|_{L^1(B_k)}} \le \lim_{k\to\infty} e^c, \tag{96}$$

i.e.

$$\begin{aligned} &e^c \le \frac{\gamma^*}{\|g\|_{L^1(V)}} \le e^c, \\ &e^{-c}\|g\|_{L^1(V)} \le \gamma^* \le e^c\|g\|_{L^1(V)}, \end{aligned} \tag{97}$$

$\forall x_1 \in V$, $\exists\ k \in \mathbb{Z}^+$, s.t. after the extension of the critical function u_k on B_k, it still satisfies the equation at x_1. i.e.

$$-\Lambda u_k(x_1) + hu_k(x_1) = \frac{ge^{u_k(x_1)}}{\int_{B_k} ge^{u_k}d\mu} + \frac{f}{u_k(x_1)+m}. \tag{98}$$

When the radius $l > k$, the corresponding critical point function still satisfies the above equation at x_1 with respect to the corresponding integral. Let $k \to \infty$, we have

$$-\Lambda u^*(x_1) + hu^*(x_1) = \frac{ge^{u^*(x_1)}}{\gamma^*} + \frac{f}{u^*(x_1)+m}. \tag{99}$$

Based on the arbitrariness of x_1, one obtains

$$-\Lambda u^* + hu^* = \frac{ge^{u^*}}{\gamma^*} + \frac{f}{u^*+m} \quad in\ V. \tag{100}$$

So far, we have got a limit function u^* that is close to the solution to the equation. Now, we will declare that $\gamma^* = \int_V ge^{u^*}d\mu$. On the one hand, for $\forall l > 1$ fixed, there holds the following formula

$$\begin{aligned} \int_{B_\ell} g\,e^{u^*}\,d\mu &= \lim_{k\to\infty} \int_{B_\ell} g\,e^{u_k}\,d\mu \\ &\le \lim_{k\to\infty} \int_{B_k} g\,e^{u_k}\,d\mu = \gamma^*. \end{aligned} \tag{101}$$

Let $l \to \infty$ to the above formula, one has

$$\int_V ge^{u^*} \le \gamma^*. \tag{102}$$

On the other hand, due to $\|u_k\|_{L^\infty(B_k)} \le C$ and $g \in L^1(V)$, for $\forall\ \eta > 0$, there exists sufficiently large $l_0 > 1$, s.t. when $l > l_0$, we have $\int_{B_k} ge^{u_k}d\mu \le \eta + \int_{B_l} ge^{u_k}d\mu$. Actually, $\|u_k\|_{L^\infty(B_k)} \le C$, we temporarily fix k, one has $|u_k(x)| \le C$, which holds true for $\forall x \in B_k$. And then, $-C \le u_k(x) \le C$, combining with $g \ge 0$, we have $ge^{u_k} \le ge^C$. Integrate the two ends of the equation on $B_k \setminus B_l$, we get

$$\int_{B_k\setminus B_l} ge^{u_k}d\mu \le e^C \int_{B_k\setminus B_l} gd\mu \le e^C \int_{V\setminus B_l} gd\mu. \tag{103}$$

Due to $\int_{V\setminus B_l} gd\mu \to 0$ when l is sufficiently large, $\int_{B_k\setminus B_l} ge^{u_k}d\mu \le o_l(1)$. Add $\int_{B_l} ge^{u_k}d\mu$ to both ends simultaneously, one has

$$\int_{B_k} ge^{u_k}d\mu \le o_l(1) + \int_{B_l} ge^{u_k}d\mu. \tag{104}$$

$\forall \eta > 0$, $\exists\ l_0 > 1$, when $l > i_0$, we have $o_l(1) < \eta$, which leads to

$$\int_{B_k} ge^{u_k}d\mu \le \eta + \int_{B_l} ge^{u_k}d\mu. \tag{105}$$

As for the equation above, we let $k \to \infty$, $l \to \infty$, $\eta \to 0^+$, and obtain taht $\gamma^* \le \int_V ge^{u^*}d\mu$. Combining with

$\int_V ge^{u^*}d\mu \leq \gamma^*$, and then, we get $\gamma^* = \int_V ge^{u^*}d\mu$. Thus, u^* is a function on the graph and satisfies the equation at every point.

$$\begin{cases} -\Lambda u^* + hu^* = \frac{1}{\gamma^* ge^{u^*}} + \frac{f}{u^*+m}, & in\ V, \\ \gamma^* = \int_V ge^{u^*}d\mu. \end{cases} \tag{106}$$

Finally, let's declare $u^* \in \mathscr{H} \cap L^\infty(V)$ and first explain $u^* \in \mathscr{H}$. We investigate $\|u_k\|^2_{\mathscr{H}}$,

$$\begin{aligned} \|u_k\|^2_{\mathscr{H}} &= \int_V (|\nabla u_k|^2 + hu_k^2)d\mu \\ &= \frac{1}{2}\sum_{x\in V}\sum_{y\sim x}\omega_{xy}(u_k(y)-u_k(x))^2 \\ &\quad + \sum_{x\in V}\mu(x)h(x)u_k^2(x) \\ &= \frac{1}{2}\sum_{x\in B_k, y\sim x}\omega_{xy}(u_k(y)-u_k(x))^2 \\ &\quad + \frac{1}{2}\sum_{x\in \partial B_k, y\sim x}\omega_{xy}(u_k(y)-u_k(x))^2 \\ &\quad + \sum_{x\in B_k}\mu(x)h(x)u_k^2(x) \\ &\leq \sum_{x\in B_k, y\sim x}\omega_{xy}(u_k(y)-u_k(x))^2 \\ &\quad + 2\sum_{x\in B_k}\mu(x)h(x)u_k^2(x) \\ &= 2(\sum_{x\in B_k}\mu(x)\cdot\frac{1}{2\mu(x)}\sum_{y\sim x}\omega_{xy}(u_k(y)-u_k(x))^2 \\ &\quad + \sum_{x\in B_k}\mu(x)h(x)u_k^2(x)) \\ &= 2\|u\|^2_{W_0^{1,2}(B_k)} \\ &\leq C \nsim k. \end{aligned} \tag{107}$$

so (u_k) is a bounded sequence in $\mathscr{H}$. Because $\mathscr{H}$ is a Hilbert space, any bounded point sequence has weakly convergent subsequences, we still label the convergent subsequence as (u_k), and $u_k \overset{weak}{\rightharpoonup} \widetilde{u}$, $u \in \mathscr{H}$. i.e. for $\forall\ \Phi \in Cc(V)$, we have $\int_V u_k\Phi d\mu \to \int_V \widetilde{u}\Phi d\mu$. Especially, $\forall x_1 \in V$, we take the characteristic function of x_1

$$\Phi(x) = \begin{cases} 1 & x = x_1, \\ 0 & x \neq x_1. \end{cases} \tag{108}$$

Then, $u_k(x_1)\mu(x_1) \to \widetilde{u}(x_1)\mu(x_1)$ as $k \to \infty$. Next, based on the multiplication property of the limit, we have $u_k(x_1) \to \widetilde{u}(x_1)$. Combining the arbitrariness of x_1, one has $u_k(x) \to \widetilde{u}(x)\ in\ V$. Also, because the subsequence of u_k maintains the convergence of the original sequence, we have $u_k(x) \to u^*(x)\ in\ V$. Due to the uniqueness of the existence of limits, we konw that $u^*(x) = \widetilde{u}(x) \in \mathscr{H}$. In the follwing step, we declare that $u^* \in L^\infty(V)$. We only need to examine $\|u^*\|^2_{\mathscr{H}}$.

$$\begin{aligned} a_0\mu_0\, u^*(x)^2 &\leq a_0\mu_0 \sum_{x\in V} u^*(x)^2 \\ &\leq \sum_{x\in V} h(x)\,\mu(x)\,u^*(x)^2 = \int_V h\, u^{*2}\, d\mu \\ &\leq \int_V \left(|\nabla u^*|^2 + h\, u^{*2}\right) d\mu. \end{aligned} \tag{109}$$

And then, $|u^*(x)| \leq \frac{1}{\sqrt{a_0\mu_0}} \cdot \|u^*\|_{\mathscr{H}}$, which holds true for $\forall\ x \in V$. Thus, $\|u^*\|_{L^\infty(V)} = \sup_{x\in V}|u^*(x)| \leq \frac{\|u^*\|_{\mathscr{H}}}{\sqrt{a_0\mu_0}} < +\infty$, which is a finite number. Moreover, $u^* \in \mathscr{H} \cap L^\infty(V)$ and u^* is the solution to the equation. As so far, the conclusion has been proven.

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