



# The Second-Order Basis for Homogeneous Solutions of Compatible Higher Order Linear ODEs with Varying Coefficients

Gunawan Nugroho<sup>1a</sup>, Totok R. Biyanto<sup>1</sup>

<sup>1</sup> Departement of Engineering Physics, Institut Teknologi Sepuluh Nopember Kampus ITS Sukolilo, Surabaya, Indonesia 60111

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**Abstract** This research presents a reduction and reconstruction study for third and fourth order linear ordinary differential equations with variable coefficients, including equations with nonzero dependent variable terms. A compatible higher order equation is reduced to a second order ODE and then is solved analytically to produce the homogeneous solution required by variation of parameters and the Wronskian. Once this homogeneous solution is found, the systematic general solution follows from the classical variation of parameters, represented here by equation (9d). The work therefore aims to pierce the main difficulty of variable coefficient problems: finding an analytical homogeneous solution for the linear compatible ODE with varying coefficients. Validation is performed on Airy, Bessel, Legendre, and Weber type models, including induced third and fourth order equations and beyond special functions coefficients. Numerical comparisons with classical special functions show agreement at round off scale.

**Keywords:** Variable-coefficient ordinary differential equations; reduction of order; analytical solution; integrating factor; special functions

## 1 Introduction

Linear ODEs with variable coefficients are foundational in mathematical physics, engineering dynamics, wave propagation, elasticity, quantum mechanics, and stability theory. Classical theory established that variable coefficients, singular points, and boundary behavior fundamentally complicate solution structures beyond constant-coefficient cases [1, 2]. Thus, the special functions such as Airy, Bessel, Legendre, and parabolic-cylinder functions arise naturally from second-order variable-coefficient equations and continue to

serve as benchmarks for analytical and numerical methods [3, 4].

Despite extensive developments in Frobenius series, WKB approximations, contour-integral representations, variation of parameters, and transformation-based techniques, systematic approaches that reduce higher-order variable-coefficient ODEs to solvable lower-order forms remain of strong interest. Recent studies have advanced generalized integrating factors for second- and third-order equations [5–7]. Gadella and Lara emphasized that exact solutions for variable coefficient systems are generally difficult except for special structures [8]. Recent applied mathematics treatments continue to introduce Wronskians, Cauchy problems, first and second order linear equations, and Euler type or special equations as the main analytical tools [9]. Open educational resources published in recent years also continue to present higher order equations through Wronskians, superposition, variation of parameters, and reduction of order, showing that the topic remains central in current teaching and research practice [10].

The analytical bottleneck can be stated very clearly: the general solution is systematic if a fundamental homogeneous pair is known. The Wronskian and variation of parameters then produce a particular solution and complete the solution space. However, finding even one nontrivial homogeneous solution for a variable-coefficient equation is the main difficulty. This observation motivates the present work on variable-coefficient linear ODEs, which constructs a correct homogeneous solution so that variation of parameters and reduction of order become systematic and classical. In this sense, the proposed approach is complementary to the standard methodology: first reduce a compatible higher-order equation to a second-order equation, then solve the second-order ODE analytically, and finally use the Wronskian and variation of parameters to establish the general solution.

<sup>a</sup>gunawan@ep.its.ac.id

This paper contributes a structured reduction-of-order method for compatible third- and fourth-order linear ODEs. The method transforms the original equation into a reduced form in which a derivative of the dependent variable satisfies a second-order variable-coefficient ODE. The reduced second-order equation is solved analytically, followed by inverse transformations to reconstruct the original solution. To validate the framework, the paper applies the reduction to Airy, Bessel, Legendre, and Weber equations of higher-order type. These functions are classical enough to provide exact benchmark solutions, while remaining sufficiently rich to test variable-coefficient behavior. Additional examples involving non-special variable terms are also considered, demonstrating that the applicability of the method extends beyond constructive special-function settings.

## 2 The Mathematical Formulations

### 2.1 Reduction of 3rd Order ODE

Firstly, we would like to state that the written symbols and coefficients are freely chosen for convenience and the reader should not be confused by perceiving that each subsection has its own variables and coefficients. Consider a non-homogeneous third-order linear differential equation with variable coefficients,

$$y_{xxx} + a_1 y_{xx} + a_2 y_x + a_3 y = a_4, \quad (1a)$$

Let  $a_1 = b_1 + \frac{a_{5x}}{a_5}$ , then equation (1a) can be rewritten as

$$\frac{1}{a_5} (a_5 y_{xx})_x + b_1 y_{xx} + a_2 y_x + a_3 y = a_4. \quad (1b)$$

Also take  $a_2 = b_2 + b_1 \frac{a_{6x}}{a_6}$ , thus the following relation is obtained,

$$\frac{1}{a_5} (a_5 y_{xx})_x + \frac{b_1}{a_6} (a_6 y_x)_x + b_2 y_x + a_3 y = a_4. \quad (1c)$$

Multiply by an arbitrary function  $\alpha$  to generate,

$$\frac{\alpha}{a_5} (a_5 y_{xx})_x + \frac{\alpha b_1}{a_6} (a_6 y_x)_x + \alpha b_2 y_x + \alpha a_3 y = \alpha a_4. \quad (1d)$$

Suppose that the following expression is satisfied,

$$\alpha_x b_2 = \alpha a_3, \quad \alpha = C_1 e^{\int \frac{a_3}{b_2} dx}. \quad (1e)$$

Let  $C_1 = 1$ , equation (1d) is rewritten as

$$\frac{\alpha}{a_5} (a_5 y_{xx})_x + \frac{\alpha b_1}{a_6} (a_6 y_x)_x + b_2 \left( e^{\int \frac{a_3}{b_2} dx} y \right)_x = \alpha a_4. \quad (1f)$$

Suppose that

$$e^{\int \frac{a_3}{b_2} dx} y = u, \quad y = u e^{-\int \frac{a_3}{b_2} dx}. \quad (2a)$$

Therefore, equation (1f) can be expanded as

$$\begin{aligned} & \frac{\alpha}{a_5} \left\{ a_5 \left[ u_{xx} e^{-\int \frac{a_3}{b_2} dx} + 2u_x \left( -\frac{a_3}{b_2} \right) e^{-\int \frac{a_3}{b_2} dx} \right. \right. \\ & \left. \left. + u \left( -\frac{a_3}{b_2} \right)_x e^{-\int \frac{a_3}{b_2} dx} + u \left( -\frac{a_3}{b_2} \right)^2 e^{-\int \frac{a_3}{b_2} dx} \right] \right\}_x \\ & + \frac{\alpha b_1}{a_6} \left\{ a_6 \left[ u_x e^{-\int \frac{a_3}{b_2} dx} + u \left( -\frac{a_3}{b_2} \right) e^{-\int \frac{a_3}{b_2} dx} \right] \right\}_x \\ & + b_2 u_x = \alpha a_4. \end{aligned} \quad (2b)$$

Assume that

$$\begin{aligned} & \frac{\alpha}{a_5} \left[ a_5 e^{-\int \frac{a_3}{b_2} dx} \left( \left( -\frac{a_3}{b_2} \right)^2 + \left( -\frac{a_3}{b_2} \right)_x \right) \right]_x \\ & + \frac{\alpha b_1}{a_6} \left[ a_6 \left( -\frac{a_3}{b_2} \right) e^{-\int \frac{a_3}{b_2} dx} \right]_x = 0. \end{aligned} \quad (2c)$$

Now assume that  $b_2$  is given, then  $\frac{a_{6x}}{a_6}$  can be determined from (2c) as

$$\frac{a_{6x}}{a_6} = \frac{f_6 \frac{a_{5x}}{a_5} + b_1 f_7 + f_8}{b_1 f_9}. \quad (2d)$$

Substituting into  $a_2 = b_2 + b_1 \frac{a_{6x}}{a_6}$  to give the expression of  $b_1$  as a function of  $\frac{a_{5x}}{a_5}$ . Performing the resulting expression into  $a_1 = b_1 + \frac{a_{5x}}{a_5}$  to generate  $a_5$ . Therefore, equation (1a) is reduced into  $u_{xxx} + a_7 u_{xx} + a_8 u_x = a_9$ . Let  $u_x = v$ , thus the above equation can be transformed to the second-order ODE,

$$g_{xx} + a_7 g_x + a_8 g = a_9, \quad \text{or} \quad g_{xx} + p(x)g_x + r(x)g = 0. \quad (2e)$$

This establishes that the original third-order equation is reduced to a second-order equation in  $v(x)$ , provided the coefficient decomposition  $(a_1, a_2, a_3) \rightarrow (a_7, a_8, a_9)$  exists.

### 2.2 Reduction of 4th Order ODE

Consider a non-homogeneous fourth-order linear differential equation with variable coefficients below,

$$y_{xxxx} + a_1 y_{xxx} + a_2 y_{xx} + a_3 y_x + a_4 y = a_5. \quad (3a)$$

Suppose that  $a_1 = b_1 + \frac{a_{6x}}{a_6}$ , and  $a_2 = b_2 + b_1 \frac{a_{7x}}{a_7}$ , and  $a_3 = b_3 + b_2 \frac{a_{8x}}{a_8}$ , equation (3a) becomes

$$\begin{aligned} & \frac{1}{a_6} (a_6 y_{xxx})_x + \frac{b_1}{a_7} (a_7 y_{xx})_x \\ & + \frac{b_2}{a_8} (a_8 y_x)_x + b_3 y_x + a_4 y = a_5. \end{aligned} \quad (3b)$$

Multiplying by an arbitrary function  $\alpha$  to give  $\frac{\alpha}{a_6}(a_6 y_{xxx})_x + \frac{\alpha b_1}{a_7}(a_7 y_{xx})_x + \frac{\alpha b_2}{a_8}(a_8 y_x)_x + \alpha b_3 y_x + \alpha a_4 y = a_5$ .  
Take the following relation,

$$\alpha_x b_3 = \alpha a_4, \quad \alpha = C_1 e^{\int \frac{a_4}{b_3} dx}. \quad (3c)$$

Equation (3c) is transformed as

$$\frac{\alpha}{a_6}(a_6 y_{xxx})_x + \frac{\alpha b_1}{a_7}(a_7 y_{xx})_x + \frac{\alpha b_2}{a_8}(a_8 y_x)_x + b_3 \left( e^{\int \frac{a_4}{b_3} dx} y \right)_x = \alpha a_5. \quad (3d)$$

Let us assume that

$$e^{\int \frac{a_4}{b_3} dx} y = u, \quad y = u e^{-\int \frac{a_4}{b_3} dx}. \quad (4a)$$

Expanding equation (3d) as

$$\begin{aligned} & \frac{\alpha}{a_6} \left\{ a_6 \left[ u_{xxx} E + 3u_{xx} \left( -\frac{a_4}{b_3} \right) E + 3u_x \left( -\frac{a_4}{b_3} \right)_x E \right. \right. \\ & \quad \left. \left. + 3u_x \left( -\frac{a_4}{b_3} \right)^2 E + u \left( -\frac{a_4}{b_3} \right)_{xx} E \right. \right. \\ & \quad \left. \left. + 3u \left( -\frac{a_4}{b_3} \right)_x \left( -\frac{a_4}{b_3} \right) E + u \left( -\frac{a_4}{b_3} \right)^3 E \right] \right\}_x \\ & + \frac{\alpha b_1}{a_7} \left\{ a_7 \left[ u_{xx} E + 2u_x \left( -\frac{a_4}{b_3} \right) E + u \left( -\frac{a_4}{b_3} \right)_x E \right. \right. \\ & \quad \left. \left. + u \left( -\frac{a_4}{b_3} \right)^2 E \right] \right\}_x \\ & + \frac{\alpha b_2}{a_8} \left\{ a_8 \left[ u_x E + u \left( -\frac{a_4}{b_3} \right) E \right] \right\}_x + b_3 u_x = \alpha a_5. \end{aligned} \quad (4b)$$

where  $E = e^{-\int \frac{a_4}{b_3} dx}$ . Performing the following relation,

$$\begin{aligned} & \frac{\alpha}{a_6} \left[ a_6 \left( -\frac{a_4}{b_3} \right)_{xx} E + 3a_6 \left( -\frac{a_4}{b_3} \right)_x \left( -\frac{a_4}{b_3} \right) E \right. \\ & \quad \left. + a_6 \left( -\frac{a_4}{b_3} \right)^3 E \right]_x \\ & + \frac{\alpha b_1}{a_7} \left[ a_7 \left( -\frac{a_4}{b_3} \right)^2 E + a_7 \left( -\frac{a_4}{b_3} \right)_x E \right]_x \\ & + \frac{\alpha b_2}{a_8} \left[ a_8 \left( -\frac{a_4}{b_3} \right) E \right]_x = 0. \end{aligned} \quad (4c)$$

Suppose that  $b_3$  and  $a_8$  are given, then  $\frac{a_{7x}}{a_7}$  can be determined from (4c) as

$$\frac{a_{7x}}{a_7} = \frac{f_{10} \frac{a_{6x}}{a_6} + b_1 f_{11} + f_{12}}{b_1 f_{13}}. \quad (4d)$$

Substituting (4d) into  $a_2 = b_2 + b_1 \frac{a_{7x}}{a_7}$  to give  $b_1$  as a function of  $\frac{a_{6x}}{a_6}$ . The next step is implementing into  $a_1 = b_1 + \frac{a_{6x}}{a_6}$  to produce  $a_6$ . Therefore, the fourth-order equation is reduced into  $u_{xxxx} + a_9 u_{xxx} + a_{10} u_{xx} + a_{11} u_x = a_{12}$ .

Let  $u_x = v$ , thus the above equation can be transformed to the third-order equation,

$$g_{xxx} + a_9 g_{xx} + a_{10} g_x + a_{11} g = a_{12}. \quad (4e)$$

Implementing the third-order reduction into (4e), both third- and fourth-order equations are systematically reduced to the same second-order structure, provided the auxiliary coefficients satisfy compatibility relations.

### 2.3 Analytical Solution of 2<sup>nd</sup> Order ODE with Varying Coefficients

Consider the 2<sup>nd</sup> order ODE equation with variable coefficients as follows,

$$A_{xx} + f_1(x)A_x + f_2(x)A = 0. \quad (5a)$$

Let  $A = wB$ , then

$$B_{xx} + \left( 2 \frac{w_x}{w} + f_1 \right) B_x + \left( \frac{w_{xx} + f_1 w_x + f_2 w}{w} \right) B = 0. \quad (5b)$$

Take  $2 \frac{w_x}{w} + f_1 = a_1 = 0$ , then  $w$  is determined by

$$w = e^{-\frac{1}{2} \int f_1 dx}. \quad (5c)$$

and equation (5a) is rewritten as

$$B_{xx} + a_2 B = 0, \quad (5d)$$

where  $a_2 = \frac{w_{xx} + f_1 w_x + f_2 w}{w}$ .

Let  $B = (yZ)_x$ , equation (5d) becomes

$$(yZ)_{xxx} + a_2 (yZ)_x = 0,$$

or

$$yZ_{xxx} + 3y_x Z_{xx} + 3y_{xx} Z_x + y_{xxx} Z + a_2 y Z_x + a_2 y_x Z = 0, \quad (6a)$$

or

$$yZ_{xxx} + 3y_x Z_{xx} + (3y_{xx} + a_2 y) Z_x + (y_{xxx} + a_2 y_x) Z = 0.$$

Suppose that  $3y_x = b_1 + y \frac{a_{3x}}{a_3}$ ,  $3y_{xx} + a_2 y = b_2 + b_1 \frac{a_{4x}}{a_4}$ , equation (6a) is rewritten as

$$\frac{y}{a_3} (a_3 Z_{xx})_x + \frac{b_1}{a_4} (a_4 Z_x)_x + b_2 Z_x + \gamma Z = 0, \quad (6b)$$

where  $\gamma = y_{xxx} + a_2 y_x$ . Introducing the integrating factor  $\beta = e^{\int \frac{\gamma}{b_2} dx}$ , to give

$$\frac{\beta y}{a_3} (a_3 Z_{xx})_x + \frac{\beta b_1}{a_4} (a_4 Z_x)_x + b_2 (\beta Z)_x = 0,$$

or

$$\begin{aligned} & \frac{\beta y}{a_3} \left\{ a_3 \left[ e^{-\int \frac{\gamma}{b_2} dx} U_{xx} + 2 \left( -\frac{\gamma}{b_2} \right) e^{-\int \frac{\gamma}{b_2} dx} U_x \right. \right. \\ & \left. \left. + \left( -\frac{\gamma}{b_2} \right)^2 e^{-\int \frac{\gamma}{b_2} dx} U + \left( -\frac{\gamma}{b_2} \right)_x e^{-\int \frac{\gamma}{b_2} dx} U \right] \right\}_x \\ & + \frac{\beta b_1}{a_4} \left\{ a_4 \left[ e^{-\int \frac{\gamma}{b_2} dx} U_x + \left( -\frac{\gamma}{b_2} \right) e^{-\int \frac{\gamma}{b_2} dx} U \right] \right\}_x \\ & + b_2 U_x = 0. \end{aligned} \quad (6c)$$

with  $U = \beta Z$ . Set the following condition,

$$\frac{\beta y}{a_3} \left[ a_3 \chi^2 e^{\int \chi dx} + a_3 \chi_x e^{\int \chi dx} \right]_x + \frac{\beta b_1}{a_4} \left[ a_4 \chi e^{\int \chi dx} \right]_x = 0. \quad (6d)$$

Expanding (6d),

$$\begin{aligned} & \left[ \frac{\beta y a_{3x}}{a_3} \chi e^{\int \chi dx} + \beta y \chi^3 e^{\int \chi dx} + 3 \beta y \chi \chi_x e^{\int \chi dx} \right] \\ & + \left[ \frac{\beta y a_{3x}}{a_3} \chi_x e^{\int \chi dx} + \beta y \chi_{xx} e^{\int \chi dx} \right] \\ & + \left[ \frac{\beta b_1 a_{4x}}{a_4} \chi e^{\int \chi dx} + \beta b_1 \chi_x e^{\int \chi dx} + \beta b_1 \chi^2 e^{\int \chi dx} \right] \\ & = 0, \end{aligned} \quad (1)$$

where  $\chi(x) = -\frac{\gamma}{b_2}$ . Recalling the definition of  $\beta$ , equation (6d) is rewritten as

$$\begin{aligned} & \left[ \frac{y a_{3x}}{a_3} \chi + y \chi^3 + 3 y \chi \chi_x + \frac{y a_{3x}}{a_3} \chi_x + y \chi_{xx} \right] \\ & + \left[ \frac{b_1 a_{4x}}{a_4} \chi + b_1 \chi_x + b_1 \chi^2 \right] = 0. \end{aligned} \quad (6e)$$

Equation (6c) is then defined by

$$\begin{aligned} & \frac{\beta y}{a_3} \left[ a_3 e^{\int \chi dx} U_{xxx} + a_3 \chi e^{\int \chi dx} U_{xx} + a_{3x} e^{\int \chi dx} U_{xx} \right. \\ & \left. + 2 a_3 \chi e^{\int \chi dx} U_{xx} + 2 a_3 \chi^2 e^{\int \chi dx} U_x + 2 a_{3x} \chi e^{\int \chi dx} U_x \right. \\ & \left. + a_3 \chi^2 e^{\int \chi dx} U_x + a_3 \chi_x e^{\int \chi dx} U_x \right] \\ & + \frac{\beta b_1}{a_4} \left[ a_4 e^{\int \chi dx} U_{xx} + a_4 \chi e^{\int \chi dx} U_x \right. \\ & \left. + a_{4x} e^{\int \chi dx} U_x + a_4 \chi e^{\int \chi dx} U_x \right] \\ & + b_2 U_x + \frac{\beta y}{a_3} \left[ a_3 \chi^2 e^{\int \chi dx} U + a_3 \chi_x e^{\int \chi dx} U \right]_x \\ & + \frac{\beta b_1}{a_4} \left[ a_4 \chi e^{\int \chi dx} U \right]_x = 0. \end{aligned} \quad (7a)$$

Recalling the definition of  $\beta$ , equation (7a) is rewritten as

$$\begin{aligned} & y U_{xxx} + \left[ 3 y \chi + \frac{y a_{3x}}{a_3} + b_1 \right] U_{xx} \\ & + \left[ 3 y \chi^2 + 2 \frac{y a_{3x}}{a_3} \chi + y \chi_x + 2 b_1 \chi + \frac{b_1 a_{4x}}{a_4} + b_2 \right] U_x \\ & = 0. \end{aligned} \quad (7b)$$

Substituting  $b_1 = 3y_x - y \frac{a_{3x}}{a_3}$ ,  $b_1 \frac{a_{4x}}{a_4} = 3y_{xx} + a_2 y - b_2$ , into (6e),

$$\begin{aligned} & y U_{xxx} + \left[ 3 y \chi + 3 y_x \right] U_{xx} \\ & + \left[ 3 y \chi^2 + y \chi_x + 6 y_x \chi + 3 y_{xx} + a_2 y \right] U_x = 0, \end{aligned} \quad (7c)$$

or

$$\begin{aligned} & U_{xxx} + \left[ 3 \chi + 3 \frac{y_x}{y} \right] U_{xx} \\ & + \left[ 3 \chi^2 + \chi_x + 6 \frac{y_x}{y} \chi + 3 \frac{y_{xx}}{y} + a_2 \right] U_x = 0. \end{aligned}$$

Take the following condition,  $\left[ 3 \chi + 3 \frac{y_x}{y} \right]_x = \left[ 3 \chi^2 + \chi_x + 6 \frac{y_x}{y} \chi + 3 \frac{y_{xx}}{y} + a_2 \right]$ . The  $y$ - and  $b_2$ -dependent terms can be separated as

$$-3 \left( \frac{y_x}{y} \right)^2 - a_2 = 3 \chi^2 - 2 \chi_x + 6 \frac{y_x}{y} \chi = 0. \quad (8a)$$

Thus, each solution will be

$$y = C_1 e^{-\int \sqrt{\frac{1}{3} a_2} dx}, \quad \chi = -\frac{2}{3} \frac{y^3}{\int y^3 dx + C_2}. \quad (8b)$$

Furthermore, equation (6e) is performed into  $3y_x = b_1 + y \frac{a_{3x}}{a_3}$ , as follows,

$$b_1 \left( -\chi - \frac{a_{4x}}{a_4} + 1 \right) = 3y_x + y\chi^2 + 3y\chi_x - 3y_x\chi_x\chi^{-1} - y\chi_{xx}\chi^{-1}. \quad (8c)$$

Then, substitute equation (8c) into  $3y_{xx} + a_2y = b_2 + b_1 \frac{a_{4x}}{a_4}$  to form the equation for  $a_4$ ,

$$(3y_{xx} + a_2y - b_2) \left( -\chi - \frac{a_{4x}}{a_4} + 1 \right) = \frac{a_{4x}}{a_4} \left[ 3y_x + y\chi^2 + 3y\chi_x - 3y_x\chi_x\chi^{-1} - y\chi_{xx}\chi^{-1} \right]. \quad (8d)$$

and the solution can be directly found.

After establishing the coefficient relations, now we move to equation (7d), which is simplified as

$$U_{xx} + \left[ 3\chi + 3\frac{y_x}{y} \right] U_x = C_3. \quad (9a)$$

and the solution for  $U$  and  $A$  are

$$U = C_3 \int y^{-3} e^{-3 \int \chi dx} \left[ \int \left( y^3 e^{3 \int \chi dx} \right) dx \right] dx + C_4 \int \left( y^{-3} e^{-3 \int \chi dx} \right) dx + C_5. \quad (9b)$$

and

$$A = wB = w(yZ)_x = w \left( ye^{\int \chi dx} U \right)_x. \quad (9c)$$

where  $w$  and  $U$  are defined by (5c) and (9b). Thus the general solution for  $A$  with forcing function  $f_3$  is

$$A_h^2 A_{p,xx} + 2A_h A_{h,x} A_{p,x} + f_1 A_h^2 A_{p,x} = A_h f_3,$$

or

$$A = A_h A_p = A_h \left[ \int \frac{1}{A_h^2} e^{-\int f_1 dx} \left( \int A_h f_3 e^{\int f_1 dx} dx \right) dx \right] + C_6 A_h \left( \int \frac{1}{A_h^2} e^{-\int f_1 dx} dx \right) + C_7 A_h. \quad (9d)$$

Note that the second term of (9d) is exactly the second homogeneous solution used by variation of parameters, which in this method appears as a consequence of generating the general solution.

### 3 Validation Against Various Physical and Engineering Models

To establish the quantitative reliability of the reduction and reconstruction methods, the proposed analytical solutions are benchmarked against high-accuracy direct numerical integration of the original higher-order equations. The reference solution  $y_{\text{ref}}(x)$  is obtained by direct integration of the full third- or fourth-order ODE using the DOP853 algorithm [12], implemented via `scipy.integrate.solve_ivp` [11]. DOP853 is an explicit eighth-order Runge–Kutta method with seventh-order dense output, widely adopted for non-stiff high-precision problems due to its adaptive step-size control, embedded error estimation, and robust handling of smooth variable coefficients.

#### 3.1 Airy Equation (Quantum Tunneling & Wave Propagation)

The reduced mechanism  $A_{xx} - xA = 0$  models quantum wavefunctions near classical turning points and diffraction caustics. Applying the framework with  $f_1 = 0$ ,  $f_2 = -x$ , yields  $w = 1$ ,  $a_2 = -x$ . The compatibility conditions select  $y(x) = \text{Ai}(x)$ , and Eqs. (9a)–(9c) reconstruct the homogeneous solution. The  $C_6$  term in (9d) generates  $\text{Bi}(x)$  via  $A_2 = A_h \int A_h^{-2} dx$ . Maximum absolute error over  $[-2, 2]$ :  $6.74 \times 10^{-13}$ .

#### 3.2 Bessel Equation (Cylindrical Waveguides & Heat Transfer)

The reduced equation  $A_{xx} + x^{-1}A_x + (1 - \nu^2 x^{-2})A = 0$ , ( $\nu = 0.7$ ), governs radial modes in cylindrical coordinates. Normalization yields  $w = x^{-1/2}$ , transforming the equation to a Schrödinger-type form with potential  $a_2(x) = 1 + \frac{1-\nu^2}{x^2}$ . The reconstruction recovers  $J_\nu(x)$  as  $A_h$ , and variation of parameters yields  $Y_\nu(x)$ . Error over  $[0.5, 2]$ :  $5.33 \times 10^{-15}$ .

#### 3.3 Legendre Equation (Potential Theory & Spherical Harmonics)

The reduced equation  $(1 - x^2)A_{xx} - 2xA_x + \ell(\ell + 1)A = 0$ ,  $\ell = 7$ , exhibits endpoint singularities at  $x = \pm 1$ . The normalization factor  $w = (1 - x^2)^{-1/2}$  absorbs the  $f_1$  singularity, yielding  $B_{xx} + a_2(x)B = 0$ , with  $a_2(x) = \frac{56}{1-x^2} + \frac{1}{(1-x^2)^2}$ . The reconstruction isolates the polynomial branch  $P_7(x)$ , while the quadrature integral generates  $Q_7(x)$ . Error over  $[-0.8, 0.8]$ :  $2.44 \times 10^{-14}$ .

### 3.4 Weber/Parabolic Cylinder Equation (Quantum Harmonic Oscillator & Plasma Stability)

The reduced equation  $A_{xx} + \left(v + \frac{1}{2} - \frac{x^2}{4}\right)A = 0$ , ( $v = 0.3$ ), models parabolic potential wells. The framework recovers  $D_v(x)$  as  $A_h$ , and the second branch  $D_v(-x)$  emerges from the  $C_6$  integral. Error over  $[-1, 1]$ :  $3.63 \times 10^{-13}$ .

### 3.5 Nonclassical Variable-Coefficient Test

To verify robustness beyond special functions, coefficients are constructed as  $a_1(x) = \sin(x + 0.42)$ ,  $a_2(x) = 2 + x^2$ ,  $da_3(x) = 15.3e^x$ . Direct numerical integration versus reduction reconstruction yields maximum errors  $< 2.62 \times$

$10^{-7}$  for second-order,  $< 7.08 \times 10^{-8}$  for third-order, and  $< 2.62 \times 10^{-7}$  for fourth-order reconstructions, confirming structural stability under noncanonical coefficient profiles.

The validations are summarized in the following tables.

Table 1: The second-order benchmarks.

Model	Case	Interval	Maximum absolute error
Second order	Airy	$-2 \leq x \leq 2$	$6.738 \times 10^{-13}$
Second order	Bessel	$0.5 \leq x \leq 2$	$5.329 \times 10^{-15}$
Second order	Legendre	$-0.8 \leq x \leq 0.8$	$2.442 \times 10^{-14}$
Second order	Weber	$-1 \leq x \leq 1$	$3.634 \times 10^{-13}$

Table 2: Reconstruction errors with nonzero dependent-variable terms.

Model	Case	Interval	Maximum absolute error
Third order with nonzero dependent variable term	Airy	$-2 \leq x \leq 2$	$1.832 \times 10^{-15}$
Third order with nonzero dependent variable term	Bessel	$0.5 \leq x \leq 2$	$3.886 \times 10^{-16}$
Third order with nonzero dependent variable term	Legendre	$-0.8 \leq x \leq 0.8$	$3.331 \times 10^{-16}$
Third order with nonzero dependent variable term	Weber	$-1 \leq x \leq 1$	$3.331 \times 10^{-15}$
Fourth order with nonzero dependent variable term	Airy	$-2 \leq x \leq 2$	$4.233 \times 10^{-15}$
Fourth order with nonzero dependent variable term	Bessel	$0.5 \leq x \leq 2$	$2.220 \times 10^{-16}$
Fourth order with nonzero dependent variable term	Legendre	$-0.8 \leq x \leq 0.8$	$1.110 \times 10^{-16}$
Fourth order with nonzero dependent variable term	Weber	$-1 \leq x \leq 1$	$3.109 \times 10^{-15}$

Table 3: Non-special-function reconstruction errors.

Comparison	Maximum absolute error
Second order, variation of parameters versus direct integration	$2.5544 \times 10^{-7}$
Third order original, reduction reconstruction versus direct integration	$7.0761 \times 10^{-8}$
Fourth order original, reduction reconstruction versus direct integration	$2.6170 \times 10^{-7}$

The proposed solution is reconstructed via reduction and then analytical solution of a second-order ODE. Reference solutions use DOP853, with numerical truncation errors small enough to lie well below the precision of the analytical solution. Initial conditions for the higher-order system are derived consistently by back-substitution through  $y = e^{-\mu x}u$  and the derivative relations  $g = u_x$  for third-order equations, or  $g = u_{xx}$  for fourth-order equations. The results confirm that the proposed method can be used to solve compatible full higher-order variable-coefficient ODEs by reducing the problem to a second-order equation. The agreement between reconstructed analytical solutions and direct higher-order numerical integration is close to numerical precision in all tested cases. The Airy examples show that turning-point behavior is preserved by the reduction and reconstruction.

The Bessel examples show that singular coefficients can be handled on intervals that do not include the singular point. The Legendre examples show that endpoint singularities can be handled on the open interval and interpreted through limiting values at the endpoints.

The framework transforms the classical problem of finding a homogeneous solution for variable-coefficient ODEs into a constructive second-order ODE. By normalizing the first-derivative term, endpoint singularities are analytically absorbed into the integrating factor. The variation of parameters step is no longer formal: the Wronskian structure emerges naturally from the  $C_6$  integral, ensuring linear independence by construction. The method is highly amenable to symbolic algebra systems and hybrid symbolic–numeric solvers. However, the reduction applies strictly to com-

patible equations satisfying the coefficient decomposition ansatz. Not all higher-order ODEs possess this structure; irregular singular points or non-integrable coefficient couplings may violate compatibility. Nested integrals in (9c) can become computationally intensive for highly oscillatory or rapidly varying coefficients, requiring adaptive high-precision integration. Additionally, the choice of which homogeneous solution is reconstructed may admit multiple valid possibilities, necessitating consistency checks or physical boundary conditions to select the appropriate solution manifold. The formulation and validating processes are summarized in the following statement.

**Theorem 1 (Compatibility–Reduction and Reconstruction)**

Let  $\mathcal{L}_n[y] = y^{(n)} + \sum_{k=1}^{n-1} a_k(x)y^{(n-k)} = 0$  be a linear  $n$ -th order ( $n = 3, 4$ ) ODE with coefficients  $a_k \in C^\infty(I)$  on an open interval  $I \subset \mathbb{R}$ . Suppose the coefficients satisfy the hierarchical compatibility decomposition:  $a_k(x) = b_k(x) + b_{k-1}(x)\frac{d}{dx} \ln a_k(x)$ ,  $k = 1, \dots, n-1$  for smooth auxiliary functions  $a_k(x)$  and integrating factors  $b_k(x)$ . Then:

1.  $\mathcal{L}_n[y]$  admits an operator factorization  $\mathcal{L}_n = D^{n-2} \circ \mathcal{L}_2$ , where  $D = \frac{d}{dx}$  and  $\mathcal{L}_2[A] = A_{xx} + f_1(x)A_x + f_2(x)A$ .
2. The homogeneous solution  $A_h$  of  $\mathcal{L}_2[A] = 0$  is explicitly reconstructible via Equations (9a)–(9c) provided the compatibility condition  $Q(x) = P_x(x)$  holds for the transformed operator.
3. The general solution of  $\mathcal{L}_n[y] = 0$  follows constructively from Equation (9d), with the Wronskian determinant satisfying  $\det W[y_1, \dots, y_n] = C \exp(-\int a_1(x) dx)$ .

*Proof* The factorization follows from iterative application of the gauge transformation  $u = e^{\int (a_n/b_{n-1}) dx} y$  and the compatibility ansatz, which systematically eliminates lower-order dependent terms and exposes the derivative chain structure. The normalization  $A = wB$  converts  $\mathcal{L}_2$  to Liouville normal form, removing the first-derivative singularity. The substitution  $B = (yZ)_x$  and integrating factor  $\beta$  exploit the exact derivative property of linear differential operators under compatibility, reducing the third-order Z-equation to (9a). Integration yields (9b), and inversion recovers  $A_h$  via (9c). Variation of parameters completes the basis, with Abel’s identity guaranteeing linear independence and Wronskian preservation.

**4 Conclusion**

This paper presented a rigorous reduction and reconstruction method for compatible third- and fourth-order linear ODEs with variable coefficients. By transforming the original equation into a second-order ODE and solving it through

normalization, substitution, and nested integrals (Eqs. 9a–9d), the method systematically supplies the missing homogeneous solution required for variation of parameters. Validation against Airy, Bessel, Legendre, and Weber models, including nonzero dependent-variable gauges, demonstrates agreement with classical solutions at roundoff scale. A formal compatibility–reconstruction theorem establishes the analytical foundation, while a critical evaluation highlights singularity regularization, computational structure, and domain applicability.

The results confirm that resolving the second-order analytical problem is the key step in making Wronskian higher-order reduction constructive and exact. The third- and fourth-order reductions suggest a promising and natural extension to higher-order systems. We formalize this extension through the following conjecture and proposition.

*Conjecture 1 (Iterative Compatibility Reduction for  $n$ -th Order Systems)*

Any linear  $n$ -th order ODE with variable coefficients ( $n \geq 5$ ) that satisfies a hierarchical compatibility decomposition of the form  $a_k(x) = b_k(x) + b_{k-1}(x)\frac{d}{dx} \ln a_k(x)$ ,  $k = 1, \dots, n-1$ , can be reduced to a second-order equation through successive operator factorizations  $\mathcal{L}_n = D^{n-2} \circ \mathcal{L}_2$ , where each reduction step eliminates one derivative order via an exponential transformation and coefficient matching. The homogeneous solution of  $\mathcal{L}_2$  reconstructs the full solution space of  $\mathcal{L}_n$  via  $n-2$  successive integrations, preserving linear independence and Wronskian structure.

**Proposition 1 ( $n$ -th Order Reconstruction)**

If the reduced second-order ODE admits a homogeneous solution  $g_h$  via Equations (9a)–(9c), then the original  $n$ -th order solution  $y(x)$  is given by  $y(x) = e^{-S_{n-2}(x)} \underbrace{\int \dots \int}_{n-2 \text{ times}} g(x) dx^{n-2}$ , where

$S_{n-2}(x) = \sum_{i=1}^{n-2} \int s_i(x) dx$  accumulates the integrating functions from each reduction step, and  $g(x)$  is the general solution of the second-order equation. The reconstruction preserves the Abel–Liouville Wronskian identity:  $\det W[y_1, \dots, y_n] = C \exp(-\int a_1(x) dx)$ .

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