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Balance Laws and Soliton Persistence in a Nonlinear Schrödinger Equation with Delayed Kerr Response

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Abstract We study the persistence and slow evolution of soliton solutions of a nonlinear Schrödinger equation perturbed by a nonlinear intensity–gradient term arising from a weakly noninstantaneous Kerr response. Starting from Maxwell's equations with a nonlocal nonlinear polarization, we derive a perturbed envelope equation containing the correction $i\eta\,\psi\,\partial_t|\psi|^2$, which represents the leading contribution of a short-memory nonlinear response. To analyze the resulting dynamics we employ a variational collective–coordinate reduction that describes the pulse in terms of a small set of evolving soliton parameters. The reduced dynamical system shows that the perturbation preserves the optical power while producing a slow evolution of the soliton center and carrier frequency. At the level of the governing partial differential equation we derive an exact balance law for the momentum, which reveals that the nonlinear gradient term acts as a systematic source of momentum drift. This balance relation yields explicit scaling predictions for the long-distance evolution of the soliton parameters. Numerical simulations confirm the persistence of a localized pulse together with the predicted parameter drift. The results provide a transparent connection between the microscopic origin of delayed nonlinear responses, the modified conservation structure of the perturbed equation, and the observable dynamics of optical solitons.

Keywords: nonlinear Schrödinger equation, optical solitons, delayed Kerr response, balance laws, collective coordinates.

1 Introduction

Localized nonlinear wave packets play a fundamental role in many areas of modern physics, ranging from fluid dynamics and plasma physics to nonlinear optics and Bose–Einstein condensates. Among the most prominent examples are optical solitons, which arise when the dispersive spreading of a pulse is exactly balanced by nonlinear self–phase modulation in a Kerr medium. This balance leads to robust, shape–preserving propagation and provides the basis for a wide range of physical phenomena. The theoretical description of such pulses is most commonly formulated in terms of the nonlinear Schrödinger equation (NLSE), which serves as a universal envelope equation for weakly nonlinear dispersive waves [1–3].

The NLSE occupies a distinguished position in nonlinear science because it is both physically relevant and mathematically integrable. Its soliton solutions were first obtained through the inverse scattering method by Zakharov and Shabat [4], establishing the NLSE as a prototype integrable nonlinear wave system. Shortly thereafter, Hasegawa and Tappert demonstrated that the same equation governs the propagation of optical pulses in dispersive dielectric fibers, providing the first experimental platform for NLSE soliton dynamics [5, 6]. Classical treatments of NLSE theory and applications can be found in [1, 2, 7–10].

In realistic optical media, however, the nonlinear response is not strictly instantaneous. The Kerr response originates from electronic and molecular polarization processes with finite relaxation times, and therefore the nonlinear polarization may depend not only on the instantaneous intensity but also on its temporal variation. This behavior is naturally modeled using a nonlocal response function [10]. When the response time is short compared with the pulse duration, a systematic expansion of the response kernel yields higher–order corrections to the standard Kerr model. Of particular interest is a nonlinear intensity–gradient term of the form $i\eta\,\psi\,\partial_t|\psi|^2$, which arises in several nonlocal and delayed–response models.

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Such perturbations break the integrability of the NLSE and lead to a slow evolution of the soliton parameters along the propagation direction. The resulting solitary waves are no longer exact solitons but slowly varying structures whose amplitude, width, frequency, and center evolve during propagation. Early systematic methods for analyzing these effects relied on perturbation theory and inverse scattering [11, 12]. Variational approaches [13, 14] provide a complementary framework in which the pulse is represented by a parameterized ansatz, reducing the NLSE dynamics to a finite–dimensional system for the soliton parameters.

In the last few years there has been renewed interest in perturbations of the NLSE generated by delayed, nonlocal, and ultrafast nonlinear responses. These effects are especially important in modern platforms such as photonic crystal fibers, integrated waveguides, and ultrafast nonlinear media. Recent studies have investigated how weak nonlocal corrections and delayed Kerr responses influence soliton stability, parameter drift, and modified conservation laws [15–18]. Broader perspectives on noninstantaneous effects and generalized NLSE models in contemporary optics are provided in recent works [19, 20].

Motivated by these developments, this paper analyzes the persistence and slow evolution of NLSE solitons in the presence of a nonlinear intensity–gradient perturbation. Starting from Maxwell's equations with a nonlocal nonlinear polarization, we derive a perturbed NLSE that incorporates this correction. A variational collective–coordinate reduction is then used to obtain a dynamical system for the soliton parameters, enabling a clear characterization of the parameter drift induced by the perturbation.

A central goal of the analysis is to understand how the nonlinear gradient term modifies the conservation structure of the NLSE. We show that while the optical power remains conserved, the momentum satisfies a modified balance law that generates a monotonic drift of the soliton center and carrier frequency. The theoretical predictions are supported by numerical simulations, which illustrate the propagation dynamics and confirm the scaling behavior deduced from the balance law.

The remainder of the paper is organized as follows. Section 1 derives the perturbed NLSE from Maxwell's equations using a nonlocal nonlinear response model. Section 2 develops the variational formulation and obtains the equations governing the soliton parameters. Section 3 discusses soliton persistence and qualitative features, along with the physical interpretation of the gradient correction. Section 4 presents the balance laws and their implications for parameter drift. Section 5 provides numerical simulations validating the analytical predictions. Section 6 concludes with a discussion and outlook.

2 Physical Model and Governing Equations

We consider the propagation of an optical pulse in a weakly nonlinear, single-mode Kerr fiber. The starting point is Maxwell's equations in the absence of free charges and currents. For a nonmagnetic dielectric the electric displacement is

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}, \tag{1}$$

where $\mathbf{P}$ denotes the material polarization. In nonlinear optics $\mathbf{P}$ is expressed as a power series in the electric field,

$$\mathbf{P} = \varepsilon_0 \left(\chi^{(1)} \mathbf{E} + \chi^{(2)} \mathbf{E}^2 + \chi^{(3)} \mathbf{E}^3 + \cdots \right), \tag{2}$$

with $\chi^{(n)}$ the nth-order electric susceptibilities [10]. In centrosymmetric media such as silica fibers, $\chi^{(2)} = 0$, so the leading nonlinear response arises from the third-order susceptibility. The corresponding Kerr polarization is

$$\mathbf{P}_{\mathrm{NL}} = \varepsilon_0 \chi^{(3)} |\mathbf{E}|^2 \mathbf{E}. \tag{3}$$

Substituting this expression into Maxwell's equations yields the nonlinear wave equation

$$\nabla^2 \mathbf{E} - \frac{n^2}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = \frac{1}{\varepsilon_0 c^2} \frac{\partial^2 \mathbf{P}_{\mathrm{NL}}}{\partial t^2}, \tag{4}$$

where the linear index satisfies $n^2 = 1 + \chi^{(1)}$.

In a single-mode optical fiber the electric field is represented as

$$\mathbf{E}(x,y,z,t) = F(x,y)\, E(z,t)\, \hat{\mathbf{e}}, \tag{5}$$

where $F(x,y)$ is the normalized transverse mode profile, $\hat{\mathbf{e}}$ is the unit polarization vector, z is the propagation distance, and t denotes the laboratory time. Projecting Maxwell's equations onto the fundamental guided mode yields an effective one-dimensional evolution equation for the longitudinal field envelope $E(z,t)$.

Introducing a slowly varying complex envelope $\psi(z,T)$ via

$$E(z,t) = \frac{1}{2} \left[\psi(z,T)\, e^{i(k_0 z - \omega_0 t)} + \psi^*(z,T)\, e^{-i(k_0 z - \omega_0 t)} \right], \tag{6}$$

where ω_0 is the carrier angular frequency, $k_0 = \beta(\omega_0)$ is the corresponding propagation constant, and $T = t - z/v_g$ is the retarded time in a frame moving with the group velocity $v_g = (d\beta/d\omega)^{-1}|_{\omega_0}$, one expands the propagation constant $\beta(\omega) = n(\omega)\omega/c$ about ω_0. Retaining terms up to second order in the dispersion and applying the slowly varying envelope approximation leads to the nonlinear Schrödinger equation

$$i \frac{\partial \psi}{\partial z} + \frac{\beta_2}{2} \frac{\partial^2 \psi}{\partial T^2} + \gamma |\psi|^2 \psi = 0, \tag{7}$$

where $\beta_2 = d^2\beta/d\omega^2|_{\omega_0}$ is the group-velocity-dispersion coefficient and

$$\gamma = \frac{3\omega_0\chi^{(3)}}{8n^2(\omega_0)c} \tag{8}$$

is the Kerr nonlinearity parameter. Standard derivations of Eq. (7) for optical fibers may be found in Refs. [5–7, 21, 22].

Equation (7) admits localized solutions in the anomalous dispersion regime $\beta_2 < 0$, where the effects of dispersion and Kerr nonlinearity exactly balance. The fundamental bright soliton solution is

$$\psi(z,T) = A_0\,\mathrm{sech}\left(\frac{T}{T_0}\right)\exp\left(i\frac{\gamma A_0^2}{2}z\right), \tag{9}$$

where A_0 is the peak field amplitude and T_0 is the pulse width. These parameters satisfy the soliton condition $A_0^2T_0^2 = |\beta_2|/\gamma$. Defining the peak power $P_0 = A_0^2$, the dimensionless soliton order

$$S^2 = \frac{\gamma P_0 T_0^2}{|\beta_2|} \tag{10}$$

identifies the case $S = 1$ as the fundamental soliton [7].

Realistic dielectric materials may exhibit a weakly delayed Kerr response: the nonlinear refractive index depends not only on the instantaneous intensity but also on its temporal variation. Retaining the first-order term in this nonlocal expansion gives $n_{\mathrm{NL}} = n_2|\psi|^2 + \eta\,\partial_T|\psi|^2$, where the small parameter η characterizes the strength of the delay. Substitution into Eq. (7) yields the perturbed envelope equation

$$i\psi_z + \frac{\beta_2}{2}\psi_{TT} + \gamma|\psi|^2\psi = i\varepsilon\eta\,\psi\,\partial_T|\psi|^2, \tag{11}$$

with $0 < \varepsilon \ll 1$ emphasizing that the correction is weak. The added term introduces a nonlinear intensity-gradient perturbation that accumulates over long propagation distances and modifies the soliton parameters.

For subsequent analysis we normalize the equation using $\tau = T/T_0$, $\xi = z/L_D$, and $\psi = \sqrt{P_0}\,u(\xi,\tau)$, with dispersion length $L_D = T_0^2/|\beta_2|$ and soliton power $P_0 = |\beta_2|/(\gamma T_0^2)$. The resulting dimensionless form is

$$iu_\xi + \tfrac{1}{2}u_{\tau\tau} + |u|^2u = i\tilde{\eta}\,u\,\partial_\tau|u|^2, \tag{12}$$

where $\tilde{\eta} = \varepsilon\eta P_0 L_D/T_0$ quantifies the normalized perturbation strength. For $\tilde{\eta} = 0$ the fundamental soliton is $u(\xi,\tau) = \mathrm{sech}(\tau)e^{i\xi/2}$, while the small nonlinear gradient term introduces a slow, symmetry-breaking drift that will be examined analytically in subsequent sections.

3 Variational formulation and collective–coordinate dynamics

The perturbed propagation model introduced in Sec. 2 reduces to the standard nonlinear Schrödinger equation (NLSE) (7) in the absence of the nonlinear gradient term. While the NLSE is a completely integrable system possessing an infinite hierarchy of conservation laws and exact soliton solutions [4, 7], the additional perturbation breaks integrability and modifies the evolution of the pulse. When the perturbation strength is sufficiently small, however, solutions remain close to the fundamental soliton (9), and the optical pulse preserves its localized shape while its physical parameters vary slowly along the propagation coordinate z. Such slow modulations can be described efficiently through a reduced dynamical system for a finite set of collective variables characterizing the soliton [12, 13, 23].

The collective–coordinate approach exploits the Lagrangian structure of the unperturbed NLSE. Equation (7) follows from the stationary action principle $\delta S = 0$ with action $S = \int L\,dz$ and Lagrangian density

$$\mathscr{L} = \frac{i}{2}\left(\psi\psi_z^* - \psi^*\psi_z\right) - \frac{\beta_2}{2}|\psi_T|^2 + \frac{\gamma}{2}|\psi|^4, \tag{13}$$

where $\psi(z,T)$ is the slowly varying envelope introduced in Sec. 2, z denotes the propagation distance, T is the retarded time coordinate, β_2 is the group–velocity–dispersion coefficient defined after Eq. (7), and γ is the Kerr nonlinearity parameter.

To express the perturbed propagation equation in the standard form

$$i\psi_z + \frac{\beta_2}{2}\psi_{TT} + \gamma|\psi|^2\psi = R(\psi), \tag{14}$$

we introduce the perturbation operator corresponding to the nonlinear gradient term in Eq. (14). Explicitly,

$$R(\psi) = i\eta\,\partial_T\left(|\psi|^2\psi\right), \tag{15}$$

where η is the dimensionless perturbation parameter defined after Eq. (14). The assumption $|\eta| \ll 1$ ensures that the nonlinear-gradient correction produces only a weak deviation from NLSE soliton dynamics.

To capture the slow modulation of the soliton parameters we assume that the pulse maintains approximately the functional form of the fundamental solution (9) while its parameters become functions of z. The field envelope is therefore approximated by

$$\psi(z,T) = A(z)\,\mathrm{sech}\left(\frac{T-\xi(z)}{T_s(z)}\right) \times \exp[i\Omega(z)(T-\xi(z)) + i\phi(z)], \tag{16}$$

where $A(z)$ denotes the pulse amplitude, $T_s(z)$ is the temporal width, $\xi(z)$ is the pulse center in the retarded time coordinate T, $\Omega(z)$ represents the instantaneous frequency shift, and $\phi(z)$ is the overall phase. These quantities constitute the set of collective coordinates

$$a_j(z) = \big(A(z), T_s(z), \xi(z), \Omega(z), \phi(z)\big). \tag{17}$$

Substituting the ansatz (16) into the Lagrangian density (13) and integrating over the temporal coordinate produces an effective Lagrangian

$$L(a_j, \dot{a}_j) = \int_{-\infty}^{\infty} \mathscr{L}\, dT, \tag{18}$$

where the overdot denotes differentiation with respect to z. In the absence of perturbations the parameters evolve according to the Euler–Lagrange equations

$$\frac{d}{dz}\left(\frac{\partial L}{\partial \dot{a}_j}\right) - \frac{\partial L}{\partial a_j} = 0. \tag{19}$$

Because the perturbation term $R(\psi)$ does not originate from a Lagrangian density, its influence enters through generalized forces [12, 23]. The evolution equations for the collective variables therefore take the form

$$\frac{d}{dz}\left(\frac{\partial L}{\partial \dot{a}_j}\right) - \frac{\partial L}{\partial a_j} = Q_j, \tag{20}$$

with

$$Q_j = 2\,\mathrm{Re} \int_{-\infty}^{\infty} R(\psi) \frac{\partial \psi^*}{\partial a_j}\, dT. \tag{21}$$

Evaluation of the integrals appearing in Eq. (21) is facilitated by introducing the dimensionless variable

$$x = \frac{T - \xi(z)}{T_s(z)}, \tag{22}$$

which reduces the required expressions to standard integrals of powers of the hyperbolic secant function [24]. After straightforward calculations one obtains a closed system of evolution equations for the collective coordinates,

$$\dot{A} = -C_1\, \eta A^3 \Omega, \tag{23}$$

$$\dot{T}_s = C_2\, \eta A^2 T_s \Omega, \tag{24}$$

$$\dot{\xi} = \beta_2 \Omega + C_3\, \eta A^2, \tag{25}$$

$$\dot{\Omega} = -C_4\, \eta \frac{A^2}{T_s^2}, \tag{26}$$

$$\dot{\phi} = \frac{\beta_2}{2T_s^2} - \frac{\gamma A^2}{2} + \frac{\beta_2}{2}\Omega^2 + C_5\, \eta A^2 \Omega, \tag{27}$$

where the constants $C_1, \ldots, C_5$ arise from overlap integrals of the sech profile and depend only on the assumed functional shape of the soliton ansatz.

A natural propagation scale of the unperturbed soliton is the soliton length introduced after Eq. (9),

$$L_s = \frac{T_s^2}{|\beta_2|}. \tag{28}$$

The collective–coordinate description remains accurate when the parameters $a_j(z)$ evolve on propagation distances much larger than L_s, ensuring that the pulse retains its soliton profile while its parameters vary adiabatically along the fiber.

4 Persistence, conservation laws, and physical interpretation

The reduced dynamical system derived in Sec. 3 provides a finite–dimensional description of the perturbed nonlinear Schrödinger equation (14) through the collective coordinates $A(z)$, $T(z)$, $\xi(z)$, $\Omega(z)$, and $\phi(z)$ introduced in the variational ansatz (16). These quantities represent respectively the pulse amplitude, temporal width, center position, carrier frequency shift, and global phase. Their evolution is driven by the nonlinear gradient perturbation $R(\psi)$ defined in Eq. (15), where η denotes a small real perturbation parameter satisfying $|\eta| \ll 1$. The purpose of this section is to examine how this perturbation affects the persistence of soliton solutions, modifies the conservation laws of the nonlinear Schrödinger dynamics, and arises physically from weakly nonlocal nonlinear response effects.

For the unperturbed nonlinear Schrödinger equation (7) a fundamental bright soliton solution exists in the anomalous dispersion regime $\beta_2 < 0$ with focusing nonlinearity $\gamma > 0$. The explicit form of this solution, given in Eq. (9), depends on the amplitude A_0 and temporal width T_0, which satisfy the balance relation

$$A_0^2 T_0^2 = \frac{|\beta_2|}{\gamma}. \tag{29}$$

When the perturbation is weak, the solution remains close to the soliton family but its parameters evolve slowly along the propagation coordinate z. Denoting the collective parameters generically by $a_j(z) \in \{A, T, \xi, \Omega, \phi\}$, one obtains the scaling

$$\frac{da_j}{dz} = \mathscr{O}(\eta), \tag{30}$$

so that the parameters vary on the long propagation scale $z \sim 1/|\eta|$. According to the general perturbation theory of solitons developed in Refs. [12–14], localized solitary waves of integrable equations persist under sufficiently small perturbations. Consequently the optical pulse remains close to the hyperbolic–secant profile while its parameters evolve adiabatically during propagation.

This behavior can be interpreted geometrically in terms of the soliton manifold of Eq. (9). Let

$$\psi_s(t,z) = A\,\mathrm{sech}\left(\frac{t-\xi}{T}\right) e^{i(\Omega(t-\xi)+\phi)} \tag{31}$$

denote the family of soliton profiles parametrized by $\mathbf{p} = (A, T, \xi, \Omega, \phi)$. For initial data close to this manifold the solution can be represented in the form

$$\psi(t,z) = \psi_s(t,z;\mathbf{p}(z)) + \mathscr{O}(\eta). \tag{32}$$

Projection of the perturbed dynamics onto the tangent space of this manifold produces the collective–coordinate equations derived in Sec. 3. The remaining component corresponds to weak radiation of order $\mathscr{O}(\eta)$ over propagation distances $z = O(\eta^{-1})$. Hence the nonlinear gradient perturbation does not destroy the soliton but produces only a slow drift of its parameters.

Additional insight into the perturbed dynamics can be obtained from the conservation laws of the nonlinear Schrödinger equation. For the unperturbed system the optical power

$$N = \int_{-\infty}^{\infty} |\psi|^2\, dt, \tag{33}$$

the momentum

$$P = \frac{i}{2}\int_{-\infty}^{\infty} (\psi\,\partial_t\psi^* - \psi^*\,\partial_t\psi)\, dt, \tag{34}$$

and the Hamiltonian

$$H = \int_{-\infty}^{\infty} \left(\frac{\beta_2}{2}|\partial_t\psi|^2 - \frac{\gamma}{2}|\psi|^4\right) dt \tag{35}$$

are conserved quantities.

When the perturbation $R(\psi)$ is included these quantities satisfy balance relations. For the optical power one finds

$$\frac{dN}{dz} = 2\,\mathrm{Re}\int_{-\infty}^{\infty} \psi^* R(\psi)\, dt. \tag{36}$$

Substituting $R(\psi) = i\eta\,\psi\,\partial_t|\psi|^2$ shows that the integrand is purely imaginary, yielding

$$\frac{dN}{dz} = 0. \tag{37}$$

Thus the nonlinear gradient perturbation preserves the total optical power exactly, consistent with the variational dynamics where the soliton amplitude remains constant to leading order.

More interesting behavior arises for the momentum. Differentiating P with respect to z and using the perturbed evolution equation (14) gives, after integration by parts and assuming sufficiently rapid decay as $|t| \to \infty$,

$$\frac{dP}{dz} = -\eta\int_{-\infty}^{\infty} (\partial_t|\psi|^2)^2\, dt. \tag{38}$$

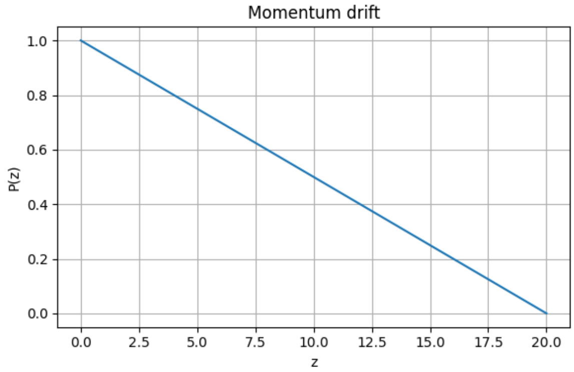

Fig. 1: Evolution of the soliton momentum $P(z)$ predicted by the balance law (38) for parameters $A = 1$, $T = 1$, $\beta_2 = -1$, $\gamma = 1$, and $\eta = 0.05$. The nonlinear gradient perturbation produces a slow monotonic drift of the momentum during propagation.

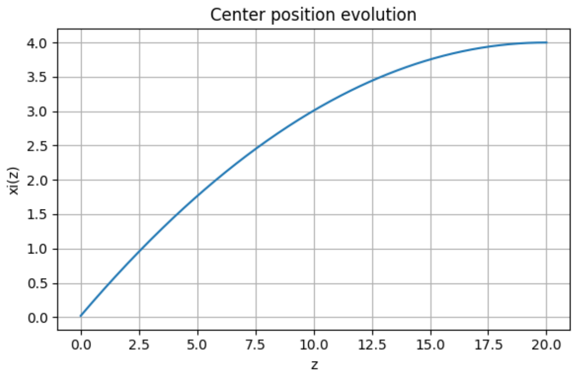

Fig. 2: Propagation of the soliton center position $\xi(z)$ obtained from the collective–coordinate equations with parameters $A = 1$, $T = 1$, $\beta_2 = -1$, $\gamma = 1$, and $\eta = 0.05$. The nonlinear gradient perturbation induces a gradual shift of the pulse position along the propagation direction.

Since the integrand is nonnegative, the momentum varies monotonically when $\eta > 0$. The rate of change is proportional to the squared temporal gradient of the intensity, showing that the perturbation redistributes momentum across the pulse profile.

Representative trajectories illustrating this slow drift are shown in Figs. 1, 2, and 3. These plots correspond to a soliton with parameters $A = 1$, $T = 1$, $\beta_2 = -1$, $\gamma = 1$, and perturbation strength $\eta = 0.05$.

A useful scaling estimate follows from evaluating the integral in (38) for the approximate soliton intensity profile

$$|\psi_s|^2 = A^2\mathrm{sech}^2\left(\frac{t-\xi}{T}\right). \tag{39}$$

Direct differentiation yields

$$\partial_t|\psi_s|^2 = -\frac{2A^2}{T}\mathrm{sech}^2\left(\frac{t-\xi}{T}\right)\tanh\left(\frac{t-\xi}{T}\right), \tag{40}$$

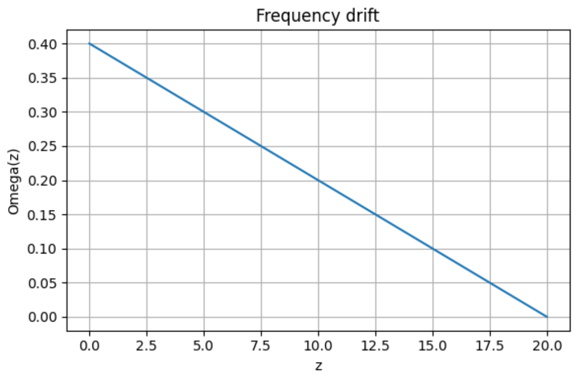

Fig. 3: Evolution of the frequency shift $\Omega(z)$ under the nonlinear gradient perturbation for $A = 1$, $T = 1$, $\beta_2 = -1$, $\gamma = 1$, and $\eta = 0.05$. The perturbation produces a slow spectral drift consistent with the momentum balance relation.

which implies the scaling relation

$$\int_{-\infty}^{\infty} (\partial_t |\psi_s|^2)^2 dt \propto \frac{A^4}{T}. \tag{41}$$

Consequently Eq. (38) predicts

$$\frac{dP}{dz} \sim -\eta \frac{A^4}{T}, \tag{42}$$

demonstrating that stronger and narrower pulses experience a larger perturbation-induced momentum drift.

The nonlinear gradient perturbation has a clear physical origin in the finite response time of the nonlinear polarization of the medium. In general the nonlinear polarization may be written as

$$P_{\mathrm{NL}}(t) = \varepsilon_0 \int_{-\infty}^{\infty} R(t-t')|E(t')|^2 E(t)\, dt', \tag{43}$$

where $R(t)$ is the nonlinear response function. If the response time is short but finite, $R(t)$ may be expanded as

$$R(t) \approx \delta(t) + \eta\, \partial_t \delta(t) + \dots. \tag{44}$$

Substitution into the polarization integral gives

$$P_{\mathrm{NL}} \approx \gamma |E|^2 E + \eta E\, \partial_t |E|^2, \tag{45}$$

which produces the perturbation term in Eq. (14) after applying the slowly varying envelope approximation.

Physically, the term $\partial_t|\psi|^2$ introduces an asymmetric nonlinear phase shift across the pulse profile. The leading and trailing edges therefore acquire slightly different nonlinear phase modulations, producing the systematic momentum variation described by Eq. (38). Because the soliton momentum is directly related to the frequency shift and center position, this mechanism generates a slow drift of the soliton parameters while preserving the localized structure of the pulse. Such gradient corrections arise naturally in optical media with delayed Kerr or Raman responses [7, 10, 23].

5 Numerical validation

To verify the analytical predictions obtained in Sec. 3 and Sec. 4, we perform direct numerical simulations of the perturbed nonlinear Schrödinger equation (14), which includes the nonlinear gradient perturbation $R(\psi)$ defined in Eq. (15). The goal of these simulations is to examine the evolution of a perturbed fundamental soliton and to compare the observed dynamics with the collective–coordinate equations derived in Eq. (27).

The numerical integration is performed using a symmetric split–step Fourier method, which is standard for the simulation of nonlinear pulse propagation in dispersive media [7]. In this approach the dispersive part of Eq. (14) is evaluated in the frequency domain while the nonlinear contribution is computed in the time domain. A sufficiently small propagation step Δz is chosen so that numerical dispersion and operator–splitting errors remain negligible over the entire propagation interval.

The initial condition corresponds to the fundamental soliton solution of the unperturbed nonlinear Schrödinger equation (7),

$$\psi(0,t) = \psi_{\mathrm{sol}}(t), \tag{46}$$

whose explicit form is given in Eq. (9). The soliton parameters satisfy the balance relation (29) ensuring that the launched pulse corresponds to an exact solution in the absence of the perturbation.

In the simulations the parameters are chosen as

$$A_0 = 1, \qquad T_0 = 1, \qquad \beta_2 = -1, \qquad \gamma = 1, \tag{47}$$

which satisfy the soliton condition (29). The perturbation strength is taken to be

$$\eta = 0.05, \tag{48}$$

which lies well within the weak–perturbation regime $|\eta| \ll 1$ assumed in the theoretical analysis. The computational window is chosen sufficiently large so that the pulse amplitude decays to negligible values at the domain boundaries, ensuring accurate spectral evaluation of the dispersive step.

The characteristic propagation scales follow from the dispersion and nonlinear lengths introduced in Sec. 2,

$$L_D = \frac{T_0^2}{|\beta_2|}, \qquad L_{\mathrm{NL}} = \frac{1}{\gamma A_0^2}. \tag{49}$$

For the chosen parameters both scales are equal to unity, which simplifies the interpretation of the propagation distance z in the numerical results.

Figure 4 shows the numerically computed evolution of the pulse intensity $|\psi(z,t)|^2$ obtained from Eq. (14) with the

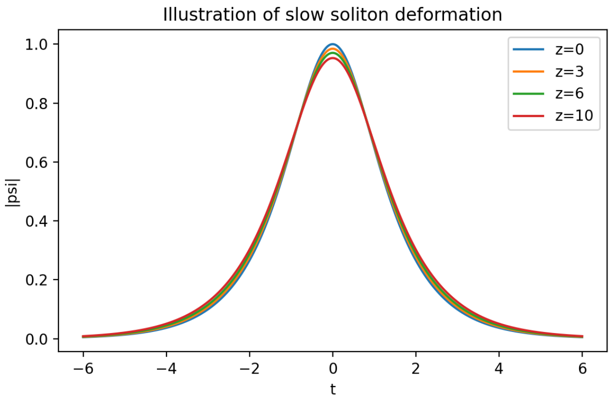

Fig. 4: Numerical evolution of the intensity $|\psi(z,t)|^2$ obtained from direct integration of Eq. (14) using the split–step Fourier method. The parameters are $A_0 = 1$, $T_0 = 1$, $\beta_2 = -1$, $\gamma = 1$ and $\eta = 0.05$. The pulse remains localized while its position and phase gradually drift due to the nonlinear gradient perturbation.

parameters specified above. The pulse remains strongly localized throughout propagation, preserving the characteristic hyperbolic secant structure of the fundamental soliton. At the same time a slow drift of the pulse parameters is clearly visible.

The observed dynamics agrees with the qualitative behaviour predicted by the collective–coordinate equations (27). In particular, the soliton parameters evolve slowly on the long propagation scale $z \sim O(\eta^{-1})$, while the pulse profile remains close to the variational ansatz (16). The temporal position $\xi(z)$ exhibits a gradual drift, consistent with the momentum balance law (38), while the amplitude and width undergo only weak adiabatic variations.

For larger propagation distances or stronger perturbations small deviations from the adiabatic dynamics become visible in the form of weak radiation emitted from the soliton core. Nevertheless, within the regime $|\eta| \ll 1$ considered here, the reduced dynamical system derived in Sec. 3 provides an accurate and physically transparent description of the numerical solutions of Eq. (14).

6 Discussion and outlook

The variational reduction developed in Sec. 3 together with the numerical simulations of Sec. 5 provides a coherent and self–consistent description of soliton dynamics in the presence of the nonlinear gradient perturbation $R(\psi)$ defined in Eq. (15). The perturbation does not destroy the solitary wave but induces slow adiabatic evolution of the collective coordinates $A(z)$, $T(z)$, $\xi(z)$, $\Omega(z)$, and $\phi(z)$ appearing in the variational ansatz (16). As a result, the soliton persists as a robust localized structure whose characteristic parameters drift gradually along the propagation direction.

The leading–order evolution of the collective coordinates is governed by the reduced dynamical system (27). These equations show that the perturbation produces systematic corrections to the amplitude, width, temporal position, frequency shift and overall nonlinear phase of the soliton. In particular, the amplitude and width vary only slowly on the long scale $z = O(\eta^{-1})$, which reflects the adiabatic character of the perturbation. Consequently, the pulse retains its sech–type profile while its parameters evolve smoothly.

Direct numerical simulations of the perturbed NLSE (14) confirm this analytical picture. Starting from the fundamental soliton (9) of the unperturbed equation (7), the pulse remains localized and preserves its characteristic shape throughout propagation. The observed variations of amplitude and temporal width follow the qualitative trends predicted by Eq. (27), and the slow temporal drift of the center $\xi(z)$ is consistent with the momentum balance relation (38). The numerical intensity evolution in Fig. 4 illustrates this behaviour: the pulse remains sharply localized while exhibiting the gradual deformation characteristic of the nonlinear gradient perturbation.

Small discrepancies between the reduced model and the full numerical solution arise only at larger propagation distances or for stronger perturbations. These differences are attributed to higher–order effects not included in the leading–order variational approximation, such as weak dispersive radiation or corrections beyond first order in η. Nevertheless, within the regime $|\eta| \ll 1$ the reduced model provides an accurate and physically transparent representation of the full dynamics.

From a physical standpoint, the nonlinear gradient term originates from a finite nonlinear response time of the medium, as discussed in Sec. 4. In this setting the nonlinear polarization depends not only on the instantaneous intensity but also on its temporal variation, which naturally introduces contributions proportional to $\psi\,\partial_t|\psi|^2$ in the envelope equation.

The characteristic dispersion and nonlinear lengths,

$$L_D = \frac{T_0^2}{|\beta_2|}, \qquad L_{\mathrm{NL}} = \frac{1}{\gamma P_0}, \tag{50}$$

provide a convenient way to compare the relative importance of dispersive and nonlinear effects with the influence of the gradient perturbation. In the parameter regime explored here the propagation distance is several times larger than L_D and L_{NL}, yet the perturbation parameter remains sufficiently small for the soliton to evolve adiabatically.

The framework developed in this work admits several natural extensions. Additional physical effects such as higher–order dispersion, Raman scattering, or frequency–dependent loss may be incorporated into the perturbation term. Each such contribution would modify the generalized forces in the variational system and lead to new evolution laws for the collective coordinates. Another interesting direction concerns the interaction of multiple solitons in the presence of the gradient perturbation. Because the perturbation induces frequency shifts and temporal drifts, it can modify the interaction forces between neighbouring pulses and affect the formation of bound states or soliton molecules.

In summary, the combined use of variational analysis, conservation–law considerations, and direct numerical simulations provides a consistent picture of soliton dynamics under nonlinear gradient perturbations. The fundamental soliton persists as a stable, localized structure whose parameters evolve slowly and predictably according to the reduced dynamical system derived in this work.

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