



Multiplicity of Homoclinic Solutions for Homogeneous Damped Vibration Systems

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Abstract. We study the existence and multiplicity of classical homoclinic solutions for a class of second-order damped vibration systems of the form

$$\ddot{u}(t) + q(t)\dot{u}(t) - L(t)u(t) = -a(t)\nabla G(u(t)) + b(t)\nabla H(u(t)) + h(t), \quad t \in \mathbb{R}.$$

where $L(t)$ is a symmetric positive definite matrix, $a(t), b(t)$ are positive functions, G and H are homogeneous potentials of different degrees, and $h(t)$ is a small external forcing term. Employing variational techniques and the Pohozaev fibering method, we establish the existence of infinitely many nontrivial homoclinic solutions in the symmetric case $h = 0$, and at least three such solutions when h is nonzero but sufficiently small. These results generalize previous findings by addressing both subcritical and supercritical homogeneous nonlinearities in a non-periodic, non-symmetric framework.

Keywords. Damped vibration systems, homogeneous functions, variational techniques, Pohozaev fibering method, implicit functions theorem.

1 Introduction

This paper is devoted to the study of the existence and multiplicity of fast homoclinic solutions (see Definition 2.1) for a class of second-order damped vibration systems described by the equation:

$$\ddot{u}(t) + q(t)\dot{u}(t) - L(t)u(t) + \nabla W(t, u(t)) = 0, \quad \forall t \in \mathbb{R}. \quad (1)$$

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where $q : \mathbb{R} \rightarrow \mathbb{R}$ is continuous, $L \in C(\mathbb{R}, \mathbb{R}^{N^2})$ is symmetric and positive definite for each t , and $W : \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}$ is continuously differentiable with respect to the second variable. A function u is called a classical homoclinic solution to zero of system (1) if $u \in C^2(\mathbb{R}, \mathbb{R}^N)$ and satisfies $u(t) \rightarrow 0$ and $\dot{u}(t) \rightarrow 0$ as $|t| \rightarrow \infty$. Such a solution is said to be nontrivial if $u \neq 0$.

When $q = 0$, the system reduces to the well-known second-order Hamiltonian system:

$$\ddot{u}(t) - L(t)u(t) + \nabla W(t, u(t)) = 0, \quad \forall t \in \mathbb{R}. \quad (2)$$

which has been extensively studied using variational methods, particularly in the context of autonomous or time-periodic systems (e.g., [1–6]). In such cases, standard compactness arguments and critical point theorems such as the Mountain Pass Theorem are applicable. However, for non-periodic systems where $L(t)$ and $W(t, x)$ do not exhibit temporal periodicity, the lack of compactness due to the unboundedness of time introduces significant analytical challenges. To address these, several authors (e.g., [7? –19]) have imposed coercivity or asymptotic growth conditions on $L(t)$ and subcritical or asymptotically linear constraints on the potential W .

In contrast, the damped case $q \neq 0$, which represents more realistic models of vibrating systems subject to energy dissipation, has received comparatively less attention (see [20? ? ? –25]). Some works, such as [23?], successfully establish the existence of infinitely many fast homoclinic solutions for superquadratic damped systems using variational techniques and compactness assumptions tailored to the properties of $L(t)$. However, most of these contributions are restricted to even or symmetric nonlinearities and do not adequately explore multiplicity in the presence of non-symmetric or non-periodic structures.

The present work aims to broaden this research direction by considering damped systems governed by a non-periodic potential of the form:

$$W(t, x) = -a(t)G(x) + b(t)H(x) + h(t) \cdot x. \quad (3)$$

where G and H are homogeneous functions of degrees ν and μ , respectively, with $1 < \nu < \mu$, and $h(t)$ is a small external forcing term. This setting allows the inclusion of both subcritical and supercritical nonlinearities, thus extending prior results to a more general and physically relevant class of systems. Importantly, we do not impose periodicity or evenness assumptions on the potentials, making the problem non-symmetric and non-autonomous. Our approach is based on the Pohozaev fibering method [?] and variational techniques developed in [23], combined with a measure-theoretic condition on $L(t)$ that ensures a suitable form of compactness:

There exists a positive constant r_0 such that $\inf_{t \in \mathbb{R}} \inf_{|\xi|=1} L(t)\xi \cdot \xi > 0$ and

$$\lim_{|s| \rightarrow \infty} \text{meas}\{t \in (s - r_0, s + r_0) : L(t) \leq MI_N\} = 0, \quad (4)$$

$$\forall M > 0.$$

Under these settings, we establish the following main results. We impose the following assumptions on the functions a, b, G, H :

Under these settings, we establish the following main results. We impose the following assumptions on the functions a, b, G, H :

(W₁) $G, H \in C^1(\mathbb{R}^N, \mathbb{R})$ and there exist constants ν, μ with $1 < \nu \leq 2 < \mu$ such that $G(sx) = |s|^\nu G(x)$ and $H(sx) = |s|^\mu H(x)$ for all $(s, x) \in \mathbb{R} \times \mathbb{R}^N$.

(W₂) $G(x) > 0$ and $H(x) > 0$ for all $|x| = 1$.

(W₃) $a \in C(\mathbb{R}, \mathbb{R}^+)$, with a bounded if $\nu \geq 2$, and $a \in L^{\frac{2}{2-\nu}}(\mathbb{R})$ if $1 < \nu < 2$.

(W₄) $b \in C(\mathbb{R}, \mathbb{R}^+)$ is bounded.

Theorem 1.1. (Symmetric Case). Let $h = 0$. Assume that conditions (4) and (W₁) – (W₄) hold. Then system (1) admits infinitely many nontrivial fast homoclinic solutions.

Remark 1.1. This result generalizes the multiplicity results obtained in [23] by accommodating potentials that combine both subcritical and supercritical homogeneous components, and by removing the requirement of symmetry.

Theorem 1.2. (Non-Symmetric Case). Let $h \in L^{\frac{\mu'}{2}}(\mathbb{R}, \mathbb{R}^N)$, where $\frac{1}{\mu} + \frac{1}{\mu'} = 1$. Suppose that assumptions (4) and (W₁) – (W₄) hold. Then there exists a

constant $\delta > 0$ such that if $\|h\|_{L^{\frac{\mu'}{2}}} < \delta$, the system (1) possesses at least three nontrivial fast homoclinic solutions.

Remark 1.2. This significantly extends earlier multiplicity results for the Hamiltonian case (see [13, 16]) to damped, non-symmetric systems, demonstrating robustness of the variational approach even under non-periodic perturbations.

To the best of our knowledge, this is the first study to address multiplicity of fast homoclinic solutions in damped systems governed by non-periodic combinations of homogeneous potentials without imposing symmetry or periodicity. These results therefore provide a meaningful extension of the existing theory and open new directions in the variational analysis of dissipative mechanical systems with complex nonlinear structures.

The structure of this paper is as follows: Section 2 presents some preliminary results and revisits the Pohozaev fibering method. Section 3 is focused on proving Theorem 1.1, while Section 4 addresses the proof of Theorem 1.2.

2 Preliminaries

In order to introduce the concept of fast homoclinic solutions for (1) conveniently, we firstly describe some properties of the weighted Sobolev space E on which the certain variational functional associated with (1) is defined and the fast homoclinic solutions of (1) are the critical points of such functional. Let $Q(t) = \int_0^t q(s)ds$, we shall use $L^2_Q(\mathbb{R})$ to denote the Hilbert space of measurable functions from \mathbb{R} into \mathbb{R}^N under the inner product We define the weighted inner product

$$\langle u, v \rangle_{L^2_Q} = \int_{\mathbb{R}} e^{Q(t)} u(t) \cdot v(t) dt, \quad (5)$$

and the induced norm

$$\|u\|_{L^2_Q} = \left(\int_{\mathbb{R}} e^{Q(t)} |u(t)|^2 dt \right)^{\frac{1}{2}}. \quad (6)$$

Similarly, for $1 \leq s < \infty$, $L^s_Q(\mathbb{R})$ denotes the Banach space of \mathbb{R}^N -valued functions on \mathbb{R} with norm

$$\|u\|_{L^s_Q} = \left(\int_{\mathbb{R}} e^{Q(t)} |u(t)|^s dt \right)^{\frac{1}{s}}, \quad (7)$$

while $L^\infty_Q(\mathbb{R})$ is endowed with the norm

$$\|u\|_{L^\infty_Q} = \text{ess sup}_{t \in \mathbb{R}} \left(e^{\frac{Q(t)}{2}} |u(t)| \right). \quad (8)$$

In this section, we assume that L satisfies (4) and introduce the Hilbert space

$$E = \left\{ u \in H^1_Q(\mathbb{R}) : \int_{\mathbb{R}} e^{Q(t)} L(t) u(t) \cdot u(t) dt < \infty \right\}. \quad (9)$$

equipped with the following inner product

$$\langle u, v \rangle = \int_{\mathbb{R}} e^{Q(t)} (\dot{u}(t) \cdot \dot{v}(t) + L(t)u(t) \cdot v(t)) dt, \quad (10)$$

and the induced norm $\|u\| = \langle u, u \rangle^{1/2}$. Here, $H_Q^1(\mathbb{R})$ denotes the Sobolev space

$$H_Q^1(\mathbb{R}) = \{u \in L_Q^2(\mathbb{R}) : \dot{u} \in L_Q^2(\mathbb{R})\}. \quad (11)$$

Evidently, E is continuously embedded into $L_Q^s(\mathbb{R})$ for $2 \leq s \leq \infty$, i.e., for all $2 \leq s \leq \infty$, there exists a constant $\eta_s > 0$ such that

$$\|u\|_{L_Q^s} \leq \eta_s \|u\|, \quad \forall u \in E. \quad (12)$$

Definition 2.1. A solution u of (1) is called a fast homoclinic solution if $u \in E$.

Lemma 2.1. [23] Assume that (4) are satisfied. Then E is compactly embedded in $L_Q^s(\mathbb{R})$ for all $2 \leq s < \infty$.

To prove our results, we will employ the spherical fibering method as introduced by Pohozaev in [??]. For the sake of completeness, we will recall this method here. Consider a real Banach space X with a norm $\|u\|_X$ that is differentiable for $u \neq 0$. Let I be a functional on X of class $C^1(X \setminus \{0\})$. We can associate I with a functional \tilde{I} defined on $\mathbb{R} \times X$ as follows:

$$\tilde{I}(t, v) = I(tv), \quad \forall (t, v) \in \mathbb{R} \times X. \quad (13)$$

Let S denote the unit sphere in X . The following result holds:

Theorem 2.1. Let X be a real Banach space with a norm differentiable on $X \setminus \{0\}$, and let $(t, v) \in (\mathbb{R} \setminus \{0\}) \times S$ be a conditionally critical point of the functional \tilde{I} considered on $\mathbb{R} \times S$. Then the vector $u = tv$ is a critical point of the functional I , that is, $I'(u) = 0$. In other words, any critical point (t, v) of \tilde{I} restricted on $(\mathbb{R} \setminus \{0\}) \times S$ generates the free non-trivial critical point u of I and vice-versa, that is, the problem $I'(u) = 0, u \neq 0$ is equivalent to

$$\begin{cases} \tilde{I}'_t(t, v) = 0, \\ \tilde{I}'_v(t, v) = 0. \end{cases} \quad (14)$$

for $\|v\| = 1$. In the following, we will call the first scalar system of the previous system the "bifurcation system".

3 Proof of Theorem 1.1.

We will proceed by successive lemmas.

Lemma 3.1. Let $K : \mathbb{R}^N \rightarrow \mathbb{R}$ be a function and $\beta > 0$ be a constant. We have the following properties:

a) Equivalence of Homogeneity and Evenness: K is homogeneous of degree β if and only if K is even and positively homogeneous of degree β .

b) Bounds on Positively Homogeneous Functions: If K is positively homogeneous of degree β , then there exist constants $m_K, M_K \in \mathbb{R}$ such that

$$m_K |x|^\beta \leq K(x) \leq M_K |x|^\beta, \quad \forall x \in \mathbb{R}^N. \quad (15)$$

c) Derivative of Positively Homogeneous Functions: If K is differentiable and positively homogeneous of degree β , then ∇K is positively homogeneous of degree $\beta - 1$. Furthermore, for all $x \in \mathbb{R}^N$, the following identity holds:

$$\nabla K(x) \cdot x = \beta K(x). \quad (16)$$

Proof (a) It suffices to observe that

$$K(-x) = K((-1)x) = |-1|^\beta K(x) = K(x). \quad (17)$$

(b) For $x \in \mathbb{R}^N \setminus \{0\}$, we have

$$K(x) = K\left(|x| \frac{x}{|x|}\right) = |x|^\beta K\left(\frac{x}{|x|}\right). \quad (18)$$

Let

$$m_K = \min_{\{|x|=1\}} K(x), \quad M_K = \max_{\{|x|=1\}} K(x), \quad (19)$$

then

$$m_K |x|^\beta \leq K(x) \leq M_K |x|^\beta, \quad \forall x \in \mathbb{R}^N. \quad (20)$$

(c) For $s > 0$ and any $y \in \mathbb{R}^N$,

$$\begin{aligned} \nabla K(sx) \cdot y &= \lim_{t \rightarrow 0^+} \frac{K(sx + ty) - K(sx)}{t} \\ &= \lim_{t \rightarrow 0^+} s^{\beta-1} \frac{K\left(x + \frac{t}{s}y\right) - K(x)}{t/s} \\ &= s^{\beta-1} \nabla K(x) \cdot y. \end{aligned} \quad (21)$$

Since y is arbitrary, it follows that $\nabla K(sx) = s^{\beta-1} \nabla K(x)$. Differentiating

$$K(sx) = s^\beta K(x), \quad (22)$$

with respect to s , we obtain

$$\nabla K(sx) \cdot x = \beta s^{\beta-1} K(x). \quad (23)$$

Setting $s = 1$ yields the desired result.

Lemma 3.2. Assume that $(W_1) - (W_4)$ are satisfied. Then, we have

a) If $u_n \rightarrow u$ in $L^2_Q(\mathbb{R})$, then $a\nabla G(u_n) \rightarrow a\nabla G(u)$ in $L^2_Q(\mathbb{R})$.

b) If $u_n \rightarrow u$ in $L^\mu_Q(\mathbb{R})$, then $a\nabla H(u_n) \rightarrow a\nabla H(u)$ in $L^{\frac{\mu}{\mu-1}}_Q(\mathbb{R})$.

Proof: a) If $u_n \rightarrow u$ in $L^2_Q(\mathbb{R})$. We claim that $a\nabla G(u_n) \rightarrow a\nabla G(u)$ in $L^2_Q(\mathbb{R})$. Arguing indirectly that there exist a subsequence (u_{n_k}) and a constant $\varepsilon_0 > 0$ such that

$$\int_{\mathbb{R}} e^{Q(t)} a^2(t) |\nabla G(u_{n_k}(t)) - \nabla G(u(t))|^2 dt \geq \varepsilon_0, \quad (24)$$

$\forall k \in \mathbb{N}$.

Up to a subsequence if necessary, we can assume that $\sum_{k=1}^{\infty} \|u_{n_k} - u\|_{L^2_Q} < \infty$ and $u_{n_k} \rightarrow u$ a.e. on \mathbb{R} . Let $v(t) = \sum_1^{\infty} |u_{n_k}(t) - u(t)|$, then $v \in L^2_Q(\mathbb{R})$ and we have

$$\begin{aligned} & a^2(t) |\nabla G(u_{n_k}(t)) - \nabla G(u(t))|^2 \\ & \leq a^2(t) M_{|\nabla G|}^2 (|u_{n_k}(t)|^{v-1} + |u(t)|^{v-1})^2 \\ & \leq 2a^2(t) M_{|\nabla G|}^2 (|u_{n_k}(t)|^{2(v-1)} + |u(t)|^{2(v-1)}) \\ & \leq 2a^2(t) M_{|\nabla G|}^2 ((|u_{n_k}(t) - u(t)| + |u(t)|)^{2(v-1)} \\ & \quad + |u(t)|^{2(v-1)}) \\ & \leq c_1 a^2(t) (v^{2(v-1)}(t) + |u(t)|^{2(v-1)}) = w(t). \end{aligned} \quad (25)$$

where c_1 is a positive constant. By (W_3) , $w \in L^1_Q(\mathbb{R})$, hence by the dominated convergence theorem, one gets

$$\int_{\mathbb{R}} e^{Q(t)} a^2(t) |\nabla G(u_{n_k}(t)) - \nabla G(u(t))|^2 dt \rightarrow 0, \quad k \rightarrow \infty, \quad (26)$$

which contradicts (24). Hence $a\nabla G(u_n) \rightarrow a\nabla G(u)$ in $L^2_Q(\mathbb{R})$.

b) Let $u_n \rightarrow u$ in $L^\mu_Q(\mathbb{R})$. We claim that $b\nabla H(u_n) \rightarrow b\nabla H(u)$ in $L^{\frac{\mu}{\mu-1}}_Q(\mathbb{R})$. Arguing indirectly that there exist a subsequence (u_{n_k}) and a constant $\varepsilon_0 > 0$ such that

$$\int_{\mathbb{R}} e^{Q(t)} b^{\frac{\mu}{\mu-1}}(t) |\nabla H(u_{n_k}(t)) - \nabla H(u(t))|^{\frac{\mu}{\mu-1}} dt \geq \varepsilon_0, \quad \forall k \in \mathbb{N}. \quad (27)$$

Taking a subsequence if necessary, we can assume that $\sum_1^{\infty} \|u_{n_k} - u\|_{L^\mu_Q} < \infty$ and $u_{n_k} \rightarrow u$ a.e. on \mathbb{R} . Let $v(t) =$

$\sum_1^{\infty} |u_{n_k}(t) - u(t)|$, then $v \in L^\mu_Q(\mathbb{R})$ and we have

$$\begin{aligned} & b^{\frac{\mu}{\mu-1}}(t) |\nabla H(u_{n_k}(t)) - \nabla H(u(t))|^{\frac{\mu}{\mu-1}} \\ & \leq b^{\frac{\mu}{\mu-1}}(t) M_{\nabla H}^{\frac{\mu}{\mu-1}} (|u_{n_k}(t)|^{\mu-1} + |u(t)|^{\mu-1})^{\frac{\mu}{\mu-1}} \\ & \leq 2^{\frac{1}{\mu-1}} b^{\frac{\mu}{\mu-1}}(t) M_{\nabla H}^{\frac{\mu}{\mu-1}} (|u_{n_k}(t)|^\mu + |u(t)|^\mu) \\ & \leq 2^{\frac{1}{\mu-1}} b^{\frac{\mu}{\mu-1}}(t) M_{\nabla H}^{\frac{\mu}{\mu-1}} ((|u_{n_k}(t) - u(t)| + |u(t)|)^\mu \\ & \quad + |u(t)|^\mu) \\ & \leq c_2 (v^\mu(t) + |u(t)|^\mu) = w(t). \end{aligned} \quad (28)$$

where c_2 is a positive constant. Since $w \in L^1_Q(\mathbb{R})$, then by the dominated convergence theorem, one gets

$$\int_{\mathbb{R}} e^{Q(t)} b^{\frac{\mu}{\mu-1}}(t) |\nabla H(u_{n_k}(t)) - \nabla H(u(t))|^{\frac{\mu}{\mu-1}} dt \rightarrow 0, \quad k \rightarrow \infty, \quad (29)$$

which contradicts (27). Hence the claim is verified. \blacksquare

Lemma 3.3. Assume that $(W_1) - (W_3)$ are satisfied. Then the functionals

$$I_1(u) = \int_{\mathbb{R}} e^{Q(t)} a(t) G(u(t)) dt, \quad (30)$$

$$I_2(u) = \int_{\mathbb{R}} e^{Q(t)} b(t) H(u(t)) dt, \quad (31)$$

are continuously differentiable respectively on $L^2_Q(\mathbb{R})$ and $L^\mu_Q(\mathbb{R})$, and we have

$$I'_1(u)v = \int_{\mathbb{R}} e^{Q(t)} a(t) \nabla G(u(t)) \cdot v(t) dt, \quad (32)$$

$\forall u, v \in L^2_Q(\mathbb{R})$,

$$I'_2(u)v = \int_{\mathbb{R}} e^{Q(t)} b(t) \nabla H(u(t)) \cdot v(t) dt, \quad (33)$$

$\forall u, v \in L^\mu_Q(\mathbb{R})$.

Proof: a) For $u, v \in L^2_Q(\mathbb{R})$, by the Mean Value Theorem and Hölder's inequality, we have

$$\begin{aligned} & |I_1(u+v) - I_1(u) \\ & \quad - \int_{\mathbb{R}} e^{Q(t)} a(t) \nabla G(u(t)) \cdot v(t) dt| \\ & = \left| \int_{\mathbb{R}} e^{Q(t)} a(t) [G(u(t)+v(t)) - G(u(t)) \right. \\ & \quad \left. - \nabla G(u(t)) \cdot v(t)] dt \right| \\ & = \left| \int_{\mathbb{R}} e^{Q(t)} a(t) [\nabla G(u(t) + \theta(t)v(t)) \right. \\ & \quad \left. - \nabla G(u(t))] \cdot v(t) dt \right| \\ & \leq \left(\int_{\mathbb{R}} e^{Q(t)} a^2(t) |\nabla G(u(t) + \theta(t)v(t)) \right. \\ & \quad \left. - \nabla G(u(t))|^2 dt \right)^{1/2} \|v\|_{L^2_Q}. \end{aligned} \quad (34)$$

where $\theta(t) \in]0, 1[$. By Lemma 3.2, the functional defined on $L^2_Q(\mathbb{R})$ by

$$v \mapsto \int_{\mathbb{R}} e^{Q(t)} a^2(t) |\nabla G(u(t) + v(t)) - \nabla G(u(t))|^2 dt \quad (35)$$

goes to zero as $v \rightarrow 0$. Hence

$$\begin{aligned} I_1(u+v) - I_1(u) & - \int_{\mathbb{R}} e^{Q(t)} a(t) \nabla G(u(t)) \cdot v(t) dt \\ & = o(\|v\|_{L^2_Q}). \end{aligned} \quad (36)$$

Thus I_1 is differentiable at u and

$$I'_1(u)v = \int_{\mathbb{R}} e^{Q(t)} a(t) \nabla G(u(t)) \cdot v(t) dt. \quad (37)$$

Let $u_n \rightarrow u$ in $L^2_Q(\mathbb{R})$. Then

$$\begin{aligned} \|I'_1(u_n) - I'_1(u)\| & = \sup_{\|v\|_{L^2_Q}=1} \left| \int_{\mathbb{R}} e^{Q(t)} a(t) \right. \\ & \quad \times [\nabla G(u_n(t)) - \nabla G(u(t))] \cdot v(t) dt \Big| \\ & \leq \left(\int_{\mathbb{R}} e^{Q(t)} a^2(t) |\nabla G(u_n(t)) - \nabla G(u(t))|^2 dt \right)^{1/2} \\ & \rightarrow 0, \quad n \rightarrow \infty. \end{aligned} \quad (38)$$

Hence I_1 is continuously differentiable on $L^2_Q(\mathbb{R})$.

b) For $u, v \in L^{\mu}_Q(\mathbb{R})$, by the Mean Value Theorem and Hölder's inequality, we have

$$\begin{aligned} |I_2(u+v) - I_2(u) & - \int_{\mathbb{R}} e^{Q(t)} b(t) \nabla H(u(t)) \cdot v(t) dt| \\ & = \left| \int_{\mathbb{R}} e^{Q(t)} b(t) [H(u(t) + v(t)) - H(u(t)) \right. \\ & \quad \left. - \nabla H(u(t)) \cdot v(t)] dt \right| \\ & = \left| \int_{\mathbb{R}} e^{Q(t)} b(t) [\nabla H(u(t) + \theta(t)v(t)) \right. \\ & \quad \left. - \nabla H(u(t))] \cdot v(t) dt \right| \\ & \leq M_b \left(\int_{\mathbb{R}} e^{Q(t)} |\nabla H(u(t) + \theta(t)v(t)) \right. \\ & \quad \left. - \nabla H(u(t))|^{\frac{\mu}{\mu-1}} dt \right)^{\frac{\mu-1}{\mu}} \|v\|_{L^{\mu}_Q}. \end{aligned} \quad (39)$$

where $\theta(t) \in (0, 1)$. By Lemma 3.2, $\int_{\mathbb{R}} e^{Q(t)} |\nabla H(u(t) + v(t)) - \nabla H(u(t))|^{\frac{\mu}{\mu-1}} dt \rightarrow 0$ as $v \rightarrow 0$. Hence

$$\begin{aligned} I_2(u+v) - I_2(u) & - \int_{\mathbb{R}} e^{Q(t)} b(t) \nabla H(u(t)) \cdot v(t) dt = o(\|v\|_{L^{\mu}_Q}). \end{aligned} \quad (40)$$

Thus I_2 is differentiable on $L^{\mu}_Q(\mathbb{R})$ and $I_2 \in C^1(L^{\mu}_Q(\mathbb{R}))$. The proof of Lemma 3.3 is completed. \blacksquare

Remark 3.1. Using Lemmas 2.2, 3.3, it is easy to see that I_1 and I_2 are continuously differentiable on E .

Associated to system (1), is the energy functional $J_0 : E \rightarrow \mathbb{R}$ defined by

$$\begin{aligned} J_0(u) & = \frac{1}{2} \int_{\mathbb{R}} e^{Q(t)} (|\dot{u}(t)|^2 + L(t)u(t) \cdot u(t)) dt \\ & \quad - \int_{\mathbb{R}} e^{Q(t)} W(t, u(t)) dt \\ & = \frac{1}{2} \|u\|^2 + \int_{\mathbb{R}} e^{Q(t)} a(t) G(u(t)) dt \\ & \quad - \int_{\mathbb{R}} e^{Q(t)} b(t) H(u(t)) dt. \end{aligned} \quad (41)$$

From Remark 3.1, J_0 is continuously differentiable on E with derivative

$$\begin{aligned} J'_0(u)v & = \int_{\mathbb{R}} e^{Q(t)} \dot{u}(t) \cdot \dot{v}(t) dt \\ & \quad + \int_{\mathbb{R}} e^{Q(t)} L(t)u(t) \cdot v(t) dt \\ & \quad + \int_{\mathbb{R}} e^{Q(t)} a(t) \nabla G(u(t)) \cdot v(t) dt \\ & \quad - \int_{\mathbb{R}} e^{Q(t)} b(t) \nabla H(u(t)) \cdot v(t) dt. \end{aligned} \quad (42)$$

for all $u, v \in E$. Moreover, the critical points of J_0 on E correspond to the solutions of (1).

According to the spherical fibering method, we look for critical points of the form

$$u = sv, \quad s \in \mathbb{R}, v \in E, \|v\| = 1. \quad (43)$$

The functional J_0 extends to $\mathbb{R} \times E$ by

$$\begin{aligned} \tilde{J}_0(s, v) & = J_0(sv) \\ & = \frac{s^2}{2} \|v\|^2 + \int_{\mathbb{R}} e^{Q(t)} a(t) G(sv(t)) dt \\ & \quad - \int_{\mathbb{R}} e^{Q(t)} b(t) H(sv(t)) dt \\ & = \frac{s^2}{2} \|v\|^2 + |s|^{\nu} \int_{\mathbb{R}} e^{Q(t)} a(t) G(v(t)) dt \\ & \quad - |s|^{\mu} \int_{\mathbb{R}} e^{Q(t)} b(t) H(v(t)) dt, \end{aligned} \quad (44)$$

for $(s, v) \in \mathbb{R} \times E$.

Its restriction to $\mathbb{R} \times S$, with $S = \{v \in E : \|v\| = 1\}$, reads

$$\tilde{J}_0(s, v) = \frac{s^2}{2} + |s|^{\nu} \int_{\mathbb{R}} e^{Q(t)} a(t) G(v(t)) dt - |s|^{\mu} \int_{\mathbb{R}} e^{Q(t)} b(t) H(v(t)) dt. \quad (45)$$

If $s \neq 0$, the bifurcation equation $\partial_s \tilde{J}_0(s, v) = 0$ becomes

$$\begin{aligned} s + \nu |s|^{\nu-2} s \int_{\mathbb{R}} e^{Q(t)} a(t) G(v(t)) dt \\ - \mu |s|^{\mu-2} s \int_{\mathbb{R}} e^{Q(t)} b(t) H(v(t)) dt = 0, \end{aligned} \quad (46)$$

which is equivalent to

$$\begin{aligned} & 1 + \nu |s|^{v-2} \int_{\mathbb{R}} e^{Q(t)} a(t) G(v(t)) dt \\ & - \mu |s|^{\mu-2} \int_{\mathbb{R}} e^{Q(t)} b(t) H(v(t)) dt = 0. \end{aligned} \quad (47)$$

Lemma 3.4. For any $v \in \tilde{E} = L_Q^2(\mathbb{R}) \cap L_Q^\mu(\mathbb{R})$, the function

$$\begin{aligned} \varphi_v(s) &= 1 + \nu |s|^{v-2} \int_{\mathbb{R}} e^{Q(t)} a(t) G(v(t)) dt \\ & - \mu |s|^{\mu-2} \int_{\mathbb{R}} e^{Q(t)} b(t) H(v(t)) dt. \end{aligned} \quad (48)$$

possesses exactly two zeros $\pm s(v)$. Moreover, the functional $v \mapsto s(v)$ is continuously differentiable on \tilde{E} .

Proof : Since $1 < \nu \leq \max\{2, \nu\} < \mu$, then it is clear that

$$\lim_{|s| \rightarrow \infty} \varphi_v(s) = -\infty. \quad (49)$$

Moreover

$$\lim_{s \rightarrow 0^+} \varphi_v(s) = \begin{cases} +\infty, & 1 < \nu < 2, \\ 1 + \nu \int_{\mathbb{R}} e^{Q(t)} a(t) G(v(t)) dt, & \nu = 2, \\ 1, & \nu > 2. \end{cases} \quad (50)$$

Since φ_v is continuous, we deduce by the Mean Value Theorem that φ_v has at least two zeros. It remains to prove that φ_v has exactly two zeros. Indeed, for $s \neq 0$, we have

$$\begin{aligned} \varphi'_v(s) &= \nu(\nu-2)|s|^{v-4} \int_{\mathbb{R}} e^{Q(t)} a(t) G(v(t)) dt \\ & - \mu(\mu-2)|s|^{\mu-4} \int_{\mathbb{R}} e^{Q(t)} b(t) H(v(t)) dt. \end{aligned} \quad (51)$$

We discuss two cases.

a) First case: $1 < \nu \leq 2$, φ'_v does not admit zeros. Hence φ_v has exactly two zeros: $\pm s(v)$.

b) Second case: $\nu > 2$. In this case, φ'_v possesses two zeros $\pm \bar{s}(v)$ with

$$\bar{s}(v) = \left(\frac{\nu(\nu-2) \int_{\mathbb{R}} e^{Q(t)} a(t) G(v(t)) dt}{\mu(\mu-2) \int_{\mathbb{R}} e^{Q(t)} b(t) H(v(t)) dt} \right)^{\frac{1}{\mu-\nu}}. \quad (52)$$

Using the variation table, we see directly that φ_v admits exactly two zeros $\pm s(v)$ with $s(v) > \bar{s}(v)$.

Now, we shall prove that the functional $v \mapsto s(v)$ obtained above is continuously differentiable on \tilde{E} . Let $v_0 \in \tilde{E}$, we consider the functional $\Phi : \tilde{E} \times I \rightarrow \mathbb{R}$ defined by $\Phi(v, s) = \varphi_v(s)$, where $I = \mathbb{R}_+^*$ if $1 < \nu \leq 2$ and $I =]\bar{s}(v), +\infty[$ if $\nu > 2$. Lemma 3.3 implies that Φ is continuously differentiable on $\tilde{E} \times I$. Moreover, we have $\Phi(v_0, s(v_0)) = 0$ and

$\frac{\partial \Phi}{\partial s}(v_0, s(v_0)) \neq 0$. By the Implicit Function Theorem, there exist an open neighborhood $V \subset \tilde{E}$ of v_0 and a unique $\theta : V \rightarrow \mathbb{R}$ which is continuously differentiable such that $\Phi(v, \theta(v)) = 0$, $\forall v \in V$. By the uniqueness of s and θ , we deduce that $s = \theta$ on V , so s is continuously differentiable on V and in particular at v_0 . Since v_0 is arbitrary, then s is continuously differentiable on \tilde{E} .

Now, consider the functional \hat{J}_0 defined on \tilde{E} by

$$\begin{aligned} \hat{J}_0(v) &= \frac{1}{2} s^2(v) + |s(v)|^\nu \int_{\mathbb{R}} e^{Q(t)} a(t) G(v(t)) dt \\ & - |s(v)|^\mu \int_{\mathbb{R}} e^{Q(t)} b(t) H(v(t)) dt. \end{aligned} \quad (53)$$

for $v \in \tilde{E}$. We deduce from Lemmas 3.3, 3.4 that $\hat{J}_0(v) = \tilde{J}_0(s(v), v)$ on S . From system $\varphi_v(s(v)) = 0$, we deduce that for all $v \in S$

$$\begin{aligned} \hat{J}_0(v) &= \left(\frac{1}{2} - \frac{1}{\mu} \right) s^2(v) \\ & + \nu \left(\frac{1}{\nu} - \frac{1}{\mu} \right) |s(v)|^\nu \int_{\mathbb{R}} e^{Q(t)} a(t) G(v(t)) dt. \end{aligned} \quad (54)$$

Since $1 < \nu \leq \max\{2, \nu\} < \mu$, $b \geq 0$ and $G \geq 0$, then \hat{J}_0 is bounded from below on S as the sum of two non-negative terms. Let $(v_n) \subset S$ be such that $\hat{J}_0(v_n) \rightarrow \inf_{v \in S} \hat{J}_0(v)$. Since (v_n) is bounded, then up to a subsequence, we can assume that $v_n \rightharpoonup \bar{v}$ weakly in E . By Theorem 2.1, we can assume after going to a subsequence, that $v_n \rightarrow \bar{v}$ in both $L_Q^2(\mathbb{R})$ and $L_Q^\mu(\mathbb{R})$. By Lemmas 3.3, 3.4, \hat{J}_0 is continuous on \tilde{E} , then $\hat{J}_0(v_n) \rightarrow \hat{J}_0(\bar{v})$. Thus \hat{J}_0 attains its minimum on S at a point \bar{v} with $\|\bar{v}\| \leq 1$. It remains to prove that $\bar{v} \in S$. Indeed, using the relation $\varphi_v(s(v)) = 0$, we obtain for all $v \in S$ and $\xi \in [0, 1]$ the following.

For convenience, set

$$A_\xi := \int_{\mathbb{R}} e^{Q(t)} a(t) G(\xi v(t)) dt, \quad (55)$$

$$B_\xi := \int_{\mathbb{R}} e^{Q(t)} b(t) H(\xi v(t)) dt. \quad (56)$$

Then

$$\begin{aligned} \frac{d}{d\xi} \hat{J}_0(\xi v) &= \frac{d}{d\xi} \left[\frac{1}{2} s^2(\xi v) + |s(\xi v)|^\nu A_\xi \right. \\ & \quad \left. - |s(\xi v)|^\mu B_\xi \right] \\ &= s(\xi v) s'(\xi v) \\ & \quad + \nu |s(\xi v)|^{\nu-2} s(\xi v) s'(\xi v) A_\xi \\ & \quad - \mu |s(\xi v)|^{\mu-2} s(\xi v) s'(\xi v) B_\xi \\ & \quad + \nu \frac{|s(\xi v)|^\nu}{\xi} A_\xi - \mu \frac{|s(\xi v)|^\mu}{\xi} B_\xi \\ &= s(\xi v) s'(\xi v) \left[1 + \nu |s(\xi v)|^{\nu-2} A_\xi \right. \\ & \quad \left. - \mu |s(\xi v)|^{\mu-2} B_\xi \right] \\ & \quad + \nu \frac{|s(\xi v)|^\nu}{\xi} A_\xi - \mu \frac{|s(\xi v)|^\mu}{\xi} B_\xi. \end{aligned} \quad (57)$$

By the defining relation $\varphi_v(s(v)) = 0$, the bracket vanishes, and we obtain Set $\sigma_\xi := s(\xi v)$. Then

$$\begin{aligned} \frac{d}{d\xi} \widehat{J}_0(\xi v) &= \frac{|\sigma_\xi|^2}{\xi} \left[v |\sigma_\xi|^{v-2} A_\xi - \mu |\sigma_\xi|^{\mu-2} B_\xi \right] \\ &= -\frac{|\sigma_\xi|^2}{\xi} < 0. \end{aligned} \quad (58)$$

Thus, $\widehat{J}_0(\xi v)$ decreases with respect to $\xi \in [0, 1]$ and reaches its minimum at $\xi = 1$. This implies that \widehat{J}_0 attains its minimum on S at $\bar{v} \in S$. According to the spherical fibering method, we obtain that $\pm s(\bar{v})\bar{v}$ are two solutions of problem (1). ■

Given that \widehat{J}_0 is an even function, bounded from below, weakly continuous, and of class C^1 on S , the Lusternik-Schnirelmann theory (as discussed in [?]) ensures that \widehat{J}_0 has a sequence of conditionally critical points $(v_n)_{n \in \mathbb{N}} \subset S$ such that $\widehat{J}_0(v_n) \rightarrow +\infty$ as $n \rightarrow \infty$. By applying Theorem 2.1, we deduce that when $h = 0$, the system (1) possesses a sequence of distinct solutions $(\pm u_n)_{n \in \mathbb{N}}$, where $u_n = s(v_n)v_n$ and $J_0(v_n) \rightarrow +\infty$ as $n \rightarrow \infty$.

4 Proof of Theorem 1.2

In the nonsymmetric case $h \neq 0$, the energy functional $J_h : E \rightarrow \mathbb{R}$ is defined by

$$\begin{aligned} J_h(u) &= \frac{1}{2} \int_{\mathbb{R}} e^{Q(t)} (|\dot{u}(t)|^2 + L(t)u(t) \cdot u(t)) dt \\ &\quad - \int_{\mathbb{R}} e^{Q(t)} W(t, u(t)) dt \\ &= \frac{1}{2} \|u\|^2 + \int_{\mathbb{R}} e^{Q(t)} a(t) G(u(t)) dt \\ &\quad - \int_{\mathbb{R}} e^{Q(t)} b(t) H(u(t)) dt \\ &\quad - \int_{\mathbb{R}} e^{Q(t)} h(t) \cdot u(t) dt \\ &= J_0(u) - \int_{\mathbb{R}} e^{Q(t)} h(t) \cdot u(t) dt. \end{aligned} \quad (59)$$

For $u \in E$, we look for critical points of the form $u = sv$, $s \in \mathbb{R}$, $v \in E$, $\|v\| = 1$.

Thus, extending J_h to $\mathbb{R} \times E$, we obtain

$$\begin{aligned} \widetilde{J}_h(s, v) &= \widetilde{J}_0(s, v) - s \int_{\mathbb{R}} e^{Q(t)} h(t) \cdot v(t) dt \\ &= \frac{s^2}{2} \|v\|^2 + |s|^v \int_{\mathbb{R}} e^{Q(t)} a(t) G(v(t)) dt \\ &\quad - |s|^\mu \int_{\mathbb{R}} e^{Q(t)} b(t) H(v(t)) dt \\ &\quad - s \int_{\mathbb{R}} e^{Q(t)} h(t) \cdot v(t) dt. \end{aligned} \quad (60)$$

for $(s, v) \in \mathbb{R} \times E$, and its restriction to $\mathbb{R} \times S$ is

$$\begin{aligned} \widetilde{J}_h(s, v) &= \frac{s^2}{2} + |s|^v \int_{\mathbb{R}} e^{Q(t)} a(t) G(v(t)) dt \\ &\quad - |s|^\mu \int_{\mathbb{R}} e^{Q(t)} b(t) H(v(t)) dt \\ &\quad - s \int_{\mathbb{R}} e^{Q(t)} h(t) \cdot v(t) dt. \end{aligned} \quad (61)$$

for $(s, v) \in \mathbb{R} \times S$. The bifurcation system $\frac{\partial \widetilde{J}_0}{\partial s}(s, v) = 0$ involves

$$\begin{aligned} s + v |s|^{v-2} s \int_{\mathbb{R}} e^{Q(t)} a(t) G(v(t)) dt \\ - \mu |s|^{\mu-2} s \int_{\mathbb{R}} e^{Q(t)} b(t) H(v(t)) dt \\ = \int_{\mathbb{R}} e^{Q(t)} h(t) \cdot v(t) dt. \end{aligned} \quad (62)$$

Let $\psi_v : \mathbb{R} \rightarrow \mathbb{R}$ be the function defined by

$$\begin{aligned} \psi_v(s) &= s + v |s|^{v-2} s \int_{\mathbb{R}} e^{Q(t)} a(t) G(v(t)) dt \\ &\quad - \mu |s|^{\mu-2} s \int_{\mathbb{R}} e^{Q(t)} b(t) H(v(t)) dt. \end{aligned} \quad (63)$$

Lemma 4.1. For every $v \in S$, the function ψ_v is odd, and it has a local minimum m_v and a local maximum M_v such that $M_v = -m_v$.

Proof: By derivation, we have

$$\begin{aligned} \psi'_v(s) &= 1 + v(v-1) |s|^{v-3} s \int_{\mathbb{R}} e^{Q(t)} a(t) G(v(t)) dt \\ &\quad - \mu(\mu-1) |s|^{\mu-3} s \int_{\mathbb{R}} e^{Q(t)} b(t) H(v(t)) dt. \end{aligned} \quad (64)$$

Since $1 < v < \mu$, we have $\lim_{s \rightarrow \infty} \psi'_v(s) = -\infty$ and

$$\lim_{s \rightarrow 0^+} \psi'_v(s) = \begin{cases} +\infty, & 1 < v < 2, \\ 1 + v(v-1) \int_{\mathbb{R}} e^{Q(t)} a(t) G(v(t)) dt, & v = 2, \\ 1, & v > 2. \end{cases} \quad (65)$$

Thus ψ'_v has at least two zeros. Moreover,

$$\begin{aligned} \psi''_v(s) &= v(v-1)(v-2) |s|^{v-4} s \int_{\mathbb{R}} e^{Q(t)} a(t) G(v(t)) dt \\ &\quad - \mu(\mu-1)(\mu-2) |s|^{\mu-4} s \int_{\mathbb{R}} e^{Q(t)} b(t) H(v(t)) dt. \end{aligned} \quad (66)$$

If $1 < v \leq 2$, then $\psi''_v(s) < 0$ for $s > 0$. If $v > 2$, then

$$\bar{s}(v) = \left(\frac{v(v-1)(v-2) \int_{\mathbb{R}} e^{Q} a G(v)}{\mu(\mu-1)(\mu-2) \int_{\mathbb{R}} e^{Q} b H(v)} \right)^{1/(\mu-v)}. \quad (67)$$

Hence ψ'_v has exactly two zeros $\pm s(v)$. As in Lemma 3.4, $s(v)$ is C^1 on \tilde{E} .

Since $\lim_{s \rightarrow \infty} \psi'_v(s) = -\infty$ and ψ_v is odd, it admits a local minimum m_v and a local maximum M_v with $M_v = -m_v$. ■

Lemma 4.2.

If $\|h\|_{L_Q^{\mu'}} < (\mu - 2) \left[\mu(\mu - 1)^{\mu-1} \eta_\mu^{2(\mu-1)} M_b M_H \right]^{-\frac{1}{\mu-2}}$, where μ' is the conjugate exponent of μ , then equation

$$\psi_v(s) = \int_{\mathbb{R}} e^{Q(t)} h(t) \cdot v(t) dt. \quad (68)$$

has three distinct solutions.

Proof: Set

$$\bar{\psi}_v(s) = s - \mu |s|^{\mu-2} \int_{\mathbb{R}} e^{Q(t)} b(t) H(v(t)) dt. \quad (69)$$

We have $\psi_v \geq \bar{\psi}_v$ and, denoting by \bar{M}_v the local maximum of $\bar{\psi}_v$,

$$\begin{aligned} \bar{M}_v &= \bar{\psi}_v \left(\left[\mu(\mu - 1) \int_{\mathbb{R}} e^{Q} b H(v) dt \right]^{-\frac{1}{\mu-2}} \right) \\ &= (\mu - 2) \left[\mu(\mu - 1)^{\mu-1} \int_{\mathbb{R}} e^{Q} b H(v) dt \right]^{-\frac{1}{\mu-2}}. \end{aligned} \quad (70)$$

Hölder's inequality implies

$$\begin{aligned} & \left| \int_{\mathbb{R}} e^{Q(t)} h(t) \cdot v(t) dt \right| \left(\int_{\mathbb{R}} e^{Q} b H(v) dt \right)^{\frac{1}{\mu-2}} \\ & \leq \|h\|_{L_Q^{\mu'}} \|v\|_{L_Q^\mu} (M_b M_H \|v\|_{L_Q^\mu}^{\mu-2})^{\frac{1}{\mu-2}} \\ & \leq \|h\|_{L_Q^{\mu'}} (M_b M_H \eta_\mu^{2(\mu-1)})^{\frac{1}{\mu-2}}. \end{aligned} \quad (71)$$

If we take

$$\|h\|_{L_Q^{\mu'}} < (\mu - 2) \left[\mu(\mu - 1)^{\mu-1} \eta_\mu^{2(\mu-1)} M_b M_H \right]^{-\frac{1}{\mu-2}}. \quad (72)$$

, then Hölder's inequality implies

$$\begin{aligned} & \sup_{v \in S} \left[\left| \int_{\mathbb{R}} e^{Q(t)} h(t) \cdot v(t) dt \right| \right. \\ & \quad \left. \times \left(\int_{\mathbb{R}} e^{Q(t)} b(t) H(v(t)) dt \right)^{\frac{1}{\mu-2}} \right] \\ & < (\mu - 2) \left[\mu(\mu - 1)^{\mu-1} \right]^{\frac{1}{\mu-2}}. \end{aligned} \quad (73)$$

Since $M_v \geq \bar{M}_v$, then (70), (73) imply

$$\left| \int_{\mathbb{R}} e^{Q(t)} h(t) \cdot v(t) dt \right| < \bar{M}_v \leq M_v,$$

which implies that system (62) has three distinct solutions. ■

Given the bifurcation system (62) with three distinct solutions $s_i(v)$, $i = 1, 2, 3$, we consider the three induced functionals

$$\begin{aligned} \widehat{J}_{h,i}(v) &= \frac{1}{2} |s_i(v)|^2 + |s_i(v)|^\nu \int_{\mathbb{R}} e^{Q(t)} a(t) G(v(t)) dt \\ &\quad - |s_i(v)|^\mu \int_{\mathbb{R}} e^{Q(t)} b(t) H(v(t)) dt \\ &\quad - s_i(v) \int_{\mathbb{R}} e^{Q(t)} h(t) \cdot v(t) dt. \end{aligned} \quad (74)$$

which are defined and distinct on $B \setminus \{0\}$, where $B = \{v \in E / \|v\| \leq 1\}$. Using Hölder's inequality and the properties of the bifurcation system (62), we obtain

$$\begin{aligned} \widehat{J}_{h,i}(v) &= \widehat{J}_{0,i}(v) - s_i(v) \int_{\mathbb{R}} e^{Q(t)} h(t) \cdot v(t) dt \\ &\geq \left(\frac{1}{2} - \frac{1}{\mu} \right) s_i(v)^2 \\ &\quad + \left(1 - \frac{\nu}{\mu} \right) |s_i(v)|^\nu \int_{\mathbb{R}} e^{Q(t)} a(t) G(v(t)) dt \\ &\quad + \left(\frac{1}{\mu} - 1 \right) \eta_\mu \|h\|_{L_Q^{\mu'}} |s_i(v)|. \end{aligned} \quad (75)$$

Since $\max\{2, \nu\} < \mu$, it follows that $\widehat{J}_{h,i}$ is bounded from below. By applying Lemmas 3.3 and 3.4, we know that $\widehat{J}_{h,i}$ is continuously differentiable on the space $\tilde{E} = L_Q^2(\mathbb{R}) \cap L_Q^\mu(\mathbb{R})$. Combining this with Lemma 2.1, we conclude that $\widehat{J}_{h,i}$ is weakly continuous on the space E . Therefore, $\widehat{J}_{h,i}$ attains its minimum on the set S at some point $\bar{v}_i \in B$ where $s_i(\bar{v}_i) \neq 0$. What remains to be shown is that $\bar{v}_i \in S$. Indeed, by utilizing equation (62), we can demonstrate for any $v \in S$ and $\xi \in [0, 1]$ that

$$\begin{aligned} \frac{d}{d\xi} (\widehat{J}_{h,i}(\xi v)) &= \frac{d}{d\xi} \left[\frac{1}{2} s_i(\xi v)^2 + |s_i(\xi v)|^\nu A_\xi \right. \\ &\quad \left. - |s_i(\xi v)|^\mu B_\xi - s_i(\xi v) H_\xi \right] \\ &= s'_i(\xi v) \left[s_i(\xi v) + \nu |s_i(\xi v)|^{\nu-2} s_i(\xi v) A_\xi \right. \\ &\quad \left. - \mu |s_i(\xi v)|^{\mu-2} s_i(\xi v) B_\xi - H_\xi \right] \\ &\quad + \frac{\nu}{\xi} |s_i(\xi v)|^\nu A_\xi - \frac{\mu}{\xi} |s_i(\xi v)|^\mu B_\xi \\ &\quad - s_i(\xi v) \int_{\mathbb{R}} e^{Q(t)} h(t) \cdot v(t) dt \\ &= \frac{s_i(\xi v)}{\xi} \left[\nu |s_i(\xi v)|^{\nu-2} s_i(\xi v) A_\xi \right. \\ &\quad \left. - \mu |s_i(\xi v)|^{\mu-2} s_i(\xi v) B_\xi - H_\xi \right] \\ &= -\frac{|s_i(\xi v)|^2}{\xi} < 0, \end{aligned} \quad (76)$$

where

$$\begin{aligned} A_\xi &:= \int_{\mathbb{R}} e^{\mathcal{Q}(t)} a(t) G(\xi v(t)) dt, \\ B_\xi &:= \int_{\mathbb{R}} e^{\mathcal{Q}(t)} b(t) H(\xi v(t)) dt, \\ H_\xi &:= \int_{\mathbb{R}} e^{\mathcal{Q}(t)} h(t) \cdot \xi v(t) dt. \end{aligned} \quad (77)$$

As ξ varies over $[0, 1]$, $\widehat{J}_{h,i}(\xi v)$ decreases, reaching its minimum when $\xi = 1$. This minimum occurs at $\bar{v}_i \in S$, indicating that $\widehat{J}_{h,i}$ achieves its minimum on S at \bar{v}_i . According to the spherical fibering method, the solutions $\pm s_i(\bar{v}_i)\bar{v}$ represent three solutions of problem (1).

References

1. M. Izydorek and J. Janczewska, Homoclinic solutions for a class of the second order Hamiltonian systems, *J. Differential Equations* 219(2), 375–389 (2005). DOI: <https://doi.org/10.1016/j.jde.2005.06.029>
2. J. Jiang, S. Lu, X. Lv, Homoclinic solutions for a class of second order Hamiltonian systems, *Nonlinear Analysis: Real World Applications* 13, 176–185 (2012). DOI: <https://doi.org/10.1016/j.nonrwa.2011.07.016>
3. S. Lu, X. Lv and P. Yan, Existence of homoclinics for a class of Hamiltonian systems, *Nonlinear Analysis* 72, 390–398 (2010). DOI: <https://doi.org/10.1016/j.na.2009.06.073>
4. P.H. Rabinowitz, Homoclinic orbits for a class of Hamiltonian systems, *Proc. Roy. Soc. Edinburgh* 114A, 33–38 (1990). DOI: <https://doi.org/10.1017/S0308210500024240>
5. X.H. Tang, L. Xiao, Homoclinic solutions for a class of second order Hamiltonian systems, *Nonlinear Analysis* 71, 1140–1152 (2009). DOI: <https://doi.org/10.1016/j.na.2008.11.039>
6. R. Yuan, Z. Zhang, Homoclinic solutions for some second order nonautonomous Hamiltonian systems without the globally superquadratic condition, *Nonlinear Analysis* 72, 1809–1819 (2010). DOI: <https://doi.org/10.1016/j.na.2009.09.011>
7. G. Chen, Homoclinic orbits for second order Hamiltonian systems with asymptotically linear terms at infinity, *Advances in Difference Equations* 2014(114), (2014). DOI: <https://doi.org/10.1186/1687-1847-2014-114>
8. G. Chen, Z. He, Infinitely many homoclinic solutions for a class of second order Hamiltonian systems, *Advances in Difference Equations* 2014(161), (2014). DOI: <https://doi.org/10.1186/1687-1847-2014-161>
9. L. Chu, Q. Zhang, Homoclinic solutions for a class of second order Hamiltonian systems with locally defined potentials, *Nonlinear Analysis* 75, 3188–3197 (2012). DOI: <https://doi.org/10.1016/j.na.2011.12.015>
10. Y. Ding, Existence and multiplicity results for homoclinic solutions to a class of Hamiltonian systems, *Nonlinear Analysis* 25(11), 1095–1113 (1995). DOI: [https://doi.org/10.1016/0362-546X\(94\)00156-Q](https://doi.org/10.1016/0362-546X(94)00156-Q)
11. X. Lin and X.H. Tang, Homoclinic solutions for a class of second order Hamiltonian systems, *J. Math. Anal. Appl.* 354, 539–549 (2009). DOI: <https://doi.org/10.1016/j.jmaa.2008.12.048>
12. X. Lin and X.H. Tang, Infinitely many homoclinic orbits for Hamiltonian systems with indefinite sign subquadratic potentials, *Nonlinear Analysis* 74, 6314–6325 (2011). DOI: <https://doi.org/10.1016/j.na.2011.06.038>
13. C. Liu, Q. Zhang, Infinitely many homoclinic solutions for second order Hamiltonian systems, *Nonlinear Analysis* 72, 894–903 (2010). DOI: <https://doi.org/10.1016/j.na.2009.07.016>
14. P.H. Rabinowitz and K. Tanaka, Some results on connecting orbits for a class of Hamiltonian systems, *Math. Z.* 206(3), 473–499 (1991). DOI: <https://doi.org/10.1007/BF02571368>
15. J. Sun, T-f. Wu, Homoclinic solutions for a second order Hamiltonian system with a positive semi-definite matrix, *Chaos, Solitons and Fractals* 76, 24–31 (2015). DOI: <https://doi.org/10.1016/j.chaos.2015.04.006>
16. J. Sun, T-f. Wu, Multiplicity and concentration of homoclinic solutions for some second order Hamiltonian systems, *Nonlinear Analysis* 114, 105–115 (2015). DOI: <https://doi.org/10.1016/j.na.2014.11.007>
17. C.L. Tang, Li-Li Wan, Existence of homoclinic orbits for second order Hamiltonian systems without (AR) condition, *Nonlinear Analysis* 74, 5303–5313 (2011). DOI: <https://doi.org/10.1016/j.na.2011.04.067>
18. J. Wei, J. Wang, Infinitely many homoclinic orbits for the second order Hamiltonian systems with general potentials, *J. Math. Anal. Appl.* 366(2), 694–699 (2010). DOI: <https://doi.org/10.1016/j.jmaa.2010.01.038>
19. Z. Zhang, Existence of homoclinic solutions for second order Hamiltonian systems with general potentials, *J. Appl. Math. Comput.* 44, 263–272 (2014). DOI: <https://doi.org/10.1007/s12190-013-0694-6>
20. R.P. Agarwal, P. Chen and X. Tang, Fast homoclinic solutions for a class of damped vibration problems, *Applied Mathematics and Computation* 219, 6053–6065

-
- (2013). DOI: <https://doi.org/10.1016/j.amc.2012.12.005>
21. P. Chen and X.H. Tang, Fast homoclinic solutions for a class of damped vibration problems with sub-quadratic potentials, *Math. Nachr.* 286(1), 4–16 (2013). DOI: <https://doi.org/10.1002/mana.201100258>
22. F. Khelifi, M. Timoumi, Even homoclinic orbits for a class of damped vibration systems, *Indagationes Mathematicae* 28(6), 1111–1125 (2017). DOI: <https://doi.org/10.1016/j.indag.2017.08.001>
23. M. Timoumi, Infinitely many fast homoclinic solutions for a class of superquadratic damped vibration systems, *Journal of Elliptic and Parabolic Equations* 6, 451–471 (2020). DOI: <https://doi.org/10.1007/s41808-020-00071-3>
24. M. Timoumi, On ground state homoclinic orbits of a class of superquadratic damped vibration systems, *Mediterr. J. Math.* 15, Article 214 (2018). DOI: <https://doi.org/10.1007/s00009-018-1212-8>
25. R. Yuan and Z. Zhang, Fast homoclinic solutions for some second order non-autonomous systems, *J. Math. Anal. Appl.* 376, 51–63 (2011). DOI: <https://doi.org/10.1016/j.jmaa.2010.10.045>