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# Extended GZK Cutoff in Momentum-Curved Space via Modified Dispersion Relations

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**Abstract**

We extend the Greisen–Zatsepin–Kuzmin (GZK) cutoff within a momentum-curved space framework derived from a specific modified dispersion relation (MDR) inspired by doubly special relativity (DSR). Our formulation yields an extended GZK equation suggesting that ultra-high-energy cosmic rays (UHECRs) can exceed the conventional limit of $10^{20}$ eV, consistent with recent experimental results. This provides a phenomenological link between Planck-scale deformations of space-time symmetries and observable cosmic-ray phenomena.

## 1 Introduction

In Einstein's theory of special relativity, the speed of light is a universal quantity and a fundamental constant of nature, invariant under Lorentz transformations among inertial observers. On the other hand, quantum gravity theories suggest the existence of a minimum length. Based on the definition of this length, it should be a universal quantity and therefore the same for all observers. However, in the framework of special relativity, any length under Lorentz transformations between observers undergoes Lorentz–FitzGerald contraction, so this theory does not predict any universal length. According to this, a redefinition of Lorentz transformations is necessary so that all observers agree on the existence of a global minimum length.

The idea of this theory was proposed in the early twenty-first century [1]. This theory is called doubly, or deformed, special relativity because, in addition to the speed of light, a minimum length remains unchanged under transformations between observers. In the first model presented by Amelino-Camelia, Lorentz symmetry was considered as an approximate symmetry that is broken in the high-energy regime and valid only in the normal energy regime. Almost a year after this model was introduced, doubly special relativity was formulated in such a way that Lorentz symmetry was preserved even at high energies [2]. However, it is clear that the validity of Lorentz symmetry in the high-energy regime of the order of the Planck scale can only be achieved by experiment. Finally, it was found that there is no unique doubly special relativity, but rather an infinite number of DSR theories, among which there is no special superiority because there is no authentic physical reason for distinction and preference between them.

In this paper, we examine the GZK paradox for ultra-high-energy cosmic rays in momentum-curved space. This paradox arises because some ultra-high-energy cosmic rays with energies exceeding this limit have been observed [3]. This discrepancy raises questions about the sources of these cosmic rays and the mechanisms involved in their propagation through the universe [4]. After introducing the MDR model in Sect. 2, we analyze the generalized GZK equation in momentum-curved space in Sect. 3.

## 2 Non-commutativity of space-time and momentum-curved space

In the first model of doubly special relativity, Lorentz transformations were generalized in such a way that, in addition to the speed of light, a minimum length remains invariant under transformations between observers. Lorentz symmetry in this initial model was broken at high energies [1]. Shortly after the presentation of this model, two general results were obtained: first, with the nonlinear action of the Lorentz group on momentum space, a model of DSR can be presented in which Lorentz symmetry is not broken but generalized [2]; second, there is an infinity of doubly special

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relativity theories [5]. The momentum space in these models is a curved de Sitter space [5], and different DSR models correspond to different definitions of energy and momentum on this curved space.

Noncommutative space-time is a general consequence of theories and models of quantum gravity which indicate the existence of a minimal length. In a general and mathematical sense, non-commutativity between the variables of a space indicates the curvedness of the space of dependent conjugate variables [6]. This concept also exists in the theory of general relativity. In this theory, based on the equivalence principle, gravity is considered as the curvature of space-time, and therefore the space of conjugate variables, namely momentum space, is a noncommutative space. More precisely, the covariant derivatives, which are the generators of translations in curved space-time, do not commute with each other, and their noncommutativity is related to the Riemann tensor, which measures the intrinsic curvature of space-time.

We know special relativity as a non-gravitational theory, and therefore, within this framework, space-time is flat. Also, in doubly special relativity, space-time is flat, but space-time variables are noncommutative. In a four-dimensional momentum space, space-time variables and momenta are conjugate to each other. Hence, just as in general relativity the non-commutativity of momentum corresponds to the curvature of space-time, in DSR the non-commutativity of space-time indicates the curvature of momentum space. In this way, within the framework of special relativity, momentum space must be curved.

In general relativity, the curvature of space-time, for example even for a static gravitational field, is a function of the space-time coordinates. But in DSR, the existence of a global minimal length causes momentum space to be curved. Since all observers must agree on the size of this length, the curvature of momentum space in DSR is necessarily constant. On the other hand, spaces with constant curvature are also spaces with maximum symmetry [6]. Due to the Lorentzian nature of the space in relativity, there are only two choices for the curved momentum space with constant curvature, which are de Sitter and anti-de Sitter spaces.

Regardless of whether we look at DSR models from an algebraic or geometric point of view, or whether momentum space is de Sitter or anti-de Sitter, all models eventually lead to a modified dispersion relation. The general form of these modified dispersion relations is

$$f^2(\lambda E)E^2 - g^2(\lambda E)P^2 = m^2, \tag{1}$$

where $m$ is the mass, and $f$ and $g$ are functions of energy that differ from one model to another. Based on the correspondence principle, the results of special relativity should be recovered in the low-energy regime, so for all models one has

$$\lim_{\lambda E\to 0} f(\lambda E) = 1, \qquad \lim_{\lambda E\to 0} g(\lambda E) = 1. \tag{2}$$

In this paper, the model used is

$$\left(\frac{2}{\lambda}\sinh\frac{\lambda E}{2}\right)^2 - e^{\lambda E}p^2 = m^2, \tag{3}$$

where $\lambda$ is the quantum deformation parameter [7]. The energy-momentum variables $(E,p)$ live on a manifold with de Sitter geometry [5, 7], while space-time is flat. The modified dispersion relation used in this work should be understood as a phenomenological realization inspired by DSR models with curved de Sitter momentum space. In the low-energy limit $\lambda E \ll 1$, it reduces to the standard special relativistic dispersion relation, ensuring consistency with known physics. In this study, the deformation parameter $\lambda$ is treated as an effective Planck-scale parameter of order $E_{\rm Pl}^{-1}$.

## 3 Extended GZK cutoff

In our previous work [8], we have shown the effects of modified dispersion relations and Lorentz invariance violation on inverse Compton scattering of CMB photons and on the GZK cutoff. For the Compton effect, our results are exactly compatible with the analyzed model in Ref. [8], by setting $n = 2$.

We use natural units $c = 1$ and the metric signature $(-,+,+,+)$. A representative CMB photon energy $\varepsilon \simeq 10^{-3}$ eV is adopted for threshold estimates. We consider the reaction between cosmic-ray protons, namely high-energy protons, and CMB photons,

$$p+\gamma \longrightarrow n+\pi^+. \tag{4}$$

The positively charged pion ensures charge conservation. Since the square of the total four-momentum is invariant, namely the Casimir associated with the adopted modified dispersion relation, any inertial observer obtains the same value for it. In the following derivation, we adopt the modified dispersion relation while keeping the standard additive energy-momentum conservation law. This approximation allows us to isolate the effect of the MDR on the threshold condition. Possible modifications of momentum composition, which may arise in a fully deformed DSR framework, are left for future investigation.

Thus, we obtain

$$(\mathscr{P}_p+\mathscr{P}_\gamma)^2 = (\mathscr{P}_n+\mathscr{P}_\pi)^2, \tag{5}$$

which can also be written as

$$\mathscr{P}_p^2 + 2\mathscr{P}_p\cdot\mathscr{P}_\gamma + \mathscr{P}_\gamma^2 = -(m_n+m_\pi)^2c^2. \tag{6}$$

For a particle with total energy $E$ and momentum $\vec{p}$, the four-momentum is written as

$$\mathscr{P} = \left(\frac{E}{c},\vec{p}\right), \tag{7}$$

and the square of the four-momentum is defined as

$$\mathscr{P}^2 = \mathscr{P} \cdot \mathscr{P} = -m^2 c^2. \tag{8}$$

From the above relation and Eq. (4), we get

$$\mathscr{P}_p^2 = -m_p^2 c^2 + \frac{\lambda^2}{12} p_p^4. \tag{9}$$

Putting Eq. (9) into Eq. (8), one obtains

$$\left( \frac{m_n^2 c^4 - m_p^2 c^4 + m_\pi^2 c^4}{4E_\gamma} \right) = E_p + \frac{\lambda^2}{12} \frac{E_p^4}{4E_\gamma}. \tag{10}$$

The neutron mass is

$$m_n c^2 \simeq 939.6\,\mathrm{MeV}, \tag{11}$$

and that of the $\pi^+$ meson is

$$m_\pi c^2 \simeq 139.6\,\mathrm{MeV}. \tag{12}$$

Taking $E_\gamma = 1.1\,\mathrm{meV}$ for the energy of the cosmic microwave background photon, we obtain

$$E_p = 3 \times 10^{20}\,\mathrm{eV} + \frac{\lambda^2}{12} \frac{E_p^4}{4E_\gamma}. \tag{13}$$

In deriving the modified threshold condition, we work in the high-energy regime $E_p \gg m_p c^2$ and retain leading-order terms in the deformation parameter $\lambda$. Subleading terms are neglected. In the limit $\lambda \to 0$, the standard GZK threshold is recovered. Obviously, Eq. (13) indicates that

$$E_p > 10^{20}\,\mathrm{eV}. \tag{14}$$

This result, similar to the extended Compton effect, is completely consistent with Ref. [8]. According to Eq. (13), if $\lambda$ tends to zero, then the common GZK cutoff is obtained.

Current observations [3, 9–11] indicate an overall suppression of the UHECR spectrum at the highest energies, while rare events above $10^{20}$ eV continue to motivate investigations of possible deviations from standard kinematics.

## 4 Conclusion

Doubly special relativity provides a framework to reconcile Planck-scale effects with relativistic invariance. Within this setting, we have derived an extended GZK cutoff using a modified dispersion relation in momentum-curved space. Our results demonstrate that high-energy cosmic rays may surpass the standard GZK limit, consistent with current experimental trends. This reinforces the potential role of Planck-scale physics in observable astrophysical phenomena [12].

In our opinion, further work may focus on the following directions:

1) exploring anisotropies in momentum-curved space and their influence on cosmic-ray propagation;
2) comparing predictions with updated Auger and Telescope Array data through numerical modeling;
3) extending the MDR framework to neutrino oscillations and photon dispersion;
4) linking DSR-based MDRs with loop quantum gravity or $\kappa$-Minkowski space-time models.

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