

Elliptic coordinates: applications from geodesy

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Received: 08 November 2025 / Accepted: 29 November 2025 / Published: 29 November 2025

Abstract In the matter of exact solutions of Einstein's field equations, coordinates play a very decisive role. Sometimes it is very difficult to understand the physical concept of a phenomenon or physical quantity in one coordinate system, while by changing the coordinate system, a simpler understanding of that quantity is obtained. In geodesy, which deals with the simulation and modeling of the gravitational field, it is common to use elliptical coordinates. In this article, we attempt to explain some computational fundamentals in elliptical coordinates and discuss its possible application to gravity.

1 Introduction

The purpose of developing elliptic coordinates in geodesy is to obtain a more accurate model of the Earth, which is not a perfect sphere. In contrast, relativists usually employ spherical coordinates, since the modeling of the Earth is not central to their concerns and spherical symmetry provides a good approximation for most astrophysical bodies under study. Nevertheless, it seems important to emphasize that elliptic coordinates can be useful for examining axisymmetric solutions in general relativity. By employing such coordinates, one may derive more elegant and potentially simpler solution methods for Einstein's field equations.

In this article, we review basic concepts of elliptic coordinates and discuss known solutions of the Laplace equation under the assumption of an ellipsoidal mass distribution acting as the gravitational source.

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2 Two-Dimensional Elliptic Coordinates

We begin by introducing two-dimensional elliptic coordinates, and in the following section the approach will be extended to three dimensions. Every point in the plane can be described using two coordinates. In polar coordinates (r, θ) , for example, specifying an angle determines a direction, and specifying a radius determines where the direction intersects a centered circle—thereby fixing the point. The same idea can be applied to elliptic coordinates: we specify a line (direction) by a known angle, and then instead of a circle, we use an ellipse to determine the position of the point by the intersection of the line and the ellipse. Therefore, we fix the first coordinate, i.e. θ , and find the second coordinate using the ellipse formula as follows:

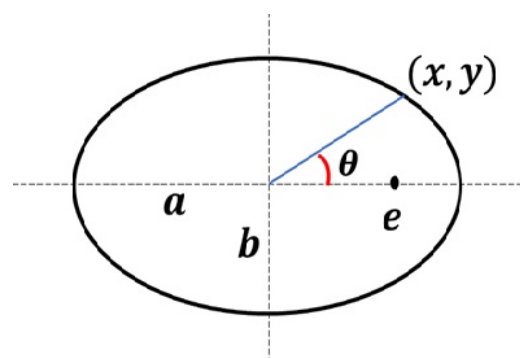


Fig. 1: ellipsoidal coordinates in two dimensions

$$y = x \tan(\theta), \quad (1)$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad (2)$$

we find out that

$$\begin{aligned} x &= a \left[1 + \frac{1}{1-e^2} \tan^2(\theta) \right]^{-1/2}, \\ y &= a \tan(\theta) \left[1 + \frac{1}{1-e^2} \tan^2(\theta) \right]^{-1/2}, \end{aligned} \quad (3)$$

where $e = \frac{c}{a}$ is the eccentricity of the ellipse and $c^2 = a^2 - b^2$. Therefore, the elliptic coordinates are determined by two parameters (a, θ) , assuming that the eccentricity is fixed and known in advance. Up to this point, the variable θ has played the role of the angle in polar coordinates. We now proceed to a different approach and introduce a new coordinate, which seems to have no direct relation to polar coordinates. Considering the equation of the ellipse, one can see that for a given angle λ , we may write:

$$\begin{aligned} x &= a \cos(\lambda), \\ y &= b \sin(\lambda), \end{aligned} \quad (4)$$

which, taking into account the eccentricity, is expressed as follows:

$$\begin{aligned} x &= a \cos(\lambda), \\ y &= a \sqrt{1-e^2} \sin(\lambda). \end{aligned} \quad (5)$$

To obtain a geometric and physical interpretation of the angle λ , one may compare relations (3) and (5) and determine its connection with the polar angle θ . Before doing so, we apply a convention commonly used in geodesy texts to relations (3) and (5). In geodesy literature, instead of treating the eccentricity as constant, the quantity c in the relation $c^2 = a^2 - b^2$ is usually taken to be constant and is denoted by E , and instead of b , the symbol u is used. Under this convention, relations (3) and (5) are rewritten in the following form:

$$\begin{aligned} x &= a \left[1 + \frac{u^2 + E^2}{u^2} \tan^2(\theta) \right]^{-1/2}, \\ y &= a \tan(\theta) \left[1 + \frac{u^2 + E^2}{u^2} \tan^2(\theta) \right]^{-1/2}, \end{aligned} \quad (6)$$

$$\begin{aligned} x &= \sqrt{u^2 + E^2} \cos(\lambda), \\ y &= u \sin(\lambda). \end{aligned} \quad (7)$$

It is easy to see that the change of variable

$$\tan(\lambda) = \frac{a}{u} \tan(\theta), \quad (8)$$

transforms relations (6) and (7) into each other. To better understand the meaning of the angles θ and λ , we consider the following figure:

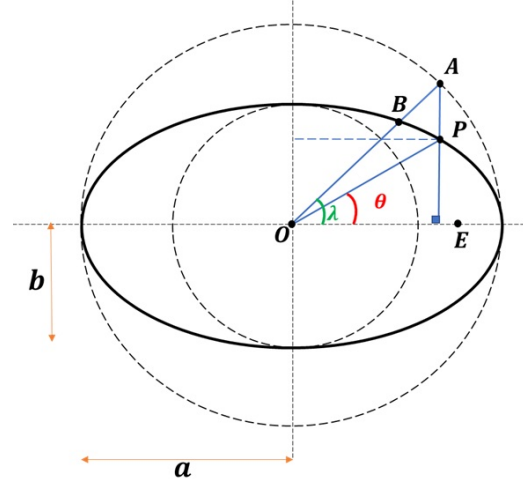


Fig. 2: relationship between the angles θ and λ

In Figure 2, the inscribed and circumscribed circles with radii a and b , respectively, are drawn as dashed curves. The point A is the vertical projection of the point P onto the circumscribed circle. Therefore, the angle λ is the polar angle corresponding to the point A . Based on these assumptions, the validity of relation (8) can be shown easily.

3 Laplacian Operator in the Two-Dimensional Coordinate System $(u, \lambda)_E$

The subscript E indicates that the parameter E is constant in this coordinate system. By applying the transformations (7) to the two-dimensional Laplacian operator

$$\nabla^2 = \partial_x^2 + \partial_y^2, \quad (9)$$

and after some algebraic manipulation, we obtain

$$\begin{aligned} (\partial_x^2 + \partial_y^2)V &= \\ \frac{1}{u^2 + E^2 \sin^2 \lambda} &\left[(u^2 + E^2) \frac{\partial^2 V}{\partial u^2} + \frac{\partial^2 V}{\partial \lambda^2} + u \frac{\partial V}{\partial u} \right]. \end{aligned} \quad (10)$$

To solve the two-dimensional Laplace equation in this elliptic coordinate system, we assume separation of variables:

$$V(u, \lambda) = \Lambda(\lambda)U(u). \quad (11)$$

Introducing the substitution

$$z = u + \sqrt{u^2 + E^2}, \quad (12)$$

the U -equation yields the solution

$$U(u) = A_0 \left(u + \sqrt{u^2 + E^2} \right)^n + B_0 \left(u + \sqrt{u^2 + E^2} \right)^{-n}, \quad (13)$$

where n is a natural number. The angular part becomes

$$\Lambda(\lambda) = A_1 \cos(n\lambda) + B_1 \sin(n\lambda). \quad (14)$$

Finally, note that if $E = 0$, then

$$u = r, \lambda = \theta, \quad (15)$$

and the usual solutions in polar coordinates are recovered.

4 Three-Dimensional Ellipsoidal Coordinate System

The procedure here follows the same approach as in the two-dimensional case. This time, instead of fixing one polar angle θ , we fix two angles θ and ϕ and consider the ellipsoid

$$\frac{x^2 + y^2}{a^2} + \frac{z^2}{c^2} = 1. \quad (16)$$

The first two semi-axes are taken equal to a , which not only simplifies the mathematics but also corresponds to axisymmetric physical bodies. The above surface is an ellipse in the z - y plane rotated about the z -axis.

A point on the ellipsoid in Cartesian coordinates (x, y, z) can be described using spherical coordinates (ρ, θ, ϕ) :

$$\begin{aligned} x &= \rho \sin \theta \cos \phi, & y &= \rho \sin \theta \sin \phi, \\ z &= \rho \cos \theta. \end{aligned} \quad (17)$$

Substituting these into (16) gives

$$\rho = c \sqrt{\frac{c^2 + E^2}{c^2 + E^2 \cos^2 \theta}}. \quad (18)$$

We also note that

$$a^2 = c^2 + E^2, \quad (19)$$

which is the standard relation between the semi-axes.

To introduce a new angular coordinate analogous to the two-dimensional case, we define

$$\cos \beta = \frac{c}{\sqrt{c^2 + E^2 \cos^2 \theta}} \sin \theta. \quad (20)$$

This gives the coordinate transformation known as the Jacobi ellipsoidal coordinates of the first type:

$$\begin{aligned} x &= \sqrt{c^2 + E^2} \cos \beta \cos \phi, \\ y &= \sqrt{c^2 + E^2} \cos \beta \sin \phi, \\ z &= c \sin \beta. \end{aligned} \quad (21)$$

Substituting $c \rightarrow u$ yields the final form of the 3-dimensional ellipsoidal coordinate system.

5 Laplace Equation in Ellipsoidal Coordinates

Using the coordinates (21) and after some algebraic manipulation, the Laplace operator becomes

$$\begin{aligned} \nabla^2 V &= \frac{1}{u^2 + E^2 \sin^2 \beta} \\ &\times \left[\frac{\partial}{\partial u} \left((u^2 + E^2) \frac{\partial V}{\partial u} \right) + \frac{1}{\cos \beta} \frac{\partial}{\partial \beta} \left(\cos \beta \frac{\partial V}{\partial \beta} \right) \right. \\ &\left. + \frac{u^2 + E^2 \sin^2 \beta}{(u^2 + E^2) \cos^2 \beta} \frac{\partial^2 V}{\partial \lambda^2} \right] = 0. \end{aligned} \quad (22)$$

To solve this equation for a static gravitational potential, we assume separation of variables:

$$V(u, \beta, \lambda) = \Lambda(\lambda) B(\beta) H(u). \quad (23)$$

This yields three ordinary differential equations:

$$\frac{d^2 \Lambda}{d\lambda^2} + c_0 \Lambda = 0, \quad (24)$$

$$\frac{d^2 B}{d\beta^2} - \tan \beta \frac{dB}{d\beta} + \left(-\frac{c_0}{\cos^2 \beta} + c_1 \right) B = 0, \quad (25)$$

$$(u^2 + E^2) \frac{d^2 H}{du^2} + 2u \frac{dH}{du} + \left(\frac{c_0 E^2}{u^2 + E^2} - c_1 \right) H = 0. \quad (26)$$

Applying boundary conditions on the potential (such as vanishing at infinity) and using relations (24)-(26), we obtain the potential outside the ellipsoid as

$$V(u, \beta, \lambda) = \sum_{n=0}^{\infty} \sum_{m=-n}^n a_{mn} Q_n^{|m|} \left(\frac{iu}{E} \right) \bar{Y}_{nm}(\lambda, \beta). \quad (27)$$

The coefficients a_{mn} are determined by the boundary conditions. $Q_n^{|m|}(iu/E)$ are the generalized Legendre functions, and $\bar{Y}_{nm}(\lambda, \beta)$ are the normalized spherical harmonics. To determine the coefficients a_{mn} , we may, for example, apply Dirichlet boundary conditions on the ellipsoidal boundary defined by $u = b$. By carrying out this procedure and using the addition theorems of spherical harmonics, we obtain the following relation:

$$a_{mn} = \frac{1}{S} \iint_E w(\beta') \bar{Y}_{nm}(\lambda', \beta') V(b, \lambda', \beta') dS. \quad (28)$$

The surface area of the ellipsoid is

$$S = 4\pi a^2 \left(\frac{1}{2} + \frac{b^2}{4aE} \ln \left(\frac{a+E}{a-E} \right) \right), \quad (29)$$

and the associated weight function is given by

$$w(\beta') = \frac{a}{\sqrt{b^2 + E^2 \sin^2 \beta'}} \left(\frac{1}{2} + \frac{b^2}{4aE} \ln \left(\frac{a+E}{a-E} \right) \right). \quad (30)$$

The generalized Legendre functions satisfy

$$\begin{aligned} Q_n^m \left(\frac{u}{E} \right) &= i^{n+1} Q_n^m \left(\frac{iu}{E} \right) \\ &= (-1)^m \frac{(n+m)!}{(2n+1)!!} \left(\frac{E}{u} \right)^{n+1} \\ &= {}_2F_1 \left(\frac{n-m+1}{2}, \frac{n+m+1}{2}, \frac{2n+3}{2}, \frac{E^2}{u^2} \right) \end{aligned} \quad (31)$$

where ${}_2F_1$ is the Gaussian hypergeometric function.

6 Review of Relativistic Results

Using the coordinate relations given in (21), the metric components of the gravitational field outside an ellipsoidal body can be rewritten in elliptic coordinates, and the unknown functions may then be determined by solving Einstein's field equations. As an example, we may take the g_{00} component of the metric of a complete ellipsoidal mass distribution in the form

$$g_{00} = 1 + \frac{1}{c^2} f(\lambda, u). \quad (32)$$

For the remaining metric components, a general form may be assumed based on the Schwarzschild metric and axial symmetry with respect to β , and the Einstein field equations can be solved accordingly. At this stage, the Newtonian limit in elliptic coordinates may be used to determine the function $f(\lambda, u)$. As mentioned earlier, this analysis is currently being developed by the authors.

7 Conclusion

In this work, we introduced the elliptic coordinate system and derived several well-known differential operators in this framework. Our motivation for employing elliptic coordinates lies in the considerable simplification they provide when solving Einstein's field equations. The authors have continued developing these solutions and have obtained a simplified derivation of the Kerr solution, which is now being prepared for publication. Since axisymmetric solutions play a central role in general relativity and astrophysics, it is natural to use axisymmetric coordinate systems such as the one defined by (21).

Acknowledgments

The first author expresses gratitude to the University of Tehran for supporting this work. Zeinab Nazari Doliskani thanks the Ministry of Education for supporting educational research. A substantial portion of this work was previously presented at the Mathematical Physics Conference held at Qom University of Technology. Borzoo Nazari sincerely thanks the organizers of that conference for their efforts and for selecting this work for publication in the present journal.

Appendix A: A Brief Overview of Generalized Legendre Functions

Equation (25) represents the angular part of the solution to Laplace's equation, and its solutions are well known and

treated in standard texts in mathematical physics. The generalized (or associated) Legendre functions arise as solutions to the associated Legendre differential equation

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + \left(\ell(\ell+1) - \frac{m^2}{1-x^2}\right)y = 0, \quad (\text{A.1})$$

where $\ell = 0, 1, 2, \dots$ is the degree and $m = 0, 1, 2, \dots, \ell$ is the order. This equation appears naturally when solving Laplace's equation in spherical coordinates and plays a central role in problems with axial symmetry [1, 6].

Appendix A.1: Legendre Polynomials and Their Generalizations

For $m = 0$, Eq. (A.1) reduces to the Legendre polynomials $P_\ell(x)$. For $m \neq 0$, the solutions generalize to the associated Legendre functions

$$P_\ell^m(x) = (1-x^2)^{m/2} \frac{d^m}{dx^m} P_\ell(x). \quad (\text{A.2})$$

These functions represent the angular dependence of fields in systems with axial (but not full spherical) symmetry.

Appendix A.2: Legendre Functions of the Second Kind

The second linearly independent solution of (A.1) is the Legendre function of the second kind, denoted $Q_\ell^m(x)$ [2]. In gravitational problems, Q_ℓ^m commonly appears in exterior (decaying) potentials, particularly when the coordinates introduce complex arguments, such as

$$Q_\ell^m\left(\frac{iu}{E}\right), \quad (\text{A.3})$$

which naturally arises in ellipsoidal coordinate geometries.

Appendix A.3: Connection to Spherical Harmonics

The generalized Legendre functions form the angular part of spherical harmonics:

$$Y_\ell^m(\theta, \phi) = N_{\ell m} P_\ell^m(\cos \theta) e^{im\phi}, \quad (\text{A.4})$$

where $N_{\ell m}$ is a normalization factor. Thus, spherical harmonics describe the angular dependence of three-dimensional solutions to Laplace's and Helmholtz's equations, while Q_ℓ^m governs the radial dependence in certain exterior potentials.

Appendix B: Normalized Spherical Harmonics

When solving Laplace's equation in spherical coordinates, the angular dependence of the solution is described by the *normalized spherical harmonics*, denoted by $Y_\ell^m(\theta, \phi)$. These functions form a complete orthonormal basis on the unit sphere and are indexed by the integers $\ell = 0, 1, 2, \dots$, $m = -\ell, \dots, \ell$.

The normalized spherical harmonics are defined in terms of the associated Legendre functions $P_\ell^m(x)$:

$$Y_\ell^m(\theta, \phi) = N_{\ell m} P_\ell^{|m|}(\cos \theta) e^{im\phi}, \quad (\text{B.5})$$

where the normalization constant is

$$N_{\ell m} = \sqrt{\frac{2\ell+1}{4\pi} \frac{(\ell-m)!}{(\ell+m)!}}. \quad (\text{B.6})$$

The normalization is chosen so that the spherical harmonics satisfy the orthonormality condition

$$\int_0^{2\pi} \int_0^\pi Y_\ell^m(\theta, \phi) Y_{\ell'}^{m'}(\theta, \phi) \sin \theta d\theta d\phi = \delta_{\ell\ell'} \delta_{mm'}. \quad (\text{B.7})$$

Spherical harmonics play a central role in quantum mechanics as the angular eigenfunctions of the orbital angular momentum operator, and in classical gravitational and electromagnetic potentials where axial or spherical symmetry is present [1, 6, 7].

References

1. J. D. Jackson, *Classical Electrodynamics, third edition*, John Wiley and Sons, (1999)
2. I. S. Gradshteyn, I. M. Ryzhik, D. Zwillinger, V. Moll, *Table of integrals, series and products, 8th edition*, Academic Press, Elsevier Inc. (2014)
3. Borzoo Nazari, 8th Iranian Conference on Mathematical Physics, (2022)
4. Borzoo Nazari, Khayyam 7th annual conference, (2024)

5. Clarence R. Wylie , Louis C. Barrett , *Advanced Engineering Mathematics, 6th edition*, McGraw-Hill College, (1995)
6. George B. Arfken, Hans J. Weber, Frank E. Harris, *Mathematical Methods for Physicists: A Comprehensive Guide, 7th edition*, Academic Press (2012)
7. A. Messiah, *Quantum Mechanics*, North-Holland, Amsterdam, (1961)