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A Discussion on the Eigenvalues of Sum Operators in the Hilbert space

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Abstract In this paper, we present rigorous proofs and counterexamples showing that the notion of partiality (reductionism) does not apply to operators in Hilbert space. We argue that the classical concepts of partiality and totality (holism) are fundamentally incompatible with the structure of quantum mechanics. This claim is supported across different interpretations of quantum theory, including cases involving nonlinear operators. Our analysis highlights the limitations of classical intuition in operator algebra and provides a refined understanding of eigenvalue behavior in quantum systems.

Keywords: Eigenvalues; Holism; Reductionism

1 Introduction

Whenever a system S is composed of components S_1 and S_2, it is often written as $S = S_1 + S_2$.

In this context, operator algebra displays properties that differ sharply from classical intuition. This is particularly evident in the behavior of inverse operators [1, 2] and nonlinear operators [3, 4]. For two arbitrary operators A and B in Hilbert space, their sum is

$$C = A + B. \tag{1}$$

A careful analysis of operator algebra reveals that even linear operators in Hilbert spaces display subtleties rarely emphasized in standard quantum mechanics textbooks. For example, the adjoint of the sum of two linear operators need not equal the sum of their adjoints. In fact, while for bounded operators one routinely has

$$(A+B)^\dagger = A^\dagger + B^\dagger,$$

the statement need not hold for densely defined (unbounded) operators without further hypothesis. This departure from classical additivity has important implications; for example, this point is treated carefully in classical perturbation theory [5] and functional analysis [6].

The spectral behavior of operator sums highlights an even deeper issue. Weyl's inequalities [7] and the min–max principle [8] provide useful upper and lower bounds on the eigenvalues of sums of Hermitian operators, but these results do not imply any simple additive law. Even if both are positive, counterexamples show that they may still admit negative eigenvalues [2, 9]. This observation undermines the reductionist expectation that the extremal eigenvalues of a composite operator must reflect those of its parts.

Furthermore, modern research has shown that estimating extremal eigenvalues remains nontrivial even in structured cases. For example, new lower bounds for the minimum singular value and minimum eigenvalue have recently been obtained in matrix analysis [3, 4, 10, 11]. Algorithmic approaches now provide sharper monotone bounds for the minimum eigenvalue of M-matrices [10], while investigations of non-Hermitian perturbations demonstrate that small changes to one operator can produce dramatic spectral shifts in their sum [11]. These studies reinforce the fact that spectral properties of operator sums cannot be reduced to a straightforward "part–whole" relation, but instead demand careful operator-theoretic analysis.

In particular, this paper investigates the behavior of the smallest eigenvalue of the sum of two operators. Counterexamples show that the minimum eigenvalue of $A + B$ (or AB) does not necessarily correspond to the eigenvalues of A or B individually. Despite the importance of this issue, such operator-theoretic details have not received the attention they deserve in standard quantum mechanics presentations. By clarifying these subtleties, our work emphasizes

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the limitations of classical reductionism and highlights the emergent character of quantum operator spectra.

2 Main Issues

Suppose the smallest eigenvalue of operator A is denoted by a, and the smallest eigenvalue of operator B is b. In this case, what is the smallest eigenvalue of the operator $C = A + B$?

At first, it seems that the answer to the above question is $c = a + b$, but we will see that we obtain a controversial result.

So, we start the discussion with the example of matrices A and B:

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 4x & -x \\ 6x & -x \end{pmatrix}. \tag{2}$$

The eigenvalues of matrix A are denoted by $\lambda_a = 0, 1$, and the eigenvalues of matrix B are denoted by $\lambda_b = x, 2x$. For all $x > 0$, these eigenvalues are non-negative. We now construct the matrix $A + B$.

$$A + B = \begin{pmatrix} 1+4x & -x \\ 6x & -x \end{pmatrix}. \tag{3}$$

The eigenvalue equation of matrix (3) is given by

$$\lambda_c^2 - \lambda_c(3x+1) + 2x^2 - x = 0. \tag{4}$$

For sufficiently small x, the eigenvalues can be expanded as

$$\begin{cases} \lambda_{c,1} = \dfrac{8x+2+o(x)}{2} = 1 + 4x + o(x), \\ \lambda_{c,2} = \dfrac{-2x + \dfrac{x^2}{2} + o(x)}{2} = -x + o(x). \end{cases} \tag{5}$$

where $o(x)$ denotes a function that converges to zero more rapidly than x.

Thus, for small positive values of x, one of the eigenvalues of $A + B$ becomes negative, even though all eigenvalues of A and B were non-negative. This demonstrates that the smallest eigenvalue of $A + B$ cannot, in general, be expressed as the sum of the smallest eigenvalues of A and B.

A special situation: Now assume that both A and B are Hermitian operators. We define the functions f and g in terms of vectors $|u\rangle$ in the Hilbert space as follows:

$$f(u) = \frac{\langle u|Au\rangle}{\langle u|u\rangle}, \qquad g(u) = \frac{\langle u|Bu\rangle}{\langle u|u\rangle}. \tag{6}$$

Because A is Hermitian, it admits an orthonormal basis of eigenvectors $\{|u_i\rangle\}$ with eigenvalues a_i.

Any vector $|u\rangle$ can be expanded as

$$A|u_i\rangle = a_i|u_i\rangle, \qquad u = \sum_i C_i|u_i\rangle. \tag{7}$$

Substituting (7) into (6), we obtain

$$f(u) = \frac{\sum_{i,j} C_i C_j^* \langle u_j|u_i\rangle}{\sum_{i,j} C_i C_j \langle u_j|u_i\rangle} = \frac{\sum_i a_i |C_i|^2}{\sum_i |C_i|^2}. \tag{8}$$

This immediately shows that $a \le f(u) \le a'$, by the same reasoning $b \le g(u) \le b'$, where b and b' are, respectively, the smallest and largest eigenvalues of the operator B.

Now consider the operator C,

$$CV = cV. \tag{9}$$

Where V is a vector in Hilbert space similar to U. We can express this as

$$c = \frac{\langle V|CV\rangle}{\langle V|V\rangle} = f(V) + g(V). \tag{10}$$

We can directly infer from equation (10) that $a + b \le c \le a' + b'$.

The lower section features compelling graphs that clearly illustrate the eigenvalues of matrices A and B and $A + B$.

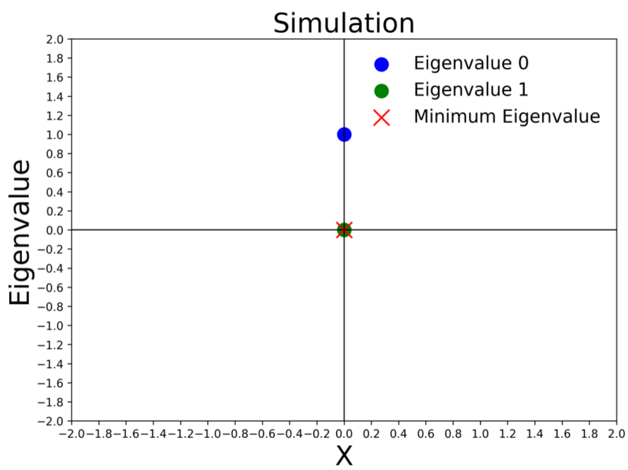

Fig. 1 This figure shows that the eigenvalues of matrix A. Direct computation yields two eigenvalues, 0 and 1, which appear as two distinct points on the axis. The smaller eigenvalue 0 is evident, contrasting with the larger eigenvalue 1.

From the x-dependence it follows that the spectrum of $A + B$ has no global minimum. For every eigenvalue identified, a smaller one appears as x increases. This graphical analysis reinforces the earlier conclusion that the smallest eigenvalue of a sum of operators cannot be simply inferred from the eigenvalues of its summands.

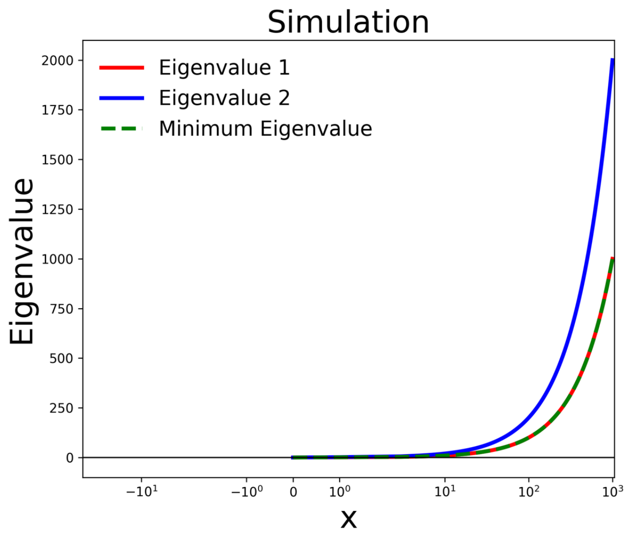

Fig. 2 This figure present the eigenvalues of matrix B. These are given by $x, 2x$. On the graph they appear as two straight lines with different slopes. Since these eigenvalues grow unbounded as $x \to \infty$, a logarithmic scale is used to display them clearly. The smaller eigenvalue x is marked with a dashed red line for emphasis.

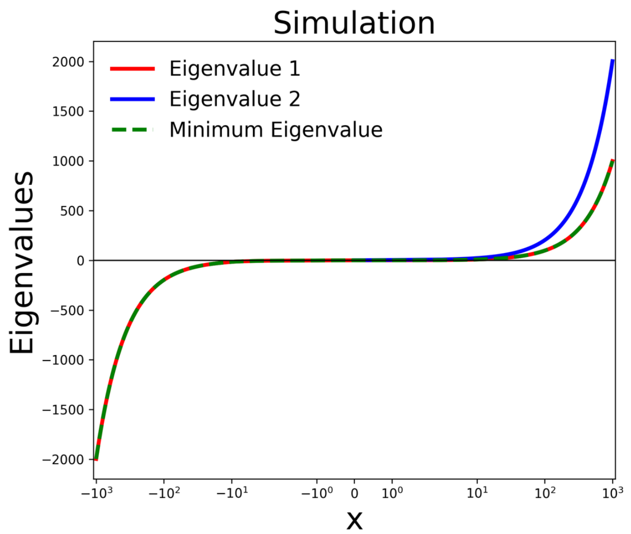

Fig. 3 This figure displays the eigenvalues of the sum $A+B$. The two eigenvalues are $1+4x$ and $-x$. The first increases linearly with x, while the second decreases without bound as $x \to \infty$. Again, a logarithmic scale is used The negative branch, $-x$, indicates that $A+B$ does not necessarily admit a non-negative minimum eigenvalue, even when both A and B possess non-negative eigenvalues.

3 Conclusion

In this paper we have shown, through explicit examples and theoretical analysis, that the spectral behavior of operator sums in Hilbert space differs fundamentally from classical expectations. In particular, the smallest eigenvalue of $A+B$ cannot, in general, be obtained as the sum of the smallest eigenvalues of A and B. Counterexamples demonstrate that even when both A and B are positive, their sum may admit negative eigenvalues. This result underscores a key distinction between classical additive structures and operator algebra in quantum mechanics.

While classical mechanics often permits decomposition of a system into independent parts, such a "part–whole" relation does not carry over straightforwardly to the spectral properties of quantum operators. Recognizing this limitation provides a clearer understanding of the mathematical foundations of quantum theory and emphasizes the need for careful operator-theoretic analysis beyond classical intuition.

References

1. J. Weidmann, *Linear Operators in Hilbert Spaces*, Springer-Verlag, New York, 1980.
2. M. H. Mortad, *Counterexamples in Operator Theory*, Birkhäuser/Springer, Cham, 2022. DOI: 10.1007/978-3-030-95750-9.
3. A. Kaur and S. H. Lui, A closer look at some new lower bounds on the minimum singular value of a matrix, *Linear Algebra and its Applications*, 2024.
4. A. Kaur and S. H. Lui, New lower bounds on the minimum singular value of a matrix, *Linear Algebra and its Applications*, 666 (2023), 62–95. DOI: 10.1016/j.laa.2023.01.017.
5. T. Kato, *Perturbation Theory for Linear Operators*, Springer, Berlin, 1995. DOI: 10.1007/978-3-642-66282-9.
6. M. Reed and B. Simon, *Methods of Modern Mathematical Physics, Vol. I: Functional Analysis*, Academic Press, New York, 1980.
7. H. Weyl, Inequalities between the two kinds of eigenvalues of a linear transformation, *Proceedings of the National Academy of Sciences of the United States of America*, 35 (1949), 408–411. DOI: 10.1073/pnas.35.7.408.
8. G. Teschl, *Mathematical Methods in Quantum Mechanics: With Applications to Schrödinger Operators*, 2nd ed., American Mathematical Society, Providence, 2014. DOI: 10.1090/gsm/157.
9. R. Bhatia, *Matrix Analysis*, Springer, New York, 1997. DOI: 10.1007/978-1-4612-0653-8.
10. Q. Zhong, An innovative algorithm for estimating the minimum eigenvalue of a general M-matrix, *Journal of Inequalities and Applications*, 2025, Article 335.
11. A. Rathore and N. Sinha, Eigenvalue behavior under non-Hermitian perturbations, *Linear Algebra and its Applications*, 670 (2024), 45–72.