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A splitting operator-based finite difference method for the solution of 2D Fokker-Planck equations

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Abstract A simple and reliable numerical approach is constructed to solve the linear and nonlinear two-dimensional Fokker-Planck equations (FPEs). Initially, the Fokker–Planck equation is reformulated by decomposing it into one-dimensional components in the x and y directions. Then, local one-dimensional sub-equations are numerically solved by the explicit and implicit finite difference methods. The convergence of the proposed schemes is proved through truncation error and von Neumann stability analyses. The effectiveness and precision of the developed numerical methods are demonstrated using test problems, and the obtained outcomes are compared against the corresponding exact solutions for validation.

1 Introduction

The Fokker–Planck equation (FPE), originally formulated by Fokker (1914) and later by Planck (1917), serves as a mathematical model for describing Brownian particle motion. Over the years, FPEs have found widespread applications in diverse areas such as solid-state physics, quantum optics, chemical physics, theoretical biology, and circuit theory [1]. Various numerical strategies have been developed for their solution, including finite volume schemes [2, 3], Galerkin-type approaches [4–6], finite difference methods [7], and particle-based techniques [8]. To overcome the instabilities of the standard finite difference methods, nonstandard finite difference schemes are used to solve the one-dimensional FPEs in [9]. The most fundamental of these methods, the finite difference technique, has also been used to solve higher-dimensional FPE (2D) [10]. Some semi-analytical techniques have also been used to solve FPEs, of which the Adomian decomposition method [11] is a well-known example.

From a theoretical physics perspective, the Fokker–Planck equation occupies a central role as the forward Kolmogorov representation of stochastic dynamics and as the macroscopic limit of Langevin and Liouville equations. It combines the conservation of probability and the relaxation toward equilibrium distributions controlled by underlying potential landscapes, and bridges the gap between tiny random processes and macroscopic transport events. In addition to being a numerical simplification, the operator splitting technique used in this study decomposes the Fokker–Planck operator into commuting drift and diffusion generators, each with unique mathematical and physical features. This decomposition parallels the Lie–Trotter and Strang formulations widely used in quantum and statistical mechanics to separate reversible and irreversible dynamics. Therefore, the current research advances our understanding of Fokker-Planck evolution from a mathematical-physics perspective.

Higher-dimensional FPEs naturally arise in systems involving multiple interacting variables, such as in population dynamics, neural networks, and chemical reactions involving multiple species. However, the increased complexity due to high dimensionality imposes significant computational challenges, often referred to as the "curse of dimensionality" [12]. To address this, researchers have developed more efficient algorithms, including operator splitting methods [13], sparse grid techniques [14], and dimensionality reduction strategies [15]. For instance, splitting methods have been employed to decouple the multidimensional FPE into simpler sub-problems, making the numerical integration more tractable. Moreover, tensor-based methods [16] and sparse spectral approaches [17] have also been proposed to reduce computational costs while maintaining accuracy in higher-dimensional settings. These advancements extend the appli-

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cability of numerical FPE solvers to realistic, multi-variable systems in physics, biology, and engineering.

In recent years, many researchers have provided numerical solutions for one-dimensional FPEs through sophisticated methods, such as structure-preserving schemes [18–20], deep KD-tree algorithm [21], stable Petrov-Galerkin discretization [22], a method based on KRnet (ADDA-KR) [23], an extension of the generalized Hermite pseudospectral method [24], and a physically guided deep learning-based method [25]. In addition to solving one-dimensional FPEs, there are existing sophisticated numerical methods for solving higher-dimensional forms[26], such as one based on wavelet theory in [27] and Chang–Cooper two-level algorithms [28]. However, only a handful of high-order finite difference schemes are available in the literature [29–32].

This paper presents a finite difference numerical method for two-dimensional FPEs using the splitting technique, for which not many studies are found in the literature. The present study was motivated by the application of the splitting operator for Burger's equation in [33]. The goal of the proposed method is to solve the higher-dimensional FPEs with simple and accurate algorithms. The splitting technique is a locally one-dimensional method for higher dimensional partial differential equations, which resolves the difficulties faced by numerical methods directly applied to higher-dimensional PDEs [34]. In this sense, this work is an extension of the simplest numerical methods in [35] to higher dimensional FPEs. The main benefits of the splitting operator method are that it is swift and straightforward to use, requiring fewer significant numerical computations to solve higher-dimensional PDEs.

The evolution of the concentration function $w(x,t)$, with respect to the spatial coordinate x and temporal variable t, is modeled by the general linear FPE, which is written in the form,

$$\frac{\partial w}{\partial t} = \left[-\frac{\partial}{\partial x} A(x) + \frac{\partial^2}{\partial x^2} B(x) \right] w(x,t), \tag{1}$$

with the initial state

$$w(x,0) = f(x), \quad x \in \mathbb{R}, \tag{2}$$

where $B(x) > 0$ denotes the diffusion coefficient and $A(x)$ represents the drift coefficient. Its extension to two variables x_1, x_2 is given by

$$\frac{\partial w}{\partial t} = \left[-\sum_{i=1}^{2} \frac{\partial}{\partial x_i} A_i(\mathbf{x}) + \sum_{i,j=1}^{2} \frac{\partial^2}{\partial x_i \partial x_j} B_{i,j}(\mathbf{x}) \right] w, \tag{3}$$

with the initial state,

$$w(\mathbf{x},0) = g(\mathbf{x}), \qquad \mathbf{x} = (x_1, x_2) \in \mathbb{R}^2. \tag{4}$$

The non-linear FPE in two variables x_1, x_2 is represented as,

$$\begin{aligned} \frac{\partial w}{\partial t} = &-\sum_{i=1}^{2} \frac{\partial}{\partial x_i} \big(A_i(\mathbf{x},t,w)\, w\big) \\ &+ \sum_{i,j=1}^{2} \frac{\partial^2}{\partial x_i \partial x_j} \big(B_{ij}(\mathbf{x},t,w)\, w\big). \end{aligned} \tag{5}$$

The rest of the paper is organized as follows. Section 2 contains the model of the problem and the formulation of the numerical method. Section 3 discusses the convergence and stability of the proposed schemes. Section 4 provides numerical examples to show the efficiency of the schemes. Section 5 presents a brief discussion and conclusion.

2 Formulation of the computational techniques

2.1 Splitting operator technique

Consider, in expanded form, the linear FPE (3) in the domain $\Omega = [0,T] \times [a,b] \times [c,d]$,

$$\begin{aligned} \frac{\partial w}{\partial t} = &-A_1 \frac{\partial w}{\partial x} - w \frac{\partial A_1}{\partial x} - A_2 \frac{\partial w}{\partial y} - w \frac{\partial A_2}{\partial y} \\ &+ B_{11} \frac{\partial^2 w}{\partial x^2} + 2 \frac{\partial B_{11}}{\partial x} \frac{\partial w}{\partial x} + w \frac{\partial^2 B_{11}}{\partial x^2} \\ &+ B_{22} \frac{\partial^2 w}{\partial y^2} + 2 \frac{\partial B_{22}}{\partial y} \frac{\partial w}{\partial y} + w \frac{\partial^2 B_{22}}{\partial y^2}. \end{aligned} \tag{6}$$

with the initial state,

$$w(\mathbf{x},0) = g(\mathbf{x}), \quad \mathbf{x} = (x,y). \tag{7}$$

The development of the scheme is as follows: First Eqn.(6) is split into two equations as

$$\begin{aligned} \frac{1}{2} \frac{\partial w}{\partial t} = &-A_1 \frac{\partial w}{\partial x} - w \frac{\partial A_1}{\partial x} + B_{11} \frac{\partial^2 w}{\partial x^2} \\ &+ 2 \frac{\partial B_{11}}{\partial x} \frac{\partial w}{\partial x} + w \frac{\partial^2 B_{11}}{\partial x^2}, \end{aligned} \tag{8a}$$

$$\begin{aligned} \frac{1}{2} \frac{\partial w}{\partial t} = &-A_2 \frac{\partial w}{\partial y} - w \frac{\partial A_2}{\partial y} + B_{22} \frac{\partial^2 w}{\partial y^2} \\ &+ 2 \frac{\partial B_{22}}{\partial y} \frac{\partial w}{\partial y} + w \frac{\partial^2 B_{22}}{\partial y^2}. \end{aligned} \tag{8b}$$

As counterparts of Eqn. (6), the formulations in (8a) and (8b) correspond to the x- and y-directions, respectively. The resulting system is given in Eqns. (8) is then approximated numerically via the finite difference method, using both explicit and implicit discretizations, while avoiding any form of linearization. The process begins by computing the solution at the $(n+\frac{1}{2})^{\text{th}}$ level along the x-direction, employing the values from the n^{th} level while treating y as constant; these intermediate solutions correspond to points on the horizontal axis. Thereafter, the $(n+1)^{\text{th}}$ level solution is obtained along the y-direction from the $(n+\frac{1}{2})^{\text{th}}$ level results, with x held constant; these solutions lie on the vertical axis. It follows that the computed approximation at the $(n+1)^{\text{th}}$ step is consistent with the two-dimensional solution of Eqn. (6) evaluated at the same step.

2.2 The explicit method via splitting operator

Consider the spatial domain $\Omega = [a,b]\times[c,d] \subset \mathbb{R}^2$, which is discretized into a grid of $(N+1)\times(M+1)$ points. he discretization in the x- and y-directions is carried out with mesh sizes defined by $h = \frac{b-a}{N}$ and $k = \frac{d-c}{M}$, respectively. The temporal interval $[0,T]$ is uniformly divided into K subintervals with step size $\tau = \frac{T}{K}$. The discrete time instants are then denoted by $t^n = n\tau$, where $n = 0,1,\dots,K-1$. The numerical solution at the n^{th} time step corresponding to the mesh point (x_i, y_j, t^n) is represented by $\mathscr{W}_{ij}^n = \mathscr{W}(x_i, y_j, t^n)$.
The time and space derivatives are explicitly discretized by the forward difference and the central difference, respectively. The discretized Eqns.(8) become

$$\begin{aligned}\frac{\mathscr{W}_{ij}^{n+\frac{1}{2}} - \mathscr{W}_{ij}^n}{\tau} &= -A_1\frac{\mathscr{W}_{i+1,j}^n - \mathscr{W}_{i-1,j}^n}{2h} - \mathscr{W}_{ij}^n A_{1x}\\ &+ B_{11}\frac{\mathscr{W}_{i+1,j}^n - 2\mathscr{W}_{ij}^n + \mathscr{W}_{i-1,j}^n}{h^2}\\ &+ 2B_{11x}\frac{\mathscr{W}_{i+1,j}^n - \mathscr{W}_{i-1,j}^n}{2h}\\ &+ \mathscr{W}_{ij}^n B_{11xx}, \end{aligned} \tag{9a}$$

$$\begin{aligned}\frac{\mathscr{W}_{ij}^{n+1} - \mathscr{W}_{ij}^{n+\frac{1}{2}}}{\tau} &= -A_2\frac{\mathscr{W}_{i,j+1}^{n+\frac{1}{2}} - \mathscr{W}_{i,j-1}^{n+\frac{1}{2}}}{2k} - \mathscr{W}_{ij}^{n+\frac{1}{2}} A_{2y}\\ &+ B_{22}\frac{\mathscr{W}_{i,j+1}^{n+\frac{1}{2}} - 2\mathscr{W}_{ij}^{n+\frac{1}{2}} + \mathscr{W}_{i,j-1}^{n+\frac{1}{2}}}{k^2}\\ &+ 2B_{22y}\frac{\mathscr{W}_{i,j+1}^{n+\frac{1}{2}} - \mathscr{W}_{i,j-1}^{n+\frac{1}{2}}}{2k}\\ &+ \mathscr{W}_{ij}^{n+\frac{1}{2}} B_{22yy}. \end{aligned} \tag{9b}$$

where $0 \le i \le N$, $0 \le j \le M$, and $0 \le n \le K-1$. The initial and boundary conditions in discrete form are

$$\begin{aligned}&\mathscr{W}_{ij}^0 = g(x_0, y_0); 0 \le i \le N, 0 \le j \le M,\\ &\mathscr{W}_a^n = f_1(y,t), \mathscr{W}_b^n = f_2(y,t),\\ &\mathscr{W}_c^n = f_3(x,t), \mathscr{W}_d^n = f_4(x,t).\end{aligned} \tag{10}$$

Eqns.(9) can be written as

$$\mathscr{W}_{ij}^{n+\frac{1}{2}} = \frac{2}{\tau}\left(\theta_i \mathscr{W}_{i+1,j}^n + \sigma_i \mathscr{W}_{i,j}^n + \omega_i \mathscr{W}_{i,j}^n\right), \tag{11a}$$

$$\mathscr{W}_{i,j}^{n+1} = \frac{2}{\tau}\left(\lambda_j \mathscr{W}_{i,j+1}^{n+\frac{1}{2}} + \mu_j \mathscr{W}_{i,j}^{n+\frac{1}{2}} + \nu_j \mathscr{W}_{i,j-1}^{n+\frac{1}{2}}\right), \tag{11b}$$

where the coefficients are defined by $\theta_i = (-A_1 h + 2B_{11} + 2h\,\partial_{x_i} B_{11})/(2h^2)$, $\sigma_i = 1 - \partial_{x_i} A_1 + \partial_{x_i}^2 B_{11} - 2B_{11}/h^2$, $\omega_i = (A_1 h + 2B_{11} - 2h\,\partial_{x_i} B_{11})/(2h^2)$, and similarly $\lambda_j = (-A_2 k + 2B_{22} + 2k\,\partial_{y_j} B_{22})/(2k^2)$, $\mu_j = 1 - \partial_{y_j} A_2 + \partial_{y_j}^2 B_{22} - 2B_{22}/k^2$, $\nu_j = (A_2 k + 2B_{22} - 2k\,\partial_{y_j} B_{22})/(2k^2)$.

The numerical scheme (11) is an explicit central difference scheme for the splitting operator Eqns. (8).

For the nonlinear FPE, the nonlinear terms in Eqn.(6) are treated in the following ways:

- $w^2 = \mathscr{W}_{i,j}^n \mathscr{W}_{i,j}^n$ in the x–direction, and
  $w^2 = \mathscr{W}_{i,j}^{n+\frac{1}{2}} \mathscr{W}_{i,j}^{n+\frac{1}{2}}$ in the y–direction
- $w w_x = \mathscr{W}_{i,j}^n \left(\frac{\mathscr{W}_{i+1,j}^n - \mathscr{W}_{i-1,j}^n}{2h}\right)$, and
  $w w_y = \mathscr{W}_{i,j}^{n+\frac{1}{2}} \left(\frac{\mathscr{W}_{i,j+1}^{n+\frac{1}{2}} - \mathscr{W}_{i,j-1}^{n+\frac{1}{2}}}{2k}\right)$
- $w w_{xx} = \mathscr{W}_{i,j}^n \left(\frac{\mathscr{W}_{i+1,j}^n - \mathscr{W}_{i,j}^n + \mathscr{W}_{i-1,j}^n}{h^2}\right)$, and
  $w w_{yy} = \mathscr{W}_{i,j}^{n+\frac{1}{2}} \left(\frac{\mathscr{W}_{i,j+1}^{n+\frac{1}{2}} - \mathscr{W}_{i,j}^{n+\frac{1}{2}} + \mathscr{W}_{i,j-1}^{n+\frac{1}{2}}}{k^2}\right)$
- $w_x^2 = \left(\frac{\mathscr{W}_{i+1,j}^n - \mathscr{W}_{i-1,j}^n}{2h}\right)^2$, and
  $w_y^2 = \left(\frac{\mathscr{W}_{i,j+1}^{n+\frac{1}{2}} - \mathscr{W}_{i,j-1}^{n+\frac{1}{2}}}{2k}\right)^2$

2.3 The implicit method via the splitting operator

In the case of the implicit scheme, the decomposed Eqns. (8) are approximated using a forward difference in time combined with central differences in the spatial variables. The resulting discretization takes the following form:

$$\frac{\mathscr{W}_{ij}^{n+\frac{1}{2}}-\mathscr{W}_{ij}^{n}}{\tau}=-A_1\frac{\mathscr{W}_{i+1,j}^{n+\frac{1}{2}}-\mathscr{W}_{i-1,j}^{n+\frac{1}{2}}}{2h}-\mathscr{W}_{ij}^{n+\frac{1}{2}}A_{1x}+B_{11}\frac{\mathscr{W}_{i+1,j}^{n+\frac{1}{2}}-2\mathscr{W}_{ij}^{n+\frac{1}{2}}+\mathscr{W}_{i-1,j}^{n+\frac{1}{2}}}{h^2}+2B_{11x}\frac{\mathscr{W}_{i+1,j}^{n+\frac{1}{2}}-\mathscr{W}_{i-1,j}^{n+\frac{1}{2}}}{2h}+\mathscr{W}_{ij}^{n+\frac{1}{2}}B_{11xx}, \tag{12a}$$

$$\frac{\mathscr{W}_{ij}^{n+1}-\mathscr{W}_{ij}^{n+\frac{1}{2}}}{\tau}=-A_2\frac{\mathscr{W}_{i,j+1}^{n+1}-\mathscr{W}_{i,j-1}^{n+1}}{2k}-\mathscr{W}_{ij}^{n+1}A_{2y}+B_{22}\frac{\mathscr{W}_{i,j+1}^{n+1}-2\mathscr{W}_{ij}^{n+1}+\mathscr{W}_{i,j-1}^{n+1}}{k^2}+2B_{22y}\frac{\mathscr{W}_{i,j+1}^{n+1}-\mathscr{W}_{i,j-1}^{n+1}}{2k}+\mathscr{W}_{ij}^{n+1}B_{22yy}. \tag{12b}$$

where $0\le i\le N$, $0\le j\le M$ and $0\le n\le K-1$. The initial and boundary conditions in Eqns.(12) are

$$\mathscr{W}_{i,j}^{0}=g(x_0,y_0);0\le i\le N,0\le j\le M,$$
$$\mathscr{W}_{a}^{n}=f_1(y,t),\mathscr{W}_{b}^{n}=f_2(y,t), \tag{13}$$
$$\mathscr{W}_{c}^{n}=f_3(x,t),\mathscr{W}_{d}^{n}=f_4(x,t).$$

The discrete forms (12) in the x and y directions, respectively, may be written as follows:

$$\frac{\tau}{2h^2}\left(\alpha_i\mathscr{W}_{i-1,j}^{n+\frac{1}{2}}+\beta_i\mathscr{W}_{i,j}^{n+\frac{1}{2}}+\gamma_i\mathscr{W}_{i+1,j}^{n+\frac{1}{2}}\right)=\mathscr{W}_{ij}^{n}, \tag{14a}$$

$$\frac{\tau}{2k^2}\left(\xi_j\mathscr{W}_{i,j-1}^{n+1}+\chi_j\mathscr{W}_{i,j}^{n+1}+\delta_j\mathscr{W}_{i,j+1}^{n+1}\right)=\mathscr{W}_{i,j}^{n+\frac{1}{2}}, \tag{14b}$$

where,

$$\alpha_i=h(-A_1+2B_{11x})-2B_{11},$$
$$\beta_i=2h^2\left(\tfrac{1}{\tau}+A_{1x}-B_{11xx}\right)+4B_{11}, \tag{15}$$
$$\gamma_i=h(A_1-2B_{11x})-2B_{11},$$

and,

$$\xi_j=k(-A_2+2B_{22y})-2B_{22},$$
$$\chi_j=2k^2\left(\tfrac{1}{\tau}+A_{2y}-B_{22yy}\right)+4B_{22}, \tag{16}$$
$$\delta_j=k(A_2-2B_{22y})-2B_{22}.$$

For the nonlinear FPE (6) nonlinear terms are discretized in the following ways:

- $w^2=\mathscr{W}_{i,j}^{n}\mathscr{W}_{i,j}^{n+\frac{1}{2}}$ in the $x-$ direction and $w^2=\mathscr{W}_{i,j}^{n+\frac{1}{2}}\mathscr{W}_{i,j}^{n+1}$ in the $y-$ direction
- $w\frac{\partial w}{\partial x}=\mathscr{W}_{i,j}^{n}\left(\frac{\mathscr{W}_{i+1,j}^{n+\frac{1}{2}}-\mathscr{W}_{i-1,j}^{n+\frac{1}{2}}}{2h}\right)$, and $w\frac{\partial w}{\partial y}=\mathscr{W}_{i,j}^{n+\frac{1}{2}}\left(\frac{\mathscr{W}_{i,j+1}^{n+1}-\mathscr{W}_{i,j-1}^{n+1}}{2k}\right)$
- $w\frac{\partial^2 w}{\partial x^2}=\mathscr{W}_{i,j}^{n}\left(\frac{\mathscr{W}_{i+1,j}^{n+\frac{1}{2}}-\mathscr{W}_{i,j}^{n+\frac{1}{2}}+\mathscr{W}_{i-1,j}^{n+\frac{1}{2}}}{h^2}\right)$, and $w\frac{\partial^2 w}{\partial y^2}=\mathscr{W}_{i,j}^{n+\frac{1}{2}}\left(\frac{\mathscr{W}_{i,j+1}^{n+1}-\mathscr{W}_{i,j}^{n+1}+\mathscr{W}_{i,j-1}^{n+1}}{k^2}\right)$
- $\left(\frac{\partial w}{\partial x}\right)^2=\left(\frac{\mathscr{W}_{i+1,j}^{n}-\mathscr{W}_{i-1,j}^{n}}{2h}\right)\left(\frac{\mathscr{W}_{i+1,j}^{n+\frac{1}{2}}-\mathscr{W}_{i-1,j}^{n+\frac{1}{2}}}{2h}\right)$, and $\left(\frac{\partial w}{\partial y}\right)^2=\left(\frac{\mathscr{W}_{i,j+1}^{n+\frac{1}{2}}-\mathscr{W}_{i,j-1}^{n+\frac{1}{2}}}{2k}\right)\left(\frac{\mathscr{W}_{i,j+1}^{n+1}-\mathscr{W}_{i,j-1}^{n+1}}{2k}\right)$

The formulation (14) is semi-implicit, achieved by linearizing the nonlinear source terms through a split evaluation at successive and previous time levels.

3 Consistency and stability

The assessment of the error and stability of the numerical schemes is given in this section. The consistency of the computational schemes is proved by the truncation error method, and their stability is derived from von Neumann stability analysis.

Theorem 1 *The numerical scheme given in (11) achieves first-order accuracy with respect to the temporal variable, while attaining second-order accuracy in the spatial directions x and y.*

Proof The fully discretized form of the explicit central difference scheme (11) is

$$\mathscr{W}_{ij}^{n+1} = \frac{4}{\tau^2}\big(\lambda_j(\theta_i\mathscr{W}_{i+1,j+1}^n + \sigma_i\mathscr{W}_{i,j+1}^n + \omega_i\mathscr{W}_{i-1,j+1}^n) + \mu_j(\theta_i\mathscr{W}_{i+1,j}^n + \sigma_i\mathscr{W}_{ij}^n + \omega_i\mathscr{W}_{i-1,j}^n) + \nu_j(\theta_i\mathscr{W}_{i+1,j-1}^n + \sigma_i\mathscr{W}_{i,j-1}^n + \omega_i\mathscr{W}_{i-1,j-1}^n)\big). \tag{17}$$

Applying Taylor expansion on all terms of Eq. (17) and simplifying results in the local truncation error (LTE) given by

$$\begin{aligned} LTE &= \mathscr{W}_{ij}^{n+1} - \frac{4}{\tau^2}\Big[\lambda_j(\theta_i W_{i+1,j+1}^n + \sigma_i W_{i,j+1}^n + \omega_i W_{i-1,j+1}^n) + \mu_j(\theta_i W_{i+1,j}^n + \sigma_i W_{ij}^n + \omega_i W_{i-1,j}^n) + \nu_j(\theta_i W_{i+1,j-1}^n + \sigma_i W_{i,j-1}^n + \omega_i W_{i-1,j-1}^n)\Big] \\ &= \tau^2 W + \tau^2 W_t + \tfrac{\tau^2}{2!}W_{tt} - 4\lambda\theta W - 4\lambda\theta h W_x - 4\lambda\theta k W_y - 2\lambda\theta h^2 W_{xx} - 2\lambda\theta k^2 W_{yy} - 8\lambda\theta hk W_{xy} - \lambda\sigma W - 4k\lambda\sigma W_y \\ &\quad - 2\lambda\sigma k^2 W_{yy} - 4\lambda\omega W + 4h\lambda\omega W_x - 4k\lambda\omega W_y - 2\lambda\omega h^2 W_{xx} - 2\lambda\omega k^2 W_{yy} + 8hk\lambda\omega W_{xy} - 4\mu\omega W + 4h\mu\omega W_x \\ &\quad - 2\mu\omega h^2 W_{xx} - 2\nu\theta k^2 W_{yy} - 4\nu\theta W + 2\nu\omega h^2 W_{xx} - 2\nu\theta k^2 W_{yy} + \cdots \\ &= \tau^2 W + t\tau^2 W_t + \tfrac{t^2\tau^2}{2!}W_{tt} - 2\lambda\omega h^2 W_{xx} - 2\nu\omega k^2 W_{yy} + \cdots \\ &= \mathscr{O}(\tau^2 + h^2 + k^2), \end{aligned} \tag{18}$$

where,

$$\begin{aligned} \theta &= (-A_1 h + 2B_{11} + 2h\tfrac{\partial B_{11}}{\partial x})/(2h^2), \\ \sigma &= \big(h^2(1 - \tfrac{\partial A_1}{\partial x} + \tfrac{\partial^2 B_{11}}{\partial x^2}) - 2B_{11}\big)/h^2, \\ \omega &= (A_1 h + 2B_{11} - 2h\tfrac{\partial B_{11}}{\partial x})/(2h^2), \\ \lambda &= (-A_2 k + 2B_{22} + 2k\tfrac{\partial B_{22}}{\partial y})/(2k^2), \\ \mu &= \big(k^2(1 - \tfrac{\partial A_2}{\partial y} + \tfrac{\partial^2 B_{22}}{\partial y^2}) - 2B_{22}\big)/k^2, \\ \nu &= (A_2 k + 2B_{22} - 2k\tfrac{\partial B_{22}}{\partial y})/(2k^2). \end{aligned} \tag{19}$$

The truncation error is

$$TE = \tau^{-1}(LTE) = \mathscr{O}(\tau) + \mathscr{O}(h^2 + k^2). \tag{20}$$

Accordingly, the numerical approximation incurs errors that are of order $\mathscr{O}(\tau)$ in time and $\mathscr{O}(h^2+k^2)$ in space, consistent with the stated result.

Theorem 2 *The explicit central difference scheme defined in (11) admits conditional stability, holding only under specific restrictions on the discretization parameters.*

Proof The stability requirement for the proposed explicit scheme is established through von Neumann stability analysis and can be formulated as follows.

Let the Fourier tranform corresponding to $\mathscr{W}^n$ at the n^{th} time level be denoted by, $\hat{\mathscr{W}}^n$. The 2D Fourier transform is

$$\hat{\mathscr{W}}^{n+1}(\zeta,\eta) = \frac{1}{\sqrt{2\pi}\sqrt{2\pi}}\sum_{j,k=-\infty}^{\infty} e^{-ij\zeta h - i\kappa\eta k}\,\mathscr{W}_{i,j}^{n+1}. \tag{21}$$

Performing a Fourier transformation of Eqn. (11) results in,

$$\begin{aligned} \hat{\mathscr{W}}^{n+\frac12}(\zeta,\eta) &= \frac{2}{\tau}\Big(\theta\, e^{ij\zeta h}\hat{\mathscr{W}}^n(\zeta,\eta) + \sigma\,\hat{\mathscr{W}}^n(\zeta,\eta) + \omega\, e^{-ij\zeta h}\hat{\mathscr{W}}^n(\zeta,\eta)\Big), \\ \hat{\mathscr{W}}^{n+1}(\zeta,\eta) &= \frac{2}{\tau}\Big(\lambda\, e^{i\kappa\eta k}\hat{\mathscr{W}}^{n+\frac12}(\zeta,\eta) + \mu\,\hat{\mathscr{W}}^{n+\frac12}(\zeta,\eta) + \nu\, e^{-i\kappa\eta k}\hat{\mathscr{W}}^{n+\frac12}(\zeta,\eta)\Big), \\ \hat{\mathscr{W}}^{n+1}(\zeta,\eta) &= \hat{\mathscr{W}}^n(\zeta,\eta)\,\frac{4}{\tau^2}\Big(\theta\, e^{ij\zeta h} + \sigma + \omega\, e^{-ij\zeta h}\Big)\Big(\lambda\, e^{i\kappa\eta k} + \mu + \nu\, e^{-i\kappa\eta k}\Big). \end{aligned} \tag{22}$$

Here the amplification factor $\rho(\zeta,\eta)$ is,

$$\rho(\zeta,\eta) = \frac{4}{\tau^2}\Big(\theta e^{ij\zeta h} + \sigma + \omega e^{-ij\zeta h}\Big) \times \Big(\lambda e^{i\kappa\eta k} + \mu + \nu e^{-i\kappa\eta k}\Big). \tag{23}$$

Rewriting the exponential terms in terms of trigonometric functions, Eqn. (23) can be expressed as

$$\begin{aligned} \rho(\zeta,\eta) = \frac{4}{\tau^2}\Big[&\big(\sigma + (\theta+\omega)\cos(\zeta h)\big) \times \big(\mu + (\lambda+\nu)\cos(\eta k)\big) - (\theta-\omega)(\lambda-\nu)\sin(\zeta h)\sin(\eta k)\Big] \\ &+ i\Big[\big(\mu + (\lambda+\nu)\cos(\eta k)\big)(\theta-\omega)\sin(\zeta h) + \big(\sigma + (\theta+\omega)\cos(\zeta h)\big) \times (\lambda-\nu)\sin(\eta k)\Big]. \end{aligned} \tag{24}$$

Therefore,

$$|\rho|^2 = \frac{4}{\tau^2}\left[(\theta-\omega)^2\, sin^2\zeta h + (\sigma+(\theta+\omega)\, cos\zeta h)^2\right] \times \left[(\lambda-\nu)^2\, sin^2\eta k + (\mu+(\lambda+\nu))\, cos\eta k)^2\right] \tag{25}$$

By the von Neumann criteria, the proposed scheme is stable under the condition $|\rho|^2 \leq 1$ and hence conditionally stable.

Theorem 3 *The implicit central difference scheme (14) is second-order convergent in temporal and spatial variables.*

Proof The LTE at the node (x_i, y_j, t^{n+1}) is given by

$$\begin{aligned}
LTE_{ij}^{n+1} &= \mathscr{W}_{i,j}^{n+1} \\
&\quad - \frac{\tau}{2k^2}\Big(\xi_j \mathscr{W}_{i,j-1}^{n+1} + \chi_j \mathscr{W}_{i,j}^{n+1} + \delta_j \mathscr{W}_{i,j+1}^{n+1}\Big) \\
&\quad - \frac{1}{\beta_i}\frac{2h^2}{\tau}\mathscr{W}_{i,j}^{n} \\
&\quad + \frac{\alpha_i}{\beta_i}\frac{\tau}{2k^2}\Big(\xi_j \mathscr{W}_{i+1,j-1}^{n+1} + \chi_j \mathscr{W}_{i+1,j}^{n+1} + \delta_j \mathscr{W}_{i+1,j+1}^{n+1}\Big) \\
&\quad + \frac{\gamma_i}{\beta_i}\frac{\tau}{2k^2}\Big(\xi_j \mathscr{W}_{i-1,j-1}^{n+1} + \chi_j \mathscr{W}_{i-1,j}^{n+1} + \delta_j \mathscr{W}_{i-1,j+1}^{n+1}\Big) \\
&= \Big(1 - \chi_j \frac{\tau}{2k^2}\Big)\mathscr{W}_{i,j}^{n+1} - \frac{1}{\beta_i}\frac{2h^2}{\tau}\mathscr{W}_{i,j}^{n} \\
&\quad + \frac{\tau}{2k^2}\Big(- \xi_j \mathscr{W}_{i,j-1}^{n+1} - \delta_j \mathscr{W}_{i,j+1}^{n+1} \\
&\qquad + \frac{\alpha_i\xi_j}{\beta_i}\mathscr{W}_{i+1,j-1}^{n+1} + \frac{\alpha_i\chi_j}{\beta_i}\mathscr{W}_{i+1,j}^{n+1}\Big) \\
&\quad + \frac{\tau}{2k^2}\Big(\frac{\alpha_i\delta_j}{\beta_i}\mathscr{W}_{i+1,j+1}^{n+1} + \frac{\gamma_i\xi_j}{\beta_i}\mathscr{W}_{i-1,j-1}^{n+1} \\
&\qquad + \frac{\gamma_i\chi_j}{\beta_i}\mathscr{W}_{i-1,j}^{n+1} + \frac{\gamma_i\delta_j}{\beta_i}\mathscr{W}_{i-1,j+1}^{n+1}\Big) \\
&= \Big(1 - \chi \frac{\tau}{2k^2}\Big)\mathscr{W}(x,y,t+\tau) \\
&\quad - \frac{1}{\beta}\frac{2h^2}{\tau}\mathscr{W}(x,y,t) \\
&\quad + \frac{\tau}{2k^2}\Big(- \xi\, \mathscr{W}(x,y-k,t+\tau) \\
&\qquad - \delta\, \mathscr{W}(x,y+k,t+\tau) \\
&\qquad + \frac{\alpha\xi}{\beta}\mathscr{W}(x+h,y-k,t+\tau)\Big) \\
&\quad + \frac{\tau}{2k^2}\Big(\frac{\alpha\chi}{\beta}\mathscr{W}(x+h,y,t+\tau) \\
&\qquad + \frac{\alpha\delta}{\beta}\mathscr{W}(x+h,y+k,t+\tau) \\
&\qquad + \frac{\gamma\xi}{\beta}\mathscr{W}(x-h,y-k,t+\tau)\Big) \\
&\quad + \frac{\tau}{2k^2}\Big(\frac{\gamma\chi}{\beta}\mathscr{W}(x-h,y,t+\tau) \\
&\qquad + \frac{\gamma\delta}{\beta}\mathscr{W}(x-h,y+k,t+\tau)\Big).
\end{aligned} \tag{26}$$

Expanding each term by Taylor's series expansion and simplifying,

$$\begin{aligned}
LTE &= \Big(\xi + \chi + \frac{\alpha\xi}{\beta} + \frac{\gamma\chi}{\beta} + \ldots\Big)\tau^3\mathscr{W} \\
&\quad + \Big(\frac{\alpha\xi}{\beta} - \frac{\gamma\chi}{\beta} + \ldots\Big)\tau^3\mathscr{W}_t \\
&\quad + \Big(-\beta + \frac{\gamma\delta}{\beta} + \ldots\Big)\tau^3\mathscr{W}_{tt} + \ldots \\
&\quad + \tau\Big(\frac{\alpha\xi}{2\beta} + \frac{\delta\chi}{4\beta} + \ldots\Big)\mathscr{W}_{xx} \\
&\quad + \tau\Big(\frac{\alpha\delta}{4\beta} + \frac{\gamma\eta}{4\beta} + \ldots\Big)\mathscr{W}_{yy} + \ldots,
\end{aligned} \tag{27}$$

where

$$\begin{aligned}
\alpha &= h\Big(-A_1 + 2\frac{\partial B_{11}}{\partial x}\Big) - 2B_{11}, \\
\beta &= 2h^2\Big(\frac{1}{\tau} + \frac{\partial A_1}{\partial x} + \frac{\partial^2 B_{11}}{\partial x^2}\Big) + 4B_{11}, \\
\gamma &= h\Big(A_1 - 2\frac{\partial B_{11}}{\partial x}\Big) - 2B_{11}, \\
\xi &= k\Big(-A_2 + 2\frac{\partial B_{22}}{\partial y^n}\Big) - 2B_{22}, \\
\chi &= 2k^2\Big(\frac{1}{\tau} + \frac{\partial A_2}{\partial y} + \frac{\partial^2 B_{22}}{\partial y^2}\Big) + 4B_{22}, \\
\delta &= k\Big(A_2 - 2\frac{\partial B_{22}}{\partial y}\Big) - 2B_{22}.
\end{aligned} \tag{28}$$

The global truncation error is

$$TE = \frac{1}{\tau}LTE = O(\tau^2 + h^2 + k^2), \tag{29}$$

which is quadratic in temporal and spatial variables, and hence the claim.

Theorem 4 *The implicit central difference scheme described in (14) possesses unconditional stability, independent of the discretization parameters.*

Proof The combined form of the decomposed Eqns. (14) is,

$$\begin{aligned}
\mathscr{W}_{i,j}^{n+1} &= \frac{4k^2h^2}{\beta_i\chi_j\tau^2}\mathscr{W}_{i,j}^{n} - \frac{\gamma_i\delta_j}{\beta_i\chi_j}\mathscr{W}_{i+1,j+1}^{n+1} - \frac{\gamma_i\chi_j}{\beta_i\chi_j}\mathscr{W}_{i+1,j}^{n+1} \\
&\quad - \frac{\gamma_i\xi_j}{\beta_i\chi_j}\mathscr{W}_{i+1,j-1}^{n+1} - \frac{\alpha_i\delta_j}{\beta_i\chi_j}\mathscr{W}_{i-1,j+1}^{n+1} \\
&\quad - \frac{\alpha_i\beta_j}{\beta_i\chi_j}\mathscr{W}_{i-1,j}^{n+1} - \frac{\alpha_i\xi_j}{\beta_i\chi_j}\mathscr{W}_{i-1,j-1}^{n+1}.
\end{aligned} \tag{30}$$

Applying the Fourier transform to each term in Eqn.(30), obtain

$$\begin{aligned}
&\hat{\mathscr{W}}^{n+1}(\zeta,\eta) \\
&+\frac{1}{\beta\chi}\left[\gamma\delta e^{ij\zeta h+i\kappa\eta k}+\gamma\chi e^{ij\zeta h}+\gamma\xi e^{ij\zeta h-i\kappa\eta k}\right] \\
&\quad\times\hat{\mathscr{W}}^{n+1}(\zeta,\eta) \\
&+\frac{1}{\beta\chi}\left[\alpha\delta e^{-ij\zeta h+i\kappa\eta k}+\alpha\xi e^{-ij\zeta h}+\alpha\xi e^{-ij\zeta h-i\kappa\eta k}\right] \\
&\quad\times\hat{\mathscr{W}}^{n+1}(\zeta,\eta) \\
&=\frac{4k^2h^2}{\beta\chi\tau}\hat{\mathscr{W}}^{n}(\zeta,\eta).
\end{aligned} \tag{31}$$

Rewriting each term in terms of trigonometric functions, Eqn. (31) may be written as

$$\begin{aligned}
&\beta\chi\hat{\mathscr{W}}^{n+1} \\
&\quad+\Big[\gamma\delta\left(\cos\zeta h+i\sin\zeta h\right)\left(\cos\eta k+i\sin\eta k\right) \\
&\qquad+\gamma\chi\left(\cos\zeta h+i\sin\zeta h\right)\Big]\hat{\mathscr{W}}^{n+1} \\
&\quad+\Big[\gamma\xi\left(\cos\zeta h+i\sin\zeta h\right)\left(\cos\eta k-i\sin\eta k\right) \\
&\qquad+\alpha\delta\left(\cos\zeta h-i\sin\zeta h\right)\left(\cos\eta k+i\sin\eta k\right)\Big] \\
&\qquad\times\hat{\mathscr{W}}^{n+1} \\
&\quad+\Big[\alpha\xi\left(\cos\zeta h-i\sin\zeta h\right) \\
&\qquad+\alpha\xi\left(\cos\zeta h-i\sin\zeta h\right)\left(\cos\eta k-i\sin\eta k\right)\Big] \\
&\qquad\times\hat{\mathscr{W}}^{n+1} \\
&=\frac{4k^2h^2}{\tau}\hat{\mathscr{W}}^{n}(\zeta,\eta).
\end{aligned} \tag{32}$$

The amplification factor ρ can be derived from (32) as

$$\rho=\frac{4k^2h^2}{\tau(\mathscr{U}+i\mathscr{V})}, \tag{33}$$

where,

$$\begin{aligned}
\mathscr{U}&=\beta\chi+(\gamma\delta+\alpha\xi)\cos(\zeta h+\eta k) \\
&\quad+(\gamma\chi+\alpha\xi)\cos(\zeta h) \\
&\quad+(\gamma\xi+\alpha\xi)\cos(\zeta h-\eta k), \\
\mathscr{V}&=(\gamma\delta-\alpha\xi)\sin(\zeta h+\eta k) \\
&\quad+(\gamma\chi-\alpha\xi)\sin(\zeta h) \\
&\quad+(\gamma\xi-\alpha\xi)\sin(\zeta h-\eta k).
\end{aligned} \tag{34}$$

Consequently, the modulus of the amplification factor takes the form,

$$\rho\overline{\rho}=|\rho|^2=\frac{16k^4h^4}{\tau^2(\mathscr{U}^2+\mathscr{V}^2)}. \tag{35}$$

Here $16h^4k^4 << \tau^2$ and $\mathscr{U}^2+\mathscr{V}^2\leq 1$

However, $16h^4k^4\leq\tau^2(\mathscr{U}^2+\mathscr{V}^2)$ and thus $|\rho|^2\leq 1$, unconditionally. Thus, by the von Neumann criteria, the numerical scheme (14) is unconditionally stable.

4 Numerical Illustrations

This section presents numerical experiments to confirm the theoretical findings, where the accuracy of the scheme is assessed using the l_2 and l_∞ error norms defined by

$$\begin{aligned}
l_2&=\frac{1}{h}\sqrt{\sum_{i=0}^{N}(\mathbf{W}_i-\mathbf{w}_i)^2}, \\
l_\infty&=\max_i;|\mathbf{W}_i-\mathbf{w}_i|.
\end{aligned} \tag{36}$$

In these expressions, $\mathbf{w}_i$ denotes the numerical approximation, $\mathbf{W}_i$ represents the exact solution at node i, and N is the total number of spatial nodes in the computational domain.

4.1 Example 1

Consider Eqn.(6) with

$$\begin{aligned}
&A_1=\frac{5x}{6}\quad ,A_2=\frac{5y}{6}, \\
&B_{11}=\frac{x^2}{6},\quad B_{12}=B_{21}=0,\quad B_{22}=\frac{y^2}{6}.
\end{aligned} \tag{37}$$

Then the following equation is obtained:

$$\frac{\partial w}{\partial t}=-w-\frac{x}{6}\frac{\partial w}{\partial x}-\frac{y}{6}\frac{\partial w}{\partial y}+\frac{x^2}{6}\frac{\partial^2 w}{\partial x^2}+\frac{y^2}{6}\frac{\partial^2 w}{\partial y^2}, \tag{38}$$

for which the exact solution can be determined,

$$w(x,y,t)=(1-x^2)(1-y^2)e^{-t},\qquad -1\leq x,y\leq 1. \tag{39}$$

A comparison of errors in l_2 and l_∞ norms for different numbers of nodes with elapsed times for the proposed schemes is presented in Table 1. Table 2 compares the proposed schemes' computed errors with existing schemes in the literature [34], and shows that the proposed simple algorithms give better results than the existing schemes.

Table 1: Comparison of errors in l_2 and l_∞ norms with $\tau = 0.0001$, $\mathscr{T} = 1$ for the Example 4.1.

	Explicit method			Implicit method			Explicit method			Implicit method		
	$\mathscr{T} = 0.1$			$\mathscr{T} = 0.1$			$\mathscr{T} = 0.25$			$\mathscr{T} = 0.25$		
Grid size	l_2	l_∞	time(s)	l_2	l_∞	time(s)	l_2	l_∞	time(s)	l_2	l_∞	time(s)
10×10	1.08×10^{-04}	5.32×10^{-05}	0.199525	1.01×10^{-05}	4.25×10^{-05}	0.118825	2.32×10^{-04}	9.74×10^{-05}	0.106984	2.32×10^{-05}	8.77×10^{-05}	1.932976
20×20	1.53×10^{-05}	4.52×10^{-05}	0.188134	2.03×10^{-05}	3.52×10^{-05}	1.596892	3.28×10^{-05}	8.21×10^{-05}	0.413535	2.57×10^{-05}	8.74×10^{-06}	3.771442
30×30	1.87×10^{-06}	4.52×10^{-07}	0.387606	1.21×10^{-07}	2.52×10^{-07}	2.949689	4.02×10^{-06}	7.74×10^{-07}	0.903345	3.02×10^{-07}	7.54×10^{-08}	6.742057
40×40	2.16×10^{-06}	5.52×10^{-07}	0.705215	3.61×10^{-07}	2.52×10^{-08}	6.945364	4.64×10^{-06}	9.24×10^{-07}	1.608009	5.64×10^{-07}	6.74×10^{-08}	17.982288
50×50	2.41×10^{-06}	4.52×10^{-07}	1.053064	1.32×10^{-07}	2.52×10^{-08}	10.32946	5.19×10^{-06}	7.74×10^{-07}	2.541735	5.16×10^{-07}	9.74×10^{-08}	26.486728
60×60	2.64×10^{-06}	3.66×10^{-07}	1.666046	1.64×10^{-07}	4.22×10^{-08}	14.307854	5.69×10^{-06}	7.22×10^{-07}	3.651856	4.69×10^{-07}	6.74×10^{-08}	44.764648
70×70	2.85×10^{-06}	4.33×10^{-07}	2.513052	2.86×10^{-07}	4.52×10^{-08}	18.132008	6.14×10^{-06}	6.74×10^{-07}	5.038307	6.11×10^{-07}	9.22×10^{-08}	53.097249
80×80	3.05×10^{-06}	5.52×10^{-07}	4.307044	2.05×10^{-07}	4.42×10^{-08}	24.037259	6.57×10^{-06}	9.01×10^{-07}	6.530545	5.75×10^{-07}	9.22×10^{-08}	6.74×10^{-08}
90×90	3.24×10^{-06}	3.52×10^{-07}	5.20769	2.24×10^{-07}	4.02×10^{-08}	43.586021	6.97×10^{-06}	8.74×10^{-07}	7.987636	5.97×10^{-07}	7.74×10^{-08}	119.364149
100×100	3.24×10^{-06}	4.52×10^{-07}	8.757617	3.41×10^{-07}	4.52×10^{-08}	54.792485	7.34×10^{-06}	9.74×10^{-07}	9.874276	7.22×10^{-07}	1.74×10^{-08}	153.558578

Table 2: Comparison of errors in l_2 and l_∞ norms of the proposed schemes with existing schemes [34] for different grid points with $\tau = 0.001$, $\mathscr{T} = 1$ for the Example 4.1.

	GFDM[34]		Explicit method		Implicit method	
Grid size	l_2	l_∞	l_2	l_∞	l_2	l_∞
55	1.14×10^{-04}	2.11×10^{-03}	1.03×10^{-04}	1.84×10^{-05}	1.03×10^{-05}	1.84×10^{-05}
197	3.19×10^{-05}	5.90×10^{-04}	1.95×10^{-05}	1.65×10^{-06}	1.87×10^{-06}	2.87×10^{-07}
743	8.45×10^{-06}	1.57×10^{-04}	2.86×10^{-06}	1.03×10^{-07}	2.56×10^{-07}	1.23×10^{-08}

4.2 Example 2

Consider Eqn.(6) with

$$A_1 = \frac{4w}{x} + \frac{x}{6}, \quad A_2 = \frac{4w}{y} + \frac{y}{6}, \qquad B_{11} = B_{22} = w, \quad B_{12} = B_{21} = 0. \tag{40}$$

The resulting FPE is

$$\begin{aligned}\frac{\partial w}{\partial t} =& \frac{4w^2}{x^2} + \frac{4w^2}{y^2} - \frac{w}{3} + \frac{\partial w}{\partial x}\left(\frac{-x}{6} - \frac{8w}{x}\right) \\ &+ \frac{\partial w}{\partial y}\left(\frac{-y}{6} - \frac{8w}{y}\right) \\ &+ 2w\frac{\partial^2 w}{\partial x^2} + 2w\frac{\partial^2 w}{\partial y^2} + 2\left(\frac{\partial w}{\partial x}\right)^2 + 2\left(\frac{\partial w}{\partial y}\right)^2,\end{aligned} \tag{41}$$

that admits an exact analytical solution

$$w(x,y,t) = x^2 y^2 e^{-t}, \qquad 0 \le x, y \le 1. \tag{42}$$

The numerical errors of the proposed schemes, evaluated using the l_2 and l_∞ norms, are summarized in Table 3. To further assess the performance and reliability of the proposed methods, their accuracy is compared with that of an established approach [34] in Table 4.

5 Conclusion

In this work, the two-dimensional FPE was systematically analyzed using finite difference schemes constructed through an operator splitting approach. Full discretizations of the model were developed, and their reliability was confirmed through both linear and nonlinear two-dimensional examples with variable coefficients. The proposed algorithms were thoroughly analyzed for stability using von Neumann's criterion. The explicit formulation proved to be temporally first-order and spatially second-order accurate, whereas the implicit formulation achieved second-order accuracy in both temporal and spatial domains. To illustrate the effectiveness and computational advantage of the proposed strategies, their outcomes were contrasted with those of a more sophisticated but resource-demanding approach documented in earlier studies. The error analysis, conducted using multiple norms, indicates that the proposed schemes not only provide high accuracy but also offer a competitive and efficient alternative for solving two-dimensional FPEs.

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Table 3: Comparison of errors in l_2 and l_∞ norms with $\tau = 0.0001$, $\mathscr{T} = 1$ for the Example 4.2.

	Explicit method		Semi-implicit method		Explicit method		Semi-implicit method	
	$\mathscr{T} = 0.25$		$\mathscr{T} = 0.0.25$		$\mathscr{T} = 0.3$		$\mathscr{T} = 0.3$	
Grid size	l_2	l_∞	l_2	l_∞	l_2	l_∞	l_2	l_∞
15×15	4.94×10^{-03}	7.79×10^{-03}	4.70×10^{-03}	7.41×10^{-03}	1.99×10^{-04}	6.75×10^{-05}	1.84×10^{-04}	6.24×10^{-05}
25×25	9.32×10^{-03}	6.29×10^{-03}	4.69×10^{-03}	4.08×10^{-03}	2.57×10^{-04}	6.79×10^{-05}	2.37×10^{-04}	6.28×10^{-05}
35×35	3.63×10^{-03}	8.80×10^{-03}	8.85×10^{-03}	8.18×10^{-04}	3.04×10^{-04}	6.80×10^{-05}	2.81×10^{-04}	6.29×10^{-05}
45×45	4.93×10^{-04}	7.83×10^{-04}	7.27×10^{-04}	7.23×10^{-04}	3.45×10^{-04}	6.81×10^{-05}	3.19×10^{-04}	6.29×10^{-05}
55×55	9.25×10^{-04}	3.07×10^{-04}	6.64×10^{-04}	1.82×10^{-04}	3.81×10^{-05}	6.81×10^{-06}	3.52×10^{-05}	6.29×10^{-06}
65×65	5.40×10^{-04}	8.01×10^{-04}	4.74×10^{-04}	2.21×10^{-04}	4.14×10^{-05}	6.81×10^{-06}	3.83×10^{-05}	6.23×10^{-06}
75×75	9.83×10^{-04}	8.80×10^{-05}	4.69×10^{-04}	7.21×10^{-05}	4.45×10^{-06}	6.81×10^{-06}	4.11×10^{-06}	6.32×10^{-06}
85×85	2.61×10^{-04}	7.83×10^{-05}	9.42×10^{-05}	1.72×10^{-05}	4.74×10^{-06}	6.81×10^{-06}	4.38×10^{-06}	6.29×10^{-06}
95×95	6.02×10^{-05}	7.83×10^{-05}	4.21×10^{-05}	1.80×10^{-05}	5.01×10^{-06}	6.81×10^{-07}	4.63×10^{-06}	6.29×10^{-07}
105×105	2.60×10^{-05}	8.01×10^{-06}	7.75×10^{-06}	7.42×10^{-06}	5.27×10^{-07}	6.81×10^{-07}	4.87×10^{-07}	6.30×10^{-07}

Table 4: Comparison of errors in l_2 and l_∞ norms of the proposed schemes with existing schemes [34] for different grid points with $\tau = 0.001$, $\mathscr{T} = 1$ for the Example 4.2.

	GFDM[34]		Explicit method		Implicit method	
Grid size	l_2	l_∞	l_2	l_∞	l_2	l_∞
55	3.61×10^{-04}	6.70×10^{-04}	1.82×10^{-04}	5.72×10^{-04}	4.81×10^{-04}	8.59×10^{-05}
197	1.01×10^{-04}	1.82×10^{-04}	2.85×10^{-06}	7.34×10^{-06}	1.06×10^{-05}	9.98×10^{-06}
743	2.01×10^{-05}	3.79×10^{-05}	5.48×10^{-06}	7.42×10^{-06}	4.03×10^{-06}	6.81×10^{-07}

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