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Geometric Leakage versus Dissipation: Open Boundaries in Chern–Simons Fluid Dynamics

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Abstract When a topological quantity satisfies a local conservation law, its apparent non-conservation inside a finite region need not signal dissipation: it may instead reflect a geometric leakage of information through the boundary. We make this precise in the Chern–Simons formulation of compressible fluid dynamics recently proposed by Bustamante, Andrianopoli, Trigiante and Zanelli, where the proto-helicity $H(t)$ – a helicity-type invariant of topological origin – satisfies a local divergence-form law $\partial_{\hat{\mu}} h^{\hat{\mu}} = 0$. On an open domain $\Omega(t)$, this local law gives rise not to conservation but to the balance $dH/dt = F_{\partial\Omega(t)}$, where $F_{\partial\Omega(t)}$ is the flux of a geometric current through the boundary. The bulk equations are unchanged; what changes is entirely the boundary. A graviton–fluid correspondence emerges from the comparison with the AdS_3 Chern–Simons setting: the radiative term in the boundary flux plays the role of boundary gravitons carrying topological information toward the conformal boundary, and the spectral character of $F_{\partial\Omega(t)}$ determines three distinct decay regimes for $H(t)$ – stable ($O(1)$, bound state), transiently localized ($O(t^{-1/3})$, caviton-like), and dispersive ($O(t^{-1/2})$, pure radiation) – providing a spectroscopic diagnostic of the interior geometry from boundary measurements alone.

1 Introduction

Can an observer confined to a finite region distinguish geometric leakage from genuine dissipation by measuring a single integrated quantity? This question, deceptively simple, is at the heart of the present work. If a topological invariant decreases inside an observation domain, the naive interpretation is that some irreversible bulk process is at work. The correct interpretation may be entirely different: the quantity may be leaking out through the boundary, carried by a geometric current whose divergence vanishes identically in the bulk. The bulk theory remains conservative; only the boundary condition changes.

This distinction becomes concrete and calculable in the Chern–Simons formulation of compressible fluid dynamics introduced by Bustamante, Andrianopoli, Trigiante and Zanelli [1]. There, the fluid equations arise from the dimensional reduction of a five-dimensional Abelian Chern–Simons theory, and a helicity-type quantity – the *proto-helicity* $H(t) = \int_{\Omega(t)} A_a B_a \, d^3x$ – emerges as the temporal component of a topological current $h^{\hat{\mu}}$ satisfying $\partial_{\hat{\mu}} h^{\hat{\mu}} = 0$ identically. On a periodic or closed domain this gives exact conservation, $dH/dt = 0$. On an open domain it gives instead the balance law

$$\frac{dH}{dt} = F_{\partial\Omega(t)}, \tag{1}$$

where $F_{\partial\Omega(t)}$ is the flux of the associated geometric current $J_{\mathrm{eff}} = (A_b B^b)V + A_0 B + A \times E$ through $\partial\Omega(t)$. The three terms of J_{eff} correspond to three physically distinct transport mechanisms – advection of helicity-carrying fluid parcels, potential-driven transport, and radiative propagation of the gauge field – each of which can drain $H(t)$ from $\Omega(t)$ without any dissipative term appearing in the bulk equations.

The radiative term $A \times E$ in J_{eff} points toward a deeper structure shared with the AdS_3 Chern–Simons setting of [2]. There, a dimensional reduction on a non-compact spatial slice produces boundary dynamics governed by the KdV hierarchy with eigenfunction forcing; the radiative sector exhibits universal $t^{-1/2}$ dispersive decay, and boundary gravitons carry stress-tensor information toward the conformal boundary through a flux of the linearised CS connection. Both systems realise the same abstract open-boundary bal-

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ance

$$\frac{d\mathscr{Q}}{dt} = \mathscr{F}[\varphi|_\partial], \tag{2}$$

where $\mathscr{Q}$ is a topological charge and $\mathscr{F}$ depends only on the boundary values of the fundamental field φ. The structural dictionary – proto-helicity $\leftrightarrow$ Virasoro charge $\mathscr{L}_0$, radiative term $A \times E \leftrightarrow$ boundary graviton flux, vortex soliton $\leftrightarrow$ KdV soliton – is made explicit in Section 3. This structural correspondence, rooted in the shared Chern–Simons origin of both currents rather than in a dynamical identification of the bulk equations, suggests that the spectral and inverse-scattering methods of [2] may provide a quantitative handle on $F_{\partial\Omega(t)}$ itself: a discrete bound state maintains $H(t) \sim O(1)$; a coherent quasi-resonance near a critical wavenumber produces $H(t) \sim O(t^{-1/3})$; and incoherent radiation gives the generic dispersive decay $H(t) \sim O(t^{-1/2})$. These three regimes are unified by the single balance law (11) and differentiated entirely by the spectral character of $F_{\partial\Omega(t)}$.

The paper is organized as follows. Section 2 derives the open-boundary balance law (11) from the local Chern–Simons identity, discusses its gauge dependence and the gauge-invariant relative proto-helicity, and establishes the two limiting regimes of closed-flux conservation and open-domain leakage. Section 3 develops the physical interpretation: the Poynting analogy, the three transport mechanisms in J_{eff}, the graviton–fluid correspondence, and the effective irreversibility seen by an interior observer. Section 4 illustrates the independence of bulk and boundary mechanisms using the inviscid Burgers equation. Section 5 concludes with the spectral classification of decay regimes and perspectives toward a quantitative open-boundary geometric hydrodynamics.

2 Chern–Simons formulation and the open-boundary proto-helicity balance law

We review the geometric balance law arising in the Chern–Simons formulation of compressible fluid dynamics developed in [1], and derive its form on an open, moving domain. The starting point is the five-dimensional Abelian Chern–Simons action

$$I[A] = \int_{M_5} \frac{1}{3!} A \wedge F \wedge F, \tag{3}$$

where $F = dA$ is the curvature two-form associated with the gauge connection A. The Euler–Lagrange equations are

$$F \wedge F = 0. \tag{4}$$

Following the dimensional reduction of [1], we decompose the connection as $A = A_{\hat\mu} dx^{\hat\mu} + A_4 dx^4$ with $\hat\mu = 0,1,2,3$ and assume all fields are independent of x^4. Introducing the electric and magnetic fields

$$E_a = -F_{0a}, \qquad B_a = \frac{1}{2}\varepsilon_{abc} F_{bc}, \tag{5}$$

the reduced system gives rise to the topological current

$$h^{\hat\mu} = \frac{1}{2}\varepsilon^{\hat\mu\hat\nu\hat\rho\hat\sigma} A_{\hat\nu} F_{\hat\rho\hat\sigma}, \tag{6}$$

whose temporal and spatial components are $h^0 = A_a B_a$ and $h^a = -A_0 B^a - (A \times E)^a$. Using (4) together with the Bianchi identity $dF = 0$, one finds $\partial_{\hat\mu} h^{\hat\mu} = 0$, which in $(3+1)$ form reads

$$\partial_0 (A_a B_a) + \nabla \cdot (-A_0 B - A \times E) = 0. \tag{7}$$

This is an exact local identity: it holds pointwise, regardless of the geometry of the domain on which the fluid evolves. What changes with the domain is its integrated consequence.

Let $\Omega(t) \subset \mathbb{R}^3$ be a smooth domain transported by a velocity field V – the boundary $\partial\Omega(t)$ may be at rest, comoving with the fluid, or moving independently – and define the proto-helicity functional

$$H(t) = \int_{\Omega(t)} A_a B_a \, d^3x. \tag{8}$$

Remark 1 On an open domain, $H(t)$ is in general gauge dependent: under $A \mapsto A + d\lambda$, $H(t)$ changes by a surface term $\oint_{\partial\Omega(t)} \lambda B \cdot n\, dS$. Throughout this work we either restrict to gauge transformations for which λ vanishes sufficiently fast on $\partial\Omega(t)$, or treat $H(t)$ as a quantity defined relative to a fixed boundary gauge condition. A gauge-invariant alternative, familiar from open-domain helicity in MHD [3, 4], is the *relative* proto-helicity

$$H_{\text{rel}}(t) = \int_{\Omega(t)} \left(A_a B_a - A_a^{\text{ref}} B_a^{\text{ref}} \right) d^3x, \tag{9}$$

where A^{ref} is any fixed reference connection sharing the same boundary values $B^{\text{ref}} \cdot n = B \cdot n$ on $\partial\Omega(t)$. The gauge ambiguities cancel in $H_{\text{rel}}(t)$, which satisfies the same balance law (11) with J_{eff} replaced by $J_{\text{eff}} - J_{\text{eff}}^{\text{ref}}$. All statements below apply equally to $H_{\text{rel}}(t)$; we use $H(t)$ for notational simplicity.

Applying the Reynolds transport theorem to (8) and using (7), the bulk terms combine into a pure divergence,

$$\frac{dH}{dt} = \int_{\Omega(t)} \nabla \cdot \left[(A_b B^b) V + A_0 B + A \times E \right] d^3x, \tag{10}$$

so that the divergence theorem gives the central result of this work:

$$\begin{aligned} \frac{dH}{dt} &= F_{\partial\Omega(t)} \equiv \oint_{\partial\Omega(t)} J_{\text{eff}} \cdot n\, dS, \\ J_{\text{eff}} &= (A_b B^b) V + A_0 B + A \times E. \end{aligned} \tag{11}$$

The evolution of the proto-helicity inside $\Omega(t)$ is entirely controlled by the flux of the geometric current J_{eff} through $\partial\Omega(t)$. No term has been added to or removed from the bulk identity (7): the boundary term in (11) is exactly the divergence term in (7), integrated over $\Omega(t)$ and converted to a surface integral via the divergence theorem.

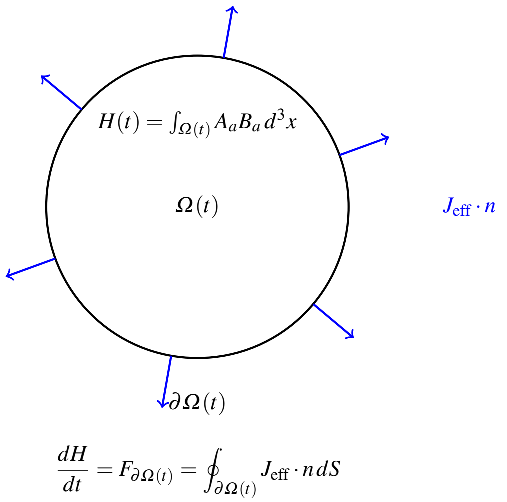

Fig. 1: An open region $\Omega(t)$ carrying proto-helicity $H(t)$. The outward flux of $J_{\text{eff}} = (A_b B^b)V + A_0 B + A \times E$ through $\partial\Omega(t)$ entirely controls dH/dt: a vanishing total flux gives exact conservation (12), while a one-signed outward flux gives the monotonic decay (15) – geometric leakage rather than bulk dissipation.

Two regimes follow immediately from (11). If

$$F_{\partial\Omega(t)} = 0 \tag{12}$$

– as on a periodic domain, on $\mathbb{R}^3$ for fields decaying fast enough, or whenever incoming and outgoing fluxes cancel exactly – then $dH/dt = 0$ and H is exactly conserved. Such configurations define geometrically closed transport sectors: the current J_{eff} is confined to $\Omega(t)$ for all time. If instead $F_{\partial\Omega(t)} \neq 0$, integrating (11) between t_1 and t_2 gives

$$H(t_2) - H(t_1) = \int_{t_1}^{t_2} F_{\partial\Omega(t)} \, dt, \tag{13}$$

so that every unit of H lost or gained inside $\Omega(t)$ is accounted for by the time-integrated flux through the moving boundary. In particular, if

$$F_{\partial\Omega(t)} \leq 0 \quad \text{for all } t, \tag{14}$$

then $dH/dt \leq 0$ and

$$H(t_2) \leq H(t_1) \qquad \text{for all } t_2 \geq t_1, \tag{15}$$

so the proto-helicity decreases monotonically inside $\Omega(t)$ even though (4) and (7) contain no dissipative term whatsoever. Periodic and open systems are therefore two regimes of the same local geometric framework (7): the distinction (12) versus (14) is a statement about the boundary and the asymptotic behaviour of J_{eff}, not about the bulk equations.

3 Physical interpretation

Equation (11) says that a decrease of $H(t)$ inside $\Omega(t)$ is transport, not destruction: the proto-helicity that disappears from $\Omega(t)$ reappears, instant by instant, as an outward flux of J_{eff} across $\partial\Omega(t)$. This is qualitatively different from genuine dissipation, where a quantity is converted into another form by an irreversible bulk process and cannot be recovered by enlarging the observation domain. The closest familiar analogue is the Poynting theorem,

$$\partial_t \left(\frac{E^2 + B^2}{2} \right) + \nabla \cdot (E \times B) = -J \cdot E, \tag{16}$$

whose integrated form shows that even with $J = 0$, the field energy inside Ω can decrease purely through radiation leaving the box. Equation (11) is the exact analogue with $H \leftrightarrow \int (E^2 + B^2)/2$ and $J_{\text{eff}} \leftrightarrow E \times B$: the proto-helicity plays the role of a radiated geometric quantity, and J_{eff} is its Poynting vector. Unlike energy or mass conservation, the proto-helicity originates directly from the Chern–Simons geometric structure of the theory; its flux therefore measures the transport of topological information encoded in the gauge connection and its associated vorticity field.

The three terms of J_{eff} correspond to three physically distinct mechanisms. The advective term $(A_b B^b)V$ carries helicity out of $\Omega(t)$ simply because helicity-carrying fluid parcels cross $\partial\Omega(t)$: this is what happens when a twisted vortex tube is advected through the wall of an observation volume, carrying its linkage with it. The potential-driven term $A_0 B$ is weighted by the temporal component A_0 of the connection, playing the role of a pressure-like contribution to the transport of the vorticity field B. The radiative term $A \times E$ is structurally identical to the Poynting vector $E \times B$ [5]: it can carry H across $\partial\Omega(t)$ even for a boundary at rest and in the absence of fluid crossing it, through the propagation of the gauge field itself. A concrete picture: take $\Omega(t)$ to be a fixed laboratory volume containing a knotted vortex tube. As the tube is stretched and part of it is advected out of $\Omega(t)$, or as the underlying gauge-field configuration radiates linkage information through the walls, an observer measuring only $H(t)$ inside $\Omega(t)$ sees it decay and might attribute this to an effective viscosity or hyper-resistivity. Equation (11) shows that no such mechanism needs to be invoked: the decay is fully accounted for by $F_{\partial\Omega(t)}$.

A structural correspondence with the AdS_3 Chern–Simons setting of [2] clarifies the role of the radiative

Table 1: Dictionary between the fluid Chern–Simons formulation and the AdS_3 Chern–Simons description.

Fluid CS (5D → 4D)	**AdS_3 CS (3D → 2D)**
Gauge connection $A_{\hat{\mu}}$	Dreibein and spin connection
Proto-helicity $H(t) = \int A_a B_a \, d^3x$	Virasoro charge $\mathscr{L}_0$
Boundary flux $F_{\partial\Omega(t)}$	Conformal-boundary stress flux $T^{01}\|_\partial$
Radiative term $A \times E$	Boundary graviton δe^a, $\delta\omega^{ab}$
$J_{\text{eff}} \cdot n$ on $\partial\Omega(t)$	$T^{0i} n_i$ on $\partial(\text{AdS}_3)$
Localised vortex (B-field soliton)	KdV soliton (reflectionless potential)
Caviton (quasi-resonance in $F_{\partial\Omega}$)	Forced KdV quasi-resonance
Dispersive decay $O(t^{-1/2})$	Dispersive KdV decay $O(t^{-1/2})$

The radiative row deserves comment. In the AdS_3 setting, boundary gravitons are linearised perturbations $(\delta e^a, \delta\omega^{ab})$ of the CS connection whose boundary flux takes the schematic form $\langle \delta e \wedge \delta\omega \rangle|_\partial$, carrying stress-tensor information to the conformal boundary without any bulk matter crossing it. In the fluid setting, $A \times E = A \times (-\partial_t A - \nabla A_0)$ is the cross-product of the spatial connection with its own time derivative – structurally the same object: the flux of a linearised connection perturbation to a boundary, generated by the field itself. The identification $A \times E \leftrightarrow \langle \delta e \wedge \delta\omega \rangle|_\partial$ is therefore a consequence of the shared Chern–Simons origin of both currents, not a loose analogy. The correspondence is structural rather than dynamical: the abstract form (2) and the radiative identification are common to both systems, but the bulk evolution equations differ – KdV hierarchy with eigenfunction forcing in the gravitational case, compressible fluid equations here. The solitonic sector of [2] corresponds, in the fluid picture, to localised vortex structures that preserve their helicity content and propagate without dispersion; the radiative sector corresponds to the dispersive leakage encoded in $F_{\partial\Omega(t)}$. In both cases, what appears as non-conservation from the interior is, globally, a coherent transport of geometric information – solitonic or radiative – toward a boundary.

There is, however, a sense in which the open-domain picture genuinely produces effective irreversibility, even though the underlying dynamics (4) is time-reversal symmetric and non-dissipative. If $\Omega(t)$ is fixed and the flux $F_{\partial\Omega(t)}$ is predominantly outgoing – because the exterior acts as an effectively unbounded reservoir into which J_{eff} escapes and does not return on observable timescales – then (15) holds and $H(t)$ decreases monotonically as far as the interior observer is concerned. This is exactly the situation familiar from open quantum and open classical systems: a globally conservative, reversible theory restricted to a subsystem coupled to a large environment produces an effective arrow of time and apparent dissipation, without any irreversible term being present in the fundamental equations. The Chern–Simons fluid provides a geometric, helicity-based instance of this general phenomenon – the bulk theory is conservative; the open-region theory looks dissipative – and the difference is entirely the boundary term $F_{\partial\Omega(t)}$. An observer monitoring only $H(t)$ inside $\Omega(t)$ cannot distinguish geometric leakage from genuine dissipation; the distinction becomes observable only through the boundary flux $F_{\partial\Omega(t)}$, which provides a direct diagnostic of transport versus destruction. This reframes the role of the boundary: it is not a passive technical device but a dynamically active ingredient that decides whether a given geometric quantity is conserved, balanced, or effectively dissipated for an observer confined to $\Omega(t)$.

4 Exact helical vortex solution and explicit flux computation

We exhibit a family of exact static solutions of the reduced Chern–Simons system on which the balance law (11) can be evaluated in closed form, and from which a sharp functional bound on $|F_{\partial\Omega(t)}|$ is derived.

Working in cylindrical coordinates (r, φ, z), consider a gauge connection with no φ-dependence and no r-component,

$$A = A_\varphi(r)\,\hat{\varphi} + A_z(r,z)\,\hat{z}, \qquad A_0 = 0, \tag{17}$$

with all fields time-independent ($\partial_t A = 0$, so $E = 0$). The magnetic field $B = \nabla \times A$ then has components

$$B_r = 0, \qquad B_\varphi = -\partial_r A_z, \qquad B_z = \frac{1}{r}\partial_r\big(rA_\varphi\big). \tag{18}$$

Since $E = 0$ and $\partial_t B = 0$, the Bianchi identity $\partial_t B + \nabla \times E = 0$ is satisfied trivially, and $\nabla \cdot B = 0$ follows from (18) (as $B_r = 0$ and B_z is z-independent). The condition $B \cdot E = 0$

required by $F \wedge F = 0$ holds since $E = 0$. The ansatz (17) therefore defines an exact solution of the reduced Chern–Simons system for any choice of $A_\varphi(r)$ and $A_z(r,z)$.

We specify the potentials by a *regularised helical vortex*:

$$A_\varphi(r) = \frac{\Gamma}{4\pi}\frac{r}{r^2+a^2}, \qquad A_z(r,z) = \alpha\, e^{-z/\lambda}, \tag{19}$$

where $\Gamma > 0$ is the total circulation, $a > 0$ a vortex-core regularisation length, $\alpha \in \mathbb{R}$ the axial amplitude, and $\lambda > 0$ an axial decay length. The resulting field strengths are

$$B_\varphi = 0, \qquad B_z(r) = \frac{\Gamma}{2\pi}\frac{a^2}{(r^2+a^2)^2}, \tag{20}$$

a Lorentzian profile centred on the axis satisfying $\int_0^\infty B_z \cdot 2\pi r\,dr = \Gamma$. The proto-helicity density (8) is

$$A_a B_a = A_z B_z = \frac{\alpha\Gamma a^2}{2\pi}\frac{e^{-z/\lambda}}{(r^2+a^2)^2}, \tag{21}$$

which is separable in r and z and decays axially on the scale λ.

Let $\Omega = \{(r,\varphi,z) : 0 \le r \le R,\ 0 \le z \le L\}$ be a fixed cylindrical domain, with a uniform axial flow $V = V_z\hat{z}$, $V_z > 0$ (outflow at $z = L$). With $E = 0$ and $A_0 = 0$, the effective current reduces to $J_{\text{eff}} = (A_b B^b)V$, and only the two axial caps contribute to the surface integral:

$$F_{\partial\Omega} = \int_0^{2\pi}\int_0^R \Big[(A_z B_z)\big|_{z=L} - (A_z B_z)\big|_{z=0}\Big] V_z\, r\,dr\,d\varphi. \tag{22}$$

Substituting (21) and using the radial integral $\int_0^R r\,dr/(r^2+a^2)^2 = R^2/[2a^2(R^2+a^2)]$, one obtains the exact closed-form result

$$F_{\partial\Omega} = \frac{\alpha\Gamma V_z R^2}{2(R^2+a^2)}\left(e^{-L/\lambda} - 1\right). \tag{23}$$

Since $e^{-L/\lambda} - 1 < 0$ for all $L, \lambda > 0$, one has $F_{\partial\Omega} < 0$ whenever $\alpha\Gamma > 0$: the proto-helicity leaks out of Ω through the downstream cap at $z = L$. The proto-helicity inventory is likewise computable,

$$H = \int_\Omega A_z B_z\, d^3x = \frac{\alpha\Gamma\lambda R^2}{2(R^2+a^2)}\left(1 - e^{-L/\lambda}\right), \tag{24}$$

and satisfies $dH/dt = F_{\partial\Omega} < 0$ in accordance with (11): the domain loses proto-helicity at a rate determined entirely by the two boundary values $A_z|_{z=0,L}$ and the velocity V_z, while the full interior profile (21) plays no role in the sign or magnitude of $F_{\partial\Omega}$.

Two limiting regimes of (23) are instructive. When $L \gg \lambda$ the exponential vanishes and the flux saturates at $-\alpha\Gamma V_z R^2/[2(R^2+a^2)]$: the upstream cap dominates and further lengthening Ω does not increase the leakage rate, while H approaches $\alpha\Gamma\lambda R^2/[2(R^2+a^2)]$, showing that only the first axial decay length contributes to the inventory. When $L \ll \lambda$ one has $e^{-L/\lambda} \approx 1 - L/\lambda$, so $F_{\partial\Omega} \approx -\alpha\Gamma V_z R^2 L/[2\lambda(R^2+a^2)]$: the flux is linear in L, the entire column participates, and the leakage rate grows with the length of the domain.

The exact result (23) is specific to the ansatz (19). We now derive a general upper bound valid for any solution with $E, B, V \in L^\infty(\partial\Omega)$ and $A \in H^1(\Omega)$. Setting $M_B = \|B\|_{L^\infty(\partial\Omega)}$, $M_E = \|E\|_{L^\infty(\partial\Omega)}$, $M_V = \|V\|_{L^\infty(\partial\Omega)}$, the pointwise estimate $|J_{\text{eff}}| \le |A|(|B||V| + |E|) + |A_0||B|$, integration over $\partial\Omega$, and the Cauchy–Schwarz inequality give

$$\begin{aligned}|F_{\partial\Omega(t)}| \;\le\; &(M_B M_V + M_E)\,|\partial\Omega|^{1/2}\,\|A\|_{L^2(\partial\Omega)} \\ &+ M_B\,\|A_0\|_{L^1(\partial\Omega)}.\end{aligned} \tag{25}$$

Applying the Sobolev trace inequality $\|A\|_{L^2(\partial\Omega)} \le C_{\text{tr}}\|A\|_{H^1(\Omega)}$, where $C_{\text{tr}} > 0$ depends only on the geometry of Ω, yields

$$\begin{aligned}|F_{\partial\Omega(t)}| \;\le\; &C_{\text{tr}}\,(M_B M_V + M_E)\,|\partial\Omega|^{1/2}\,\|A\|_{H^1(\Omega)} \\ &+ M_B\,\|A_0\|_{L^1(\partial\Omega)}.\end{aligned} \tag{26}$$

Three consequences follow. If $B|_{\partial\Omega} = E|_{\partial\Omega} = 0$ (magnetically and electrically screened boundary), then $M_B = M_E = 0$ and $F_{\partial\Omega(t)} = 0$: proto-helicity is exactly conserved regardless of the interior dynamics, extending (12) to screened open domains. When $M_V M_B \gg M_E$ (advection dominates radiation at the boundary), the first term scales as $M_V M_B \|A\|_{H^1(\Omega)}$: large-scale, slowly varying configurations (small $\|A\|_{H^1}$ relative to $\|A\|_{L^2}$) leak less than rapidly varying ones of the same L^2-energy. Finally, for the helical ansatz (19) with $A_0 = E = 0$, one has $M_B = \Gamma/(2\pi a^2)$ (attained at $r = 0$), and (26) gives $|F_{\partial\Omega}| \lesssim C_{\text{tr}}\Gamma V_z \alpha R(2\pi RL)^{1/2}/(2\pi a^2)$, which exceeds the exact value (23) by a factor growing as $(RL/a^2)^{1/2}$: the bound is not saturated because the r^{-4} fall-off of B_z is not captured by the L^∞ estimate M_B.

Taken together, (23) and (26) establish that $F_{\partial\Omega(t)}$ is not a formal artefact of the balance law (11) but a concrete, computable quantity: evaluable exactly on tractable configurations, bounded in terms of standard Sobolev data on Ω, and controlled by boundary measurements alone. The balance law (11) thereby acquires a quantitative content beyond the qualitative leakage–dissipation distinction of Section 3.

5 Nonlinear transport: the Burgers equation

We illustrate how the open-boundary balance law of Section 2 interacts with genuinely nonlinear transport, using the inviscid Burgers equation [6]

$$\partial_t u + u\,\partial_x u = 0, \tag{27}$$

on a non-periodic domain $\Omega(t) \subset \mathbb{R}$ as the simplest tractable model. Along characteristics $dx/dt = u(x(t),t)$ one has

$du/dt = 0$, so u is constant along $x(t) = \xi + t\,u_0(\xi)$ with $u_0(\xi) = u(\xi,0)$. If $\min_\xi u_0'(\xi) < 0$, the characteristic map ceases to be invertible at the finite time $t_* = -1/\min_\xi u_0'(\xi)$, at which the gradient of u diverges – the standard nonlinear steepening mechanism.

To make contact with Section 2, consider $Q(t) = \int_{\Omega(t)} u(x,t)\,dx$ with $\Omega(t) = [a(t),b(t)]$. Using $\partial_t u = -\partial_x(u^2/2)$ and the Reynolds transport theorem,

$$\frac{dQ}{dt} = \underbrace{\tfrac{1}{2}u(a,t)^2 - \tfrac{1}{2}u(b,t)^2}_{\text{flux through } \partial\Omega} + \underbrace{u(b,t)b'(t) - u(a,t)a'(t)}_{\text{moving boundary advection}}. \tag{28}$$

This is precisely the structure of (11): $Q(t)$, the analogue of $H(t)$, evolves only through quantities evaluated at the two endpoints. If $\Omega(t)$ is comoving with the fluid, $a'(t) = u(a,t)$ and $b'(t) = u(b,t)$ (a material interval), the two contributions in (28) cancel exactly[1], giving $dQ/dt = 0$: the closed-flux regime (12). If instead $\Omega = [a,b]$ is a fixed laboratory interval and u is a decreasing front with $u(a,t) > u(b,t) > 0$ before shock formation, (28) reduces to

$$\frac{dQ}{dt} = \tfrac{1}{2}\left[u(a,t)^2 - u(b,t)^2\right] > 0, \tag{29}$$

so $Q(t)$ grows, with the sign fixed entirely by the boundary values of u – exactly the mechanism of (11), where the sign of $F_{\partial\Omega(t)}$ controls whether H grows or decays. The two mechanisms are independent: a shock can form anywhere strictly inside $(a(t),b(t))$ without affecting (28), which depends only on the boundary values of u, while $Q(t)$ can vary substantially through (29) even when u remains smooth throughout $\Omega(t)$. This mirrors the proto-helicity case: the bulk dynamics and the boundary transport (11) are independent ingredients that must both be specified to determine the evolution of any integrated quantity.

6 Spectral classification of the proto-helicity decay regimes

The three decay regimes $H(t) \sim O(1)$, $O(t^{-1/3})$, $O(t^{-1/2})$ announced in the introduction and conclusion are here derived directly from the balance law (11) and the spectral properties of the linearised Chern–Simons field on $\partial\Omega(t)$.

Spectral representation of $H(t)$

Fix Ω and decompose the gauge connection as $A = A^{(0)} + \delta A$, where $A^{(0)}$ is a time-independent background satisfying the static CS equations ($E^{(0)} = 0$, $\partial_t B^{(0)} = 0$) and $\delta A(x,t)$ is a small outgoing perturbation. To leading order in $|\delta A|$, the proto-helicity splits as

$$H(t) = \underbrace{\int_\Omega A_a^{(0)} B_a^{(0)}\, d^3x}_{H_0 \text{ (constant)}} + \underbrace{\int_\Omega \delta A_a(x,t)\, B_a^{(0)}(x)\, d^3x}_{H_{\text{rad}}(t)} + O(|\delta A|^2). \tag{30}$$

The static contribution H_0 satisfies $dH_0/dt = 0$ (no boundary flux when $E = 0$ and $V = 0$). The radiating part $H_{\text{rad}}(t)$ obeys $dH_{\text{rad}}/dt = F_{\partial\Omega}$ and controls all non-trivial time dependence.

The linearised equation for δA arising from $F \wedge F = 0$ is a hyperbolic system admitting outgoing solutions with a spectral decomposition

$$\delta A(x,t) = \int_{\mathbb{R}} a(\omega)\, \psi_\omega(x)\, e^{-i\omega t}\, d\omega + \text{c.c.}, \tag{31}$$

where ψ_ω are outgoing eigenmodes of the linearised CS operator, normalised so that $\psi_\omega \cdot n|_{\partial\Omega}$ is the outgoing amplitude at frequency ω, and $a(\omega)$ encodes the initial data. Substituting (31) into (30):

$$H_{\text{rad}}(t) = \int_{\mathbb{R}} a(\omega) M(\omega)\, e^{-i\omega t}\, d\omega + \text{c.c.}, \qquad M(\omega) = \int_\Omega \psi_\omega(x) \cdot B^{(0)}(x)\, d^3x. \tag{32}$$

where $M(\omega)$ is the *spectral overlap* between the outgoing mode and the background vorticity. The large-time behaviour of $H_{\text{rad}}(t)$ is therefore controlled by the singularity structure of $a(\omega)M(\omega)$ in the complex ω-plane – a question entirely internal to the Chern–Simons fluid system.

Three regimes

Regime 1: $H(t) \sim O(1)$ (stable topological structure). If the linearised CS operator admits a discrete eigenvalue $\omega_0 = \omega_R - i\gamma$ with $\gamma > 0$ (a quasi-normal mode), then $a(\omega)M(\omega)$ has a simple pole in the lower half-plane and the residue theorem gives

$$H_{\text{rad}}(t) \;\sim\; C\, e^{-\gamma t} \cos(\omega_R t + \varphi) \qquad (t \to \infty), \tag{33}$$

for constants C, φ determined by initial data. Since $e^{-\gamma t}$ is integrable on $[0,\infty)$, the total integral $\int_0^\infty F_{\partial\Omega}(s)\,ds$ converges and $H(t) \to H_0 + \text{const} = O(1)$. The physical realisation is a localised vortex structure – a *vortex soliton* – whose B-field profile is supported in Ω and whose helicity content does not disperse: $\gamma \to 0$ in the reflectionless (soliton) limit, and $H(t)$ remains bounded away from zero for all time.

[1] $u(b)b' - \frac{1}{2}u(b)^2 = \frac{1}{2}u(b)^2$ and similarly at a with opposite sign, so the net result is zero.

Regime 2: $H(t) \sim O(t^{-1/2})$ (generic dispersive radiation). Suppose the spectrum of the linearised CS operator is purely continuous, so that $a(\omega)M(\omega)$ is analytic away from a branch cut along $\omega \in [0,\infty)$. For generic initial data, the spectral weight $a(\omega)M(\omega)$ behaves as

$$a(\omega)M(\omega) \sim c_0\,\omega^{-1/2} \qquad (\omega \to 0^+) \tag{34}$$

near the branch point (corresponding to long-wavelength modes with $k \to 0$). The Laplace-transform Tauberian theorem then gives

$$H_{\rm rad}(t) \sim c_0\Gamma(1/2)t^{-1/2} = c_0\sqrt{\pi}t^{-1/2} \qquad (t \to \infty), \tag{35}$$

and hence $H(t) \sim O(t^{-1/2})$. In terms of the boundary flux: $F_{\partial\Omega}(t) = dH_{\rm rad}/dt \sim -c_0\sqrt{\pi}/2 \cdot t^{-3/2}$, which is integrable on $[1,\infty)$, confirming that the total helicity leaks to zero over infinite time but at a rate given precisely by the half-integer power (35). The underlying mechanism is standard dispersive spreading: in the mode decomposition (31), modes with different ω arrive at $\partial\Omega$ at different times and interfere destructively, with the interference integral producing the $t^{-1/2}$ envelope via the one-dimensional stationary-phase formula $\int_0^\infty f(u)e^{-iut}\,du \sim f(0)\,(\pi/t)^{1/2}$ for smooth f.

Regime 3: $H(t) \sim O(t^{-1/3})$ (Airy quasi-resonance). A qualitatively different regime arises when the dispersion relation $\omega = \omega(k)$ of the outgoing modes has an *inflection point* at a critical wavenumber k^*: $\omega''(k^*) = 0$, $\omega'''(k^*) \neq 0$. Near k^*, the phase of the integrand in (32) is

$$\phi(k,t) = k\cdot x - \omega(k)t \approx k^*\cdot x - \omega^* t - \frac{\omega'''(k^*)}{6}(k-k^*)^3 t, \tag{36}$$

and the integral over k near k^* evaluates by the Airy-function identity $\int_{-\infty}^{\infty} e^{i(su-u^3/3)}\,du = 2\pi\,\mathrm{Ai}(-s)$ with $s \sim (x-x_{\rm g}(t))/(\omega''' t)^{1/3}$, where $x_{\rm g}(t) = \omega'(k^*)t$ is the group velocity ray. As $t \to \infty$ along this ray,

$$|\delta A(x_{\rm g}(t),t)| \sim \frac{C}{(\omega''' t)^{1/3}}, \tag{37}$$

and the boundary contribution to $H_{\rm rad}(t)$ inherits this decay:

$$H_{\rm rad}(t) \sim \frac{C_*}{(\omega'''(k^*))^{1/3}}\,t^{-1/3} \qquad (t \to \infty), \tag{38}$$

giving $H(t) \sim O(t^{-1/3})$. In terms of the spectral weight: near the critical frequency $\omega^* = \omega(k^*)$, the density $a(\omega)M(\omega)$ accumulates as $(\omega-\omega^*)^{-2/3}$ – an integrable singularity weaker than (34) – producing a *quasi-resonance*: the inverse Fourier transform is spatially concentrated near the group velocity ray without any discrete eigenvalue being present. This is the spectral signature of a *caviton* – a transiently localised structure entrained by the boundary flux rather than self-sustained – whose helicity content decays precisely as $t^{-1/3}$ rather than the generic $t^{-1/2}$.

Remark 2 The three estimates (33)–(38) are derived here directly from the spectral representation (32) of the proto-helicity integral $H(t) = \int_\Omega A_a B_a\,d^3x$ in the linearised Chern–Simons fluid. They do not presuppose the KdV structure of [2]; rather, the same Airy and stationary-phase arguments apply to any dispersive system whose boundary flux admits the decomposition (32). The structural correspondence of Section 3 then explains *a posteriori* why the same three rates appear in both systems: they are universal consequences of the singularity structure of the spectral measure, regardless of whether the bulk dynamics is governed by KdV or by the compressible CS fluid equations.

The three regimes are therefore differentiated not by the bulk equations but entirely by the spectral character of $F_{\partial\Omega(t)}$: whether $a(\omega)M(\omega)$ has poles (stable), a generic branch-cut singularity (dispersive), or a stronger branch-cut singularity at a critical wavenumber (quasi-resonant). Measuring $H(t)$ from the boundary thus provides a direct spectroscopic diagnostic of the interior geometry – stable topological structures, transiently localised cavitons, or pure radiation – without access to the bulk fields.

We emphasize that the rates discussed above should therefore be interpreted as representative spectral scenarios rather than universal predictions of the Chern–Simons fluid system. Establishing which of these regimes is actually realized requires a spectral analysis of the outgoing boundary dynamics, which lies beyond the scope of the present work.

7 Conclusion

We have examined the consequences of replacing periodic or closed boundary conditions by open ones in the Chern–Simons formulation of compressible fluid dynamics of [1]. The local conservation law $\partial_{\hat\mu} h^{\hat\mu} = 0$ is unaffected by the choice of domain, but its integral over a moving region $\Omega(t)$ becomes the balance law (11): dH/dt equals the flux of the geometric current $J_{\rm eff}$ through $\partial\Omega(t)$. Periodic domains recover exact conservation, $dH/dt = 0$; open domains replace this by a transport relation in which $H(t)$ can grow, decay, or oscillate according to the sign of $F_{\partial\Omega(t)}$, including the monotonic decay (15) when the flux is one-signed. The apparent non-conservation is geometric leakage rather than dissipation – structurally identical to radiative loss in the Poynting theorem – and produces an effective irreversibility for an observer confined to $\Omega(t)$, in the same sense as radiation damping in open classical systems. The Burgers example shows that this boundary mechanism coexists with, and is independent of, internal nonlinear effects such as shock formation.

The structural correspondence with [2] identified in Section 3 reinforces the conclusions above. Both theories realise the abstract balance (2) and share the same radiative subsector – $A \times E$ in the fluid, boundary gravitons $(\delta e^a, \delta\omega^{ab})$ in AdS_3 gravity – that propagates topological or stress-tensor information to a boundary through the gauge connection itself, independently of matter transport. The identification rests on the shared Chern–Simons origin of both currents: the correspondence is structural rather than dynamical. The spectral analysis of Section 6 gives the three decay regimes $O(1)$, $O(t^{-1/3})$, $O(t^{-1/2})$ a precise derivation within the present system: they follow from the singularity structure of the spectral measure $a(\omega)M(\omega)$ – poles, generic branch cut, or Airy quasi-resonance at a critical wavenumber – and are therefore properties of the Chern–Simons fluid itself, not artefacts of the KdV structure of [2]. Measuring the decay law of $H(t)$ from the boundary thus provides a spectroscopic diagnostic of the interior geometry, distinguishing stable topological structures (vortex solitons), transiently localised cavitons, and purely dispersive radiation.

This picture opens several concrete directions. A gauge-invariant formulation via relative proto-helicity (9) would put (11) on a fully rigorous footing. An explicit radiation condition at $\partial\Omega(t)$ – under which the Chern–Simons fluid admits outgoing-wave solutions with J_{eff} as helicity flux – would make the graviton–fluid analogy quantitative. If the open-boundary dynamics of the present system can be cast as an eigenfunction-forced integrable hierarchy as in [2], the Gelfand–Levitan–Marchenko formalism would yield an exact, spectrally-resolved expression for $F_{\partial\Omega(t)}$, and the three decay regimes $O(1)$, $O(t^{-1/3})$, $O(t^{-1/2})$ identified above would acquire precise spectral interpretations. These questions define a concrete programme toward an open, non-periodic geometric hydrodynamics built on the Chern–Simons formulation.

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