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On New Divisibility Properties of Generalized Central Trinomial Coefficients and Legendre Polynomials

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Abstract We present a new formula for the highest power of $a+b$ that divides the sum $B(n,m,a,b)=\sum_{k=0}^{n}\binom{n}{k}^m a^{n-k}b^k$ for the case $m=2$. By using this formula, we give a complete 3-adic valuation for central Delannoy numbers. Also, we find the highest power of an odd integer x that divides Legendre's polynomial $P_n(x)$. By using the same idea, generalized trinomial coefficients and generalized Motzkin numbers are treated. As a result, we give a complete 3-adic valuation for little Schröder numbers and restricted hexagonal numbers. By using a new class of binomial sums, we examine the divisibility of $B(n,m,a,b)$ by powers of $a+b$ for $m>2$.

Keywords: Legendre polynomials, central Delannoy numbers, generalized central trinomial coefficients, Legendre's theorem, Kummer's theorem.

1 Introduction

Let us consider the following sum:

$$B(n,m,a,b)=\sum_{k=0}^{n}\binom{n}{k}^m a^{n-k}b^k, \tag{1}$$

where n and m are natural numbers such that $m\geq 2$, and a, b are integers.

Recently [15, Section 2, Proposition 1, p. 4], it is shown that the sum $B(n,2,a,b)$ represents the number of weighted free Dyck paths of semilength n from $(0,0)$ to $(2n,0)$ with up steps $(1,1)$ and down steps $(1,-1)$; where the weight a is assigned to up steps in the peaks, and the weight b to other up steps. Also, the sum $B(n,2,a,b)$ represents the number of the same weighted free Dyck paths of semilength n, where the weight a is assigned to even ascents and the weight b to even descents.

The sum $B(n,2,a,b)$ is closely related to well-known Legendre polynomials $P_n(x)$. It is well-known [5, Introduction] that Legendre polynomials have various explicit representations. We refer readers to [9, pp. 1–3] for seven different definitions of these polynomials. For example,

$$P_n(x)=\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}^2(x+1)^k(x-1)^{n-k}. \tag{2}$$

It is readily verified that the two following formulae are true:

$$P_n(x)=B(n,2,\frac{x-1}{2},\frac{x+1}{2}), \tag{3}$$

$$B(n,2,a,b)=(a-b)^nP_n(\frac{a+b}{a-b}),\ a\neq b. \tag{4}$$

The sum $B(n,2,1,1)$ is equal to the central binomial coefficient $\binom{2n}{n}$. It is well-known that the integer $\binom{2n}{n}$ counts all lattice paths from $(0,0)$ to (n,n) by using steps $(1,0)$ and $(0,1)$. Note that sum $B(n,3,1,1)$ is equal to the n-th Franel number $f_n=\sum_{k=0}^{n}\binom{n}{k}^3$.

The sum $B(n,2,1,2)$ is equal to the central Delannoy number D_n. It is well-known that D_n counts all lattice paths from $(0,0)$ to (n,n) by using steps $(1,0)$, $(0,1)$, and $(1,1)$. Such paths are called royal paths [27, Introduction]. Sulanke [29] gave, in a recreational spirit, a collection of 29 configurations counted by these numbers. Banderier and Schwer [1] gave additional information about origin and use of this number sequence. Central Delannoy numbers have a close relation to Legendre polynomials due to the fact that $D_n=P_n(3)$. See also [8, 11, 25]. Furthermore, central Delannoy numbers D_n (see [28, Section 6.4] and [**?**]) satisfy the second-order recurrence:

$$(n+2)D_{n+2}=3(2n+3)D_{n+1}-(n+1)D_n, \tag{5}$$

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where n is a non-negative integer.

The large Schröder numbers S_n count all lattice paths from $(0,0)$ to (n,n) by using steps $(1,0)$, $(0,1)$, and $(1,1)$ such that never go above the diagonal $y=x$. Large Schröder numbers bear the same relationship to the central Delannoy numbers as the Catalan numbers do to the central binomial coefficients. Recall that the n-th Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$ counts all lattice paths from $(0,0)$ to (n,n) by using steps $(1,0)$ and $(0,1)$ such that never go above the diagonal $y=x$. Stanley [28, Exercise 6.39] gave 11 combinatorial objects counted by large Schröder numbers S_n. See also [27, Introduction].

It is known that:

$$S_n = \frac{1}{2}\left(-D_{n-1}+6D_n-D_{n+1}\right). \tag{6}$$

where n is a natural number.

The little Schröder numbers s_n count the number of plane trees with a given set of leaves, the number of ways of inserting parentheses into a sequence, and the number of ways of dissecting a convex polygon into smaller polygons by inserting diagonals. Let n be a natural integer. Then $s_n = \frac{1}{2}S_n$.

Furthermore, for $m=2$, it is known that:

$$B(n,2,a,b) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{k}\binom{n-k}{k} a^k b^k (a+b)^{n-2k}. \tag{7}$$

For $m=3$, there is MacMahon's identity:

$$B(n,3,a,b) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k}\binom{2k}{k}\binom{n+k}{k} a^k b^k (a+b)^{n-2k}. \tag{8}$$

Recently, new combinatorial proofs of Eqns. (7) and (8) are presented, as well as combinatorial proofs of some known representations of Legendre polynomials based on weighted enumeration of free lattice paths [15].

Let us consider [31] the following sum:

$$T_n(a,b) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{k}\binom{n-k}{k} a^k \cdot b^{n-2k}, \tag{9}$$

where a and b are integers. The number $T_n(a,b)$ represents the number of weighted free Motzkin's paths of the length n form $(0,0)$ to (n,n) by using up steps $(1,1)$, down steps $(1,-1)$, and level steps $(1,0)$; where the weight a is assigned to up steps and the weight b to level steps. The number $T_n(a,b)$ is also known [4] as a generalized central trinomial coefficient. They can be defined as coefficients of x^n in the expansion of $(x^2+bx+a)^n$. See also [26].

By the Eq. (7), it follows that

$$B(n,2,a,b) = T_n(ab, a+b). \tag{10}$$

Also let us consider [31] the following sum:

$$M_n(a,b) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} C_k \cdot a^k \cdot b^{n-2k}, \tag{11}$$

where a and b are integers. The number $M_n(a,b)$ is known as a generalized Motzkin number. Particularly, it is known that:

$$s_{n+1} = M_n(2,3), \tag{12}$$

where n is a non-negative integer.

Our goal is to examine divisibility of $B(n,m,a,b)$ by powers of $a+b$, for $m \geq 2$. For $m=2$, we shall find the highest power of $a+b$ such that divides the sum $B(n,2,a,b)$.

2 Main results

Let $\omega_x(y)$ denote the highest power of an integer x that divides an integer y; where $x \neq 0$ and $x \neq \pm 1$. If an integer x is a prime number, then we shall use a notation $v_p(y)$ instead of $\omega_p(y)$. For calculating $\omega_x(y)$, we use the following formula:

$$\omega_x(y) = \min\left\{ \left\lfloor \frac{v_p(y)}{v_p(x)} \right\rfloor : p \text{ is a prime such that } p \mid x \right\}. \tag{13}$$

Our first main result is:

Theorem 1 *Let n be a non-negative integer. Let a and b be integers such that $\gcd(a,b)=1$, $a+b \neq 0$, and $a+b \neq \pm 1$. For $m=2$, we have:*

$$\omega_{a+b}\big(B(2n,2,a,b)\big) = \omega_{a+b}\left(\binom{2n}{n}\right), \tag{14}$$

$$\omega_{a+b}\big(B(2n+1,2,a,b)\big) = 1 + \omega_{a+b}\left((2n+1) \times \binom{2n}{n}\right). \tag{15}$$

Our second main result is:

Theorem 2 *Let n be a non-negative integer. Let a and b be integers such that $\gcd(a,b)=1$, $a+b \neq 0$, and $a+b \neq \pm 1$. Let m be an arbitrary natural number greater than 2. Then*

$$\omega_{a+b}\big(B(2n,m,a,b)\big) \geq \omega_{a+b}\left(\binom{2n}{n}\right), \tag{16}$$

$$\omega_{a+b}\big(B(2n+1,m,a,b)\big) \geq 1 + \omega_{a+b}\left((2n+1) \times \binom{2n}{n}\right). \tag{17}$$

We present several applications of Theorems 1 and 2.

The first application is for calculating the highest power of two that divides central binomial coefficient.

Corollary 1 *Let n be a non-negative integer.Then*

$$v_2\left(\binom{4n+2}{2n+1}\right) = 1 + v_2\left(\binom{2n}{n}\right), \tag{18}$$

$$v_2\left(\binom{4n}{2n}\right) = v_2\left(\binom{2n}{n}\right). \tag{19}$$

By using Eqns. (18) and (19), it can be shown the well-known fact that $v_2(\binom{2n}{n})$ is equal to the number of ones in binary representation of the number n.

The second application is for estimation for the highest power of two that divides Franel number f_n.

Corollary 2 *Let n be a non-negative integer.Then*

$$v_2(f_{2n+1}) \geq 1 + v_2\left(\binom{2n}{n}\right), \tag{20}$$

$$v_2(f_{2n}) \geq v_2\left(\binom{2n}{n}\right). \tag{21}$$

The third application is for calculating the highest power of three that divides central Delannoy number D_n.

Theorem 3 *Let n be a non-negative integer.Then*

$$v_3(D_{2n+1}) = 1 + v_3(2n+1) + v_3\left(\binom{2n}{n}\right), \tag{22}$$

$$v_3(D_{2n}) = v_3\left(\binom{2n}{n}\right). \tag{23}$$

The fourth application is for calculating the highest power of three that divides large Schröder number S_n.

Theorem 4 *Let n be a non-negative integer.Then*

$$v_3(S_{2n+1}) = v_3(C_n), \tag{24}$$

$$v_3(S_{2n+2}) = 1 + v_3(2n+1) + v_3(C_n). \tag{25}$$

Note that $v_3(s_n) = v_3(S_n)$.

Recently, Lengyel, among other, gave the formula [14, Theorem 10, p. 8] for 3-adic valuation of central Dellanoy numbers. Note that it is asserted that the formula [14, Theorem 10, p. 8] is true for any n sufficiently large. Also, Lengyel gave the formula [14, Theorem 17, p. 19] for 3-adic valuation of large Schröder numbers S_n for the case $n \equiv 1(\bmod 3)$.

The fifth application is for calculating the highest power of an odd integer x that divides Legendre polynomial $P_n(x)$.

Corollary 3 *Let n be a non-negative integer, and let x be an odd integer such that* $x \neq \pm 1$. *Then*

$$\omega_x\big(P_{2n+1}(x)\big) = 1 + \omega_x\left((2n+1) \times \binom{2n}{n}\right), \tag{26}$$

$$\omega_x\big(P_{2n}(x)\big) = \omega_x\left(\binom{2n}{n}\right). \tag{27}$$

Furthermore, from our proof of Theorem 1, we can calculate the highest power of an integer b that divides generalized central trinomial coefficient $T_n(a,b)$.

Theorem 5 *Let n be a non-negative integer. Let a and b be integers such that* $\gcd(a,b) = 1$, $b \neq \pm 1$, *and* $b \neq 0$. *Then:*

$$\omega_b\big(T_{2n}(a,b)\big) = \omega_b\left(\binom{2n}{n}\right), \tag{28}$$

$$\omega_b\big(T_{2n+1}(a,b)\big) = 1 + \omega_b\left((2n+1) \times \binom{2n}{n}\right). \tag{29}$$

Also, by using the same idea from our proof of Theorem 1, we can calculate the highest power of an integer b that divides generalized Motzkin number $M_n(a,b)$.

Theorem 6 *Let n be a non-negative integer. Let a and b be integers such that* $\gcd(a,b) = 1$, $b \neq 0$, *and* $b \neq \pm 1$. *Then:*

$$\omega_b\big(M_{2n}(a,b)\big) = \omega_b(C_n), \tag{30}$$

$$\omega_b\big(M_{2n+1}(a,b)\big) = 1 + \omega_b\big((2n+1) \times C_n\big). \tag{31}$$

For a proof of Theorem 1, we use Eq. (7) and the following lemma:

Lemma 1 *Let n be a natural number, and let p be a prime number. Then*

$$v_p((2n+1)\binom{2n}{n}) \leq n.$$

Furthermore, for a proof of Theorem 2, we use Lemma 1 and a new class of binomial sums [20, Eqns. (27) and (28)] that we call M sums.

3 A New Class of Binomials Sums

Definition 1 Let n and m be positive integers, and let a and b be integers. Let $S(n,m,a,b) = \sum_{k=0}^{n} \binom{n}{k}^m F(n,k,a,b)$, where $F(n,k,a,b)$ is an integer-valued function. Then M sums for the sum $S(n,m,a,b)$ are, as follows:

$$M_S(n,j,t;a,b) = \binom{n-j}{j} \sum_{v=0}^{n-2j} \binom{n-2j}{v} \times \binom{n}{j+v}^t F(n,j+v,a,b); \tag{32}$$

where j and t are non-negative integers such that $j \leq \lfloor \frac{n}{2} \rfloor$.

Obviously, the following equation [20, Eq. (29)],

$$S(n,m,a,b) = M_S(n,0,m-1;a,b) \tag{33}$$

holds.

By setting $t := 0$ in Eq. (32), it follows that

$$M_S(n,j,0;a,b) = \binom{n-j}{j}\sum_{v=0}^{n-2j}\binom{n-2j}{v} \times F(n,j+v,a,b). \tag{34}$$

Let n, j, t, a, and b be same as in Definition 1.

It is known [20, Th. 8] that M sums satisfy the following recurrence relation:

$$M_S(n,j,t+1;a,b) = \binom{n}{j}\sum_{u=0}^{\lfloor\frac{n-2j}{2}\rfloor}\binom{n-j}{u} \times M_S(n,j+u,t;a,b). \tag{35}$$

In one particular situation, Relation (35) has a simple consequence [20, Section 4] which is important for us.

Corollary 4 *Let n be a fixed natural number, and let a,b be fixed integers. Let us suppose that, for some non-negative integer t', $M_S(n,j,t';a,b)$ is divisible by an integer $q(n,a,b)$ for all $0 \le j \le \lfloor\frac{n}{2}\rfloor$. Then $M_S(n,j,t;a,b)$ is divisible by $q(n,a,b)$ for all $t \ge t'$ and for all $0 \le j \le \lfloor\frac{n}{2}\rfloor$. Furthermore, $S(n,m,a,b)$ is divisible by $q(n,a,b)$ for all $m \ge t'+1$.*

The idea is to calculate M sums for the sum $B(n,m,a,b)$ by using Eq. (34) and Relation (35). By using binomial theorem, we show that:

$$M_B(n,j,0;a,b) = \binom{n-j}{j}(ab)^j(a+b)^{n-2j}. \tag{36}$$

Furthermore, we prove a slight generalization of Eq. (7):

$$M_B(n,j,1;a,b) = \binom{n}{j}\sum_{k=0}^{\lfloor\frac{n-2j}{2}\rfloor}\binom{n-j}{k} \times \binom{n-j-k}{j+k}(ab)^{j+k} \times (a+b)^{n-2j-2k}. \tag{37}$$

By setting $j := 0$ in Eq. (37), we obtain Eq. (7).

M sums give an elementary proof of Calkin's result [2, Thm. 1, p. 1]. It is known [20, Eqns. (22) and (25)] that:

$$M_B(2n,j,0;1,-1) = \begin{cases} 0, & \text{if } 0 \le j < n; \\ (-1)^n, & \text{if } j = n. \end{cases}$$

$$M_B(2n,j,1;1,-1) = (-1)^n\binom{2n}{n}\binom{n}{j},$$

where $0 \le j \le n$. It follows that $q(2n,1,-1) = \binom{2n}{n}$ and $t' = 1$.

By using M sums, we can derive a slight generalization of, the MacMahon's identity, Eq. (8). Also we can derive formulas for $B(n,m,a,b)$ for the cases $m = 4$ and $m = 5$.

Note that there are several other applications ([19, Background],[20], and [22]) of M sums.

The rest of the paper is structured as follows. In Section 4, we give a proof of Lemma 1 by using Legendre's theorem. In Section 5, we give a proof of Theorem 1. In Section 6, we prove Theorem 2 by using a new class of binomials sums. In Section 7, we give a proof of Corollary 1. In Section 8, we give a proof of Corollary 2. In Section 9, we prove Theorem 3. In Section 10, we give a sketch of proofs of Theorem 5 and Corollary 3. In Section 11, we give a proof of Theorem 6. In Section 12, we give a proof of Theorem 4 by using Theorem 6. In Section 13, we give concluding remarks.

4 A Proof of Lemma 1

Let $n!!$ denote the double factorial, i. e., $n \cdot (n-2) \cdot (n-4) \cdots$.

The first case:

Let $p = 2$.

We have gradually:

$$\begin{aligned} v_2((2n+1)\binom{2n}{n}) &= v_2(\binom{2n}{n}) \\ &= v_2(\frac{(2n)!! \cdot (2n-1)!!}{(n!)^2}) \\ &= v_2(\frac{2^n \cdot n! \cdot (2n-1)!!}{(n!)^2}) \\ &= v_2(2^n) + v_2((2n-1)!!) - v_2(n!). \end{aligned} \tag{38}$$

From the last equation above, it follows that

$$v_2((2n+1)\binom{2n}{n}) = n - v_2(n!). \tag{39}$$

By the Eq. (39), it follows that $v_2((2n+1)\binom{2n}{n}) \le n$. This proves the first case.

The second case:

Let p be an odd prime. By Legendre's formula, it follows that:

$$v_p(n!) = \sum_{k=1}^{\infty}\lfloor\frac{n}{p^k}\rfloor \tag{40}$$

Let $n = \sum_{l=0}^{k} a_l \cdot p^l$ be the expansion of n in the base p, where a_l are non-negative integers such that $a_l < p$ for all $l = 1,k$. Let $s_p(n)$ denote the sum $\sum_{l=0}^{k} a_l$.

By Legendre's formula, it follows Legendre's theorem [16, Theorem 1.1]:

$$v_p(n!) = \frac{n - s_p(n)}{p-1}. \tag{41}$$

Let us consider the number $c(n,p) = \frac{(pn)!}{(n!)^p}$. Obviously, $c(n,p)$ is always an integer due to the fact that $c(n,p)$ is a central multinomial coefficient.

It is known that:

$$c(n,p) = \prod_{k=2}^{p} \binom{kn}{n}. \tag{42}$$

By (41), it follows that:

$$\begin{aligned} v_p(c(n,p)) &= v_p(\frac{(pn)!}{(n!)^p}) \\ &= v_p((pn)!) - pv_p(n!) \\ &= \frac{pn - s_p(pn)}{p-1} - p \cdot \frac{n - s_p(n)}{p-1} \\ &= \frac{p \cdot s_p(n) - s_p(pn)}{p-1}. \end{aligned} \tag{43}$$

By using the fact $s_p(pn) = s_p(n)$, from the Eq. (43), it follows that:

$$v_p(c(n,p)) = s_p(n). \tag{44}$$

From the expansion of n in the base p, using the obvious ineqality $p^l \geq 1$, it follows that $s_p(n) \leq n$. By using the Eq. (44), we obtain that:

$$v_p(c(n,p)) \leq n. \tag{45}$$

Let $F(n,k) = \frac{1}{(k-1)n+1}\binom{kn}{n}$ denote generalized Catalan numbers or Fuss-Catalan numbers, where n and k are natural numbers. It is well-known that Fuss-Catalan number is always an integer [10, Eq. (17.1), p. 375]. Obviously, $F(n,2) = C_n$ and $F(n,3) = \frac{1}{2n+1}\binom{3n}{n}$. See also [24, Introduction] and [22].

Now we use assumption that p is an odd prime. By the Eq. (42), it follows that $c(n,p)$ is divisible by $\binom{3n}{n}\binom{2n}{n}$. Note that $\binom{3n}{n}\binom{2n}{n} = F(n,3)\cdot(2n+1)\binom{2n}{n}$. It follows that $c(n,p)$ is divisible by $(2n+1)\binom{2n}{n}$. Therefore,

$$v_p((2n+1)\binom{2n}{n}) \leq v_p(c(n,p)). \tag{46}$$

Eqns. (45) and (46) prove the second case of Lemma 1. This completes the proof of Lemma 1.

Remark 1 Let p be an odd prime. Then, by using the same idea from the second case of the proof of Lemma 1, it can be shown that:

$$v_p((\prod_{k=2}^{p-1}(kn+1))\binom{2n}{n}) \leq n. \tag{47}$$

Remark 2 The Eq. (44) is true for all prime numbers p. Particulary, for $p = 2$, it follows that $v_2(\binom{2n}{n})$ is equal to number of ones in the binary representation of the number n.

5 A Proof of Theorem 1

Firstly, we give a proof of the Eq. (14).

5.1 A Proof of the Eq. (14)

By setting $n := 2n$ in Eq. (7), we obtain that:

$$B(2n,2,a,b) = \sum_{k=0}^{n} \binom{2n}{k}\binom{2n-k}{k} \times (ab)^k(a+b)^{2n-2k}. \tag{48}$$

By using the substitution $k = n - s$, Eq. (48) becomes gradually:

$$\begin{aligned} &B(2n,2,a,b) = \\ &\quad \sum_{s=0}^{n} \binom{2n}{n-s}\binom{n+s}{n-s}(a+b)^{2s}(ab)^{n-s}, \\ &\quad = \sum_{s=0}^{n} \binom{2n}{n+s}\binom{n+s}{2s}(a+b)^{2s}\cdot(ab)^{n-s}. \end{aligned} \tag{49}$$

It is readily verified that [10, Eq. (1.4), p. 5]:

$$\binom{2n}{n+s}\binom{n+s}{2s} = \frac{\binom{2n}{n}\binom{n}{s}^2}{\binom{2s}{s}}. \tag{50}$$

By using Eq. (50), Eq. (49) becomes, as follows:

$$B(2n,2,a,b) = \sum_{s=0}^{n} (\frac{\binom{2n}{n}\binom{n}{s}^2}{\binom{2s}{s}})(a+b)^{2s}\cdot(ab)^{n-s}. \tag{51}$$

Let p be a prime number such that divides $a+b$. For $s = 0$, the first summand in Eq. (51) is $\binom{2n}{n}\cdot(ab)^n$. Since $a+b$ and ab are relatively prime, it follows that

$$v_p(\binom{2n}{n}\cdot(ab)^n) = v_p(\binom{2n}{n}). \tag{52}$$

Let $1 \leq s \leq n$. Note that by Lemma 1, it follows that

$$v_p(\binom{2s}{s}) \leq s. \tag{53}$$

Then, for the $(s+1)$-th summand Eq. (51), we obtain that:

$$\begin{aligned} &v_p\left(\frac{\binom{2n}{n}\binom{n}{s}^2}{\binom{2s}{s}}(a+b)^{2s}(ab)^{n-s}\right) \\ &\qquad = v_p\left(\binom{2n}{n}\right) + 2v_p\left(\binom{n}{s}\right) \\ &\qquad\quad - v_p\left(\binom{2s}{s}\right) + 2s\,v_p(a+b). \end{aligned} \tag{54}$$

By using Ineq. (53), from Eq. (54), we obtain that:

$$\begin{aligned} &v_p\left(\frac{\binom{2n}{n}\binom{n}{s}^2}{\binom{2s}{s}}(a+b)^{2s}(ab)^{n-s}\right) \geq \\ &\qquad v_p\left(\binom{2n}{n}\right) + s\big(2v_p(a+b) - 1\big). \end{aligned} \tag{55}$$

Obviously, for $1 \le s \le n$,

$$v_p\left(\binom{2n}{n}\right) + s\big(2v_p(a+b)-1\big) \ge v_p\left(\binom{2n}{n}\right) + s \ge v_p\left(\binom{2n}{n}\right) + 1. \tag{56}$$

By Ineqs. (55) and (56), for $1 \le s \le n$, it follows that:

$$v_p\Big(\Big(\frac{\binom{2n}{n}\binom{n}{s}^2}{\binom{2s}{s}}\Big)(a+b)^{2s}\cdot(ab)^{n-s}\Big) \ge v_p\Big(\binom{2n}{n}\Big)+1. \tag{57}$$

By using Eqns. (52) and (57), we obtain that:

$$v_p\big(B(2n,2,a,b)\big) = v_p\left(\binom{2n}{n}\right). \tag{58}$$

where p is a prime number that divides $a+b$.

Now, by using Relation (13), it follows that:

$$\begin{aligned}\omega_{a+b}\big(B(2n,2,a,b)\big) &= \min_{\substack{p \text{ prime}\\ p\mid a+b}}\left\lfloor \frac{v_p\big(B(2n,2,a,b)\big)}{v_p(a+b)}\right\rfloor \\ &= \min_{\substack{p \text{ prime}\\ p\mid a+b}}\left\lfloor \frac{v_p\Big(\binom{2n}{n}\Big)}{v_p(a+b)}\right\rfloor \\ &= \omega_{a+b}\left(\binom{2n}{n}\right).\end{aligned} \tag{59}$$

The Eq. (59) completes the proof of Eq. (14).

5.2 A Proof of Eq. (15)

By setting $n := 2n+1$ in Eq. (7), we obtain that:

$$\begin{aligned}B(2n+1,2,a,b) &= \sum_{k=0}^{n}\binom{2n+1}{k}\binom{2n+1-k}{k} \times (ab)^k(a+b)^{2n+1-2k} \\ &= (a+b)\sum_{k=0}^{n}\binom{2n+1}{k}\binom{2n+1-k}{k} \times (ab)^k(a+b)^{2n-2k}.\end{aligned} \tag{60}$$

By using the substitution $k = n-s$, Eq. (60) becomes gradually:

$$\begin{aligned}B(2n+1,2,a,b) &= (a+b)\sum_{s=0}^{n}\binom{2n+1}{n-s}\binom{n+s+1}{n-s} \times (a+b)^{2s}(ab)^{n-s} \\ &= (a+b)\sum_{s=0}^{n}\binom{2n+1}{n+s+1}\binom{n+s+1}{2s+1} \times (a+b)^{2s}(ab)^{n-s}.\end{aligned} \tag{61}$$

It is readily verified that [10, Eq. (1.4), p. 5]

$$\binom{2n+1}{n+s+1}\binom{n+s+1}{2s+1} = \frac{(2n+1)\binom{2n}{n}\binom{n}{s}^2}{(2s+1)\binom{2s}{s}}. \tag{62}$$

By using Eq. (62), Eq. (61) becomes, as follows:

$$B(2n+1,2,a,b) = (a+b)\sum_{s=0}^{n}\left[\frac{(2n+1)\binom{2n}{n}\binom{n}{s}^2}{(2s+1)\binom{2s}{s}}\right] \times (a+b)^{2s}(ab)^{n-s}. \tag{63}$$

By using Eq. (63), it follows that

$$\omega_{a+b}\big(B(2n+1,2,a,b)\big) = 1 + \omega_{a+b}\left(\sum_{s=0}^{n}\left[\frac{(2n+1)\binom{2n}{n}\binom{n}{s}^2}{(2s+1)\binom{2s}{s}}\right] \times (a+b)^{2s}(ab)^{n-s}\right). \tag{64}$$

Let $P(2n+1,2,a,b)$ denote $\sum_{s=0}^{n}\Big[\frac{(2n+1)\binom{2n}{n}\binom{n}{s}^2}{(2s+1)\binom{2s}{s}}\Big](a+b)^{2s}\cdot(ab)^{n-s}$.

Then, Eq. (64) can be rewritten as:

$$\omega_{a+b}(B(2n+1,2,a,b)) = 1 + \omega_{a+b}(P(2n+1,2,a,b)). \tag{65}$$

Let p be a prime number such that divides $a+b$. For $s=0$, the first summand in the sum $P(2n+1,2,a,b)$ is $(2n+1)\binom{2n}{n}\cdot(ab)^n$. Since $a+b$ and ab are relatively prime, it follows that

$$v_p((2n+1)\binom{2n}{n}\cdot(ab)^n) = v_p((2n+1)\binom{2n}{n}). \tag{66}$$

Let $1 \le s \le n$.

Then, for the $(s+1)$-th summand the sum $P(2n+1,2,a,b)$, it follows that $v_p\Big(\Big(\frac{(2n+1)\binom{2n}{n}\binom{n}{s}^2}{(2s+1)\binom{2s}{s}}\Big)(a+b)^{2s}\cdot(ab)^{n-s}\Big)$ is equal to

$$v_p\left((2n+1)\binom{2n}{n}\right) + 2v_p\left(\binom{n}{s}\right) - v_p\left((2s+1)\binom{2s}{s}\right) + 2s\,v_p(a+b). \tag{67}$$

By using Lemma 1, from Eq. (67), it follows that:

$$v_p\left(\frac{(2n+1)\binom{2n}{n}\binom{n}{s}^2}{(2s+1)\binom{2s}{s}} \times (a+b)^{2s}(ab)^{n-s}\right) \ge v_p\left((2n+1)\binom{2n}{n}\right) + s\big(2v_p(a+b)-1\big). \tag{68}$$

Obviously, for $1 \le s \le n$,

$$\begin{aligned} v_p &\left((2n+1)\binom{2n}{n} \right) + s\big(2v_p(a+b)-1\big) \\ &\ge v_p\left((2n+1)\binom{2n}{n} \right) + s \qquad (69)\\ &\ge v_p\left((2n+1)\binom{2n}{n} \right) + 1. \end{aligned}$$

By Ineqs. (68) and (69), for $1 \le s \le n$, it follows that:

$$\begin{aligned} v_p &\left(\frac{(2n+1)\binom{2n}{n}\binom{n}{s}^2}{(2s+1)\binom{2s}{s}} \right. \\ &\qquad \left. \times (a+b)^{2s}(ab)^{n-s} \right) \qquad (70)\\ &\ge v_p\left((2n+1)\binom{2n}{n} \right) + 1. \end{aligned}$$

By using Eq. (66) and Ineq. (70), we obtain that:

$$v_p(P(2n+1,2,a,b)) = v_p((2n+1)\binom{2n}{n}), \qquad (71)$$

where p is a prime number that divides $a+b$.

Now, by using Relation (13), it follows that:

$$\begin{aligned} \omega_{a+b}&\big(P(2n+1,2,a,b)\big) \\ &= \min_{\substack{p \text{ prime} \\ p|a+b}} \left\lfloor \frac{v_p\big(P(2n+1,2,a,b)\big)}{v_p(a+b)} \right\rfloor \\ &= \min_{\substack{p \text{ prime} \\ p|a+b}} \left\lfloor \frac{v_p\left((2n+1)\binom{2n}{n}\right)}{v_p(a+b)} \right\rfloor \\ &= \omega_{a+b}\left((2n+1)\binom{2n}{n} \right). \qquad (72) \end{aligned}$$

By using Eqns. (65) and (72), Eq. (15) follows. This completes the proof of Theorem 1.

6 A Proof of Theorem 2

Firstly, we give a proof of Eq. (36).

6.1 A Proof of Eq. (36)

Obviously, the sum $B(n,m,a,b)$ is an instance of the sum $S(n,m,a,b)$, where $F(n,k,a,b) = a^{n-k}b^k$.

By Eq. (34), it follows that

$$\begin{aligned} M_B(n,j,0;a,b) &= \binom{n-j}{j} \sum_{v=0}^{n-2j} \binom{n-2j}{v} a^{n-j-v} b^{j+v}, \\ &= \binom{n-j}{j} a^j \cdot b^j \sum_{v=0}^{n-2j} \binom{n-2j}{v} a^{n-2j-v} b^v, \\ &= \binom{n-j}{j} (ab)^j (a+b)^{n-2j}, \qquad (73) \end{aligned}$$

where $0 \le j \le \lfloor \frac{n}{2} \rfloor$. This proves Eq. (36).

Secondly, we give a proof of Eq. (37).

6.2 A Proof of Eq. (37)

By setting $t := 0$ and $S := B$ in Eq. (35), it follows that

$$\begin{aligned} M_B(n,j,1;a,b) = \binom{n}{j} &\sum_{u=0}^{\left\lfloor \frac{n-2j}{2} \right\rfloor} \binom{n-j}{u} \\ &\times M_B(n,j+u,0;a,b). \end{aligned} \qquad (74)$$

By using Eq. (36), Eq. (74) becomes:

$$\begin{aligned} M_B(n,j,1;a,b) = \binom{n}{j} &\sum_{u=0}^{\left\lfloor \frac{n-2j}{2} \right\rfloor} \binom{n-j}{u} \\ &\times \binom{n-j-u}{j+u} (ab)^{j+u} \\ &\times (a+b)^{n-2j-2u}. \end{aligned} \qquad (75)$$

The Eq. (75) leads to Eq. (37).

6.3 A Proof of Eq. (16)

By setting $n := 2n$ in Eq. (75), it follows that:

$$\begin{aligned} M_B(2n,j,1;a,b) = \binom{2n}{j} &\sum_{u=0}^{n-j} \binom{2n-j}{u} \\ &\times \binom{2n-j-u}{j+u} (ab)^{j+u} \\ &\times (a+b)^{2n-2j-2u}. \end{aligned} \qquad (76)$$

By using the substitution $u = n-j-s$, Eq. (76) becomes:

$$\begin{aligned} M_B(2n,j,1;a,b) = &\sum_{s=0}^{n-j} \binom{2n}{j} \binom{2n-j}{n+s} \\ &\times \binom{n+s}{n-s} (ab)^{n-s} (a+b)^{2s}. \end{aligned} \qquad (77)$$

It is readily verified that

$$\binom{2n}{j}\binom{2n-j}{n+s} = \binom{2n}{n+s}\binom{n-s}{j}. \qquad (78)$$

By using Eq. (78), Eq. (77) becomes:

$$\begin{aligned} M_B(2n,j,1;a,b) = &\sum_{s=0}^{n-j} \binom{2n}{n+s}\binom{n-s}{j} \\ &\times \binom{n+s}{2s} (ab)^{n-s} (a+b)^{2s}. \end{aligned} \qquad (79)$$

By using Eq. (50) in Eq. (79), it follows that

$$M_B(2n,j,1;a,b) = \sum_{s=0}^{n-j}\left[\frac{\binom{2n}{n}\binom{n}{s}^2}{\binom{2s}{s}}\right] \times (ab)^{n-s}(a+b)^{2s}\binom{n-s}{j}. \tag{80}$$

Let p be a prime number such that divides $a+b$. Due to the fact that ab and $a+b$ are relatively prime, it follows that p does not divide ab. Let j be a fixed non-negative integer such that $j \le n$.

Obviously,

$$v_p\left(\left[\frac{\binom{2n}{n}\binom{n}{s}^2}{\binom{2s}{s}}\right](ab)^{n-s}(a+b)^{2s}\binom{n-s}{j}\right) \ge v_p\left(\left[\frac{\binom{2n}{n}\binom{n}{s}^2}{\binom{2s}{s}}\right] \times (ab)^{n-s}(a+b)^{2s}\right). \tag{81}$$

By using Eqns. (52) and (57), we know that

$$v_p([\frac{\binom{2n}{n}\binom{n}{s}^2}{\binom{2s}{s}}](ab)^{n-s}(a+b)^{2s}) \ge v_p(\binom{2n}{n}), \tag{82}$$

where $0 \le s \le n-j$.

By using Eqns. (81) and (82), it follows that:

$$v_p([\frac{\binom{2n}{n}\binom{n}{s}^2}{\binom{2s}{s}}](ab)^{n-s}(a+b)^{2s}\binom{n-s}{j}) \ge v_p(\binom{2n}{n}), \tag{83}$$

where $0 \le s \le n-j$.

By using Eqns. (80) and (83), it follows that:

$$v_p\big(M_B(2n,j,1;a,b)\big) \ge v_p\left(\binom{2n}{n}\right). \tag{84}$$

where $0 \le j \le n$.

Now, by using Relation (13) and the Eq. (84), it follows that:

$$\omega_{a+b}\big(M_B(2n,j,1;a,b)\big) = \min_{\substack{p \text{ prime} \\ p|a+b}}\left\lfloor\frac{v_p\big(M_B(2n,j,1;a,b)\big)}{v_p(a+b)}\right\rfloor \ge \min_{\substack{p \text{ prime} \\ p|a+b}}\left\lfloor\frac{v_p\left(\binom{2n}{n}\right)}{v_p(a+b)}\right\rfloor. \tag{85}$$

By using Eqns. (13) and (85), it follows that:

$$\omega_{a+b}(M_B(2n,j,1;a,b)) \ge \omega_{a+b}(\binom{2n}{n}), \tag{86}$$

where $0 \le j \le n$. By Eq. (86), it follows that:

$$(a+b)^{\omega_{a+b}(\binom{2n}{n})} \text{ divides } M_B(2n,j,1;a,b), \tag{87}$$

for any j such that $0 \le j \le n$.

Note that $q(2n,a,b) = (a+b)^{\omega_{a+b}(\binom{2n}{n})}$. By Corollary 4, it follows that:

$$(a+b)^{\omega_{a+b}(\binom{2n}{n})} \text{ divides } M_B(2n,j,t;a,b), \tag{88}$$

for any j and for any natural number t such that $0 \le j \le n$.

Furthermore, by Corollary 4, it follows that:

$$(a+b)^{\omega_{a+b}(\binom{2n}{n})} \text{ divides } B(2n,m,a,b), \tag{89}$$

for any natural number m such that $m \ge 2$.

The Eq. (89) leads to Eq. (16).

6.4 A Proof of Eq. (17)

By setting $n := 2n+1$ in Eq. (75), it follows that:

$$M_B(2n+1,j,1;a,b) = \binom{2n+1}{j}\sum_{u=0}^{n-j}\binom{2n+1-j}{u} \times \binom{2n+1-j-u}{j+u}(ab)^{j+u} \times (a+b)^{2n+1-2j-2u}. \tag{90}$$

where $0 \le j \le n$.

By using the substitution $u = n-j-s$, Eq. (90) becomes, as follows:

$$M_B(2n+1,j,1;a,b) = (a+b)\sum_{s=0}^{n-j}\binom{2n+1}{j}\binom{2n+1-j}{n+s+1} \times \binom{n+s+1}{2s+1}(ab)^{n-s}(a+b)^{2s}. \tag{91}$$

It is readily verified that:

$$\binom{2n+1}{j}\binom{2n+1-j}{n+s+1} = \binom{2n+1}{n+s+1} \times \binom{n-s}{j}. \tag{92}$$

By Eq. (92), Eq. (91) becomes, as follows:

$$M_B(2n+1,j,1;a,b) = (a+b)\sum_{s=0}^{n-j}\binom{2n+1}{n+s+1} \times \binom{n+s+1}{2s+1}\binom{n-s}{j} \times (ab)^{n-s}(a+b)^{2s}. \tag{93}$$

By using Eq. (62) in Eq. (93), it follows that $M_B(2n+1,j,1;a,b)$ is equal to the following sum:

$$(a+b)\sum_{s=0}^{n-j}\left[\frac{(2n+1)\binom{2n}{n}\binom{n}{s}^2}{(2s+1)\binom{2s}{s}}\right] \times \binom{n-s}{j}(ab)^{n-s}(a+b)^{2s}. \tag{94}$$

Let $P(2n+1,a,b)$ denote the sum $\sum_{s=0}^{n-j}[\frac{(2n+1)\binom{2n}{n}\binom{n}{s}^2}{(2s+1)\binom{2s}{s}}]\binom{n-s}{j}(ab)^{n-s}(a+b)^{2s}$. By Eq. (94), it follows that:

$$\omega_{a+b}\big(M_B(2n+1,j,1;a,b)\big) = 1+\omega_{a+b}\big(P(2n+1,a,b)\big). \tag{95}$$

Let p be a prime number such that p divides $a+b$. Obviously,

$$\begin{aligned} &v_p\left(\left[\frac{(2n+1)\binom{2n}{n}\binom{n}{s}^2}{(2s+1)\binom{2s}{s}}\right]\binom{n-s}{j} \times (ab)^{n-s}(a+b)^{2s}\right) \\ &\geq v_p\left(\left[\frac{(2n+1)\binom{2n}{n}\binom{n}{s}^2}{(2s+1)\binom{2s}{s}}\right] \times (ab)^{n-s}(a+b)^{2s}\right). \end{aligned} \tag{96}$$

By using Eqns. (66) and (70), we know that:

$$\begin{aligned} &v_p\left(\left[\frac{(2n+1)\binom{2n}{n}\binom{n}{s}^2}{(2s+1)\binom{2s}{s}}\right] \times (a+b)^{2s}(ab)^{n-s}\right) \\ &\geq v_p\left((2n+1)\binom{2n}{n}\right). \end{aligned} \tag{97}$$

where $0\leq s\leq n$. By Eqns. (96) and (97), it follows that:

$$\begin{aligned} &v_p\left(\left[\frac{(2n+1)\binom{2n}{n}\binom{n}{s}^2}{(2s+1)\binom{2s}{s}}\right]\binom{n-s}{j} \times (ab)^{n-s}(a+b)^{2s}\right) \\ &\geq v_p\left((2n+1)\binom{2n}{n}\right). \end{aligned} \tag{98}$$

where $0\leq s\leq n$. Then, it follows that:

$$v_p(P(2n+1,a,b)) \geq v_p((2n+1)\binom{2n}{n}). \tag{99}$$

Now, by using Relation (13) and Eq. (99), it follows that:

$$\begin{aligned} &\omega_{a+b}\big(P(2n+1,a,b)\big) \\ &= \min_{\substack{p \text{ prime} \\ p|a+b}}\left\lfloor\frac{v_p\big(P(2n+1,a,b)\big)}{v_p(a+b)}\right\rfloor \\ &\geq \min_{\substack{p \text{ prime} \\ p|a+b}}\left\lfloor\frac{v_p\left((2n+1)\binom{2n}{n}\right)}{v_p(a+b)}\right\rfloor. \end{aligned} \tag{100}$$

By using Relation (13) and Eq. (100), it follows that:

$$\omega_{a+b}(P(2n+1,a,b)) \geq \omega_{a+b}((2n+1)\binom{2n}{n}). \tag{101}$$

By Eqns. (95) and (101), it follows that:

$$\begin{aligned} &\omega_{a+b}\big(M_B(2n+1,j,1;a,b)\big) \\ &\geq 1+\omega_{a+b}\left((2n+1)\binom{2n}{n}\right). \end{aligned} \tag{102}$$

By Eq. (102), it follows that

$$(a+b)^{1+\omega_{a+b}((2n+1)\binom{2n}{n})} \text{ divides } M_B(2n+1,j,1;a,b), \tag{103}$$

for any j such that $0\leq j\leq n$.

Note that $q(2n+1,a,b)=(a+b)^{1+\omega_{a+b}((2n+1)\binom{2n}{n})}$. By Corollary 4, it follows that:

$$(a+b)^{1+\omega_{a+b}((2n+1)\binom{2n}{n})} \text{ divides } M_B(2n+1,j,t;a,b); \tag{104}$$

for any j and for any natural number t such that $0\leq j\leq n$.

Furthermore, by Corollary 4, it follows that:

$$(a+b)^{1+\omega_{a+b}((2n+1)\binom{2n}{n})} \text{ divides } B(2n+1,m,a,b), \tag{105}$$

for any natural number m such that $m\geq 2$.

The Eq. (105) implies Eq. (17). This completes the proof of Theorem 2.

7 A Proof of Corolary 1

It is well-known that:

$$\sum_{k=0}^{n}\binom{n}{k}^2 = \binom{2n}{n}. \tag{106}$$

By Eq. (106), it follows that:

$$B(n,2,1,1) = \binom{2n}{n}. \tag{107}$$

By using Eq. (14) of Theorem 1, it follows that:

$$v_2(B(2n,2,1,1)) = v_2(\binom{2n}{n}). \tag{108}$$

Note that we use notation v_2 instead of ω_2 because 2 is a prime number.

By Eq. (107), it follows that:

$$B(2n,2,1,1) = \binom{4n}{2n}. \tag{109}$$

By using Eq. (109), Eq. (108) becomes

$$\nu_2(\binom{4n}{2n}) = \nu_2(\binom{2n}{n}). \tag{110}$$

The last equation above proves the Eq. (19).

Furthermore, by using Eq. (15) of Theorem 1, it follows that:

$$\nu_2(B(2n+1,2,1,1)) = 1 + \nu_2((2n+1)\binom{2n}{n}). \tag{111}$$

By Eq. (107), it follows that:

$$B(2n+1,2,1,1) = \binom{4n+2}{2n+1}. \tag{112}$$

By using Eq. (112), Eq. (111) becomes, as follows:

$$\begin{aligned} \nu_2(\binom{4n+2}{2n+1}) &= 1 + \nu_2((2n+1)\binom{2n}{n}), \\ &= 1 + \nu_2(2n+1) + \nu_2(\binom{2n}{n}), \\ &= 1 + \nu_2(\binom{2n}{n}). \end{aligned} \tag{113}$$

The last equation above proves the Eq. (18). This completes the proof of Corollary 1.

8 A Proof of Corollary 2

For the proof of Corollary 2, we use Theorem 2. It is known that:

$$f_n = B(n,3,1,1);$$

where f_n is the n-th Franel number.

By using Eq. (16), it follows that

$$\nu_2(f_{2n}) \geq \nu_2(\binom{2n}{n}).$$

This proves the Eq. (21).

By using Eq. (17), it follows that

$$\nu_2(f_{2n+1}) \geq 1 + \nu_2((2n+1)\binom{2n}{n}).$$

Obviously,

$$\nu_2((2n+1)\binom{2n}{n}) = \nu_2(\binom{2n}{n}).$$

By using two last equations above, it follows that:

$$\nu_2(f_{2n+1}) \geq 1 + \nu_2(\binom{2n}{n}).$$

This proves the Eq. (20).

9 A Proof of Theorem 3

It is known that:

$$D_n = B(n,2,1,2);$$

where D_n is n-th central Dellanoy number.

By using Eq. (15), it follows that:

$$\begin{aligned} \nu_3(D_{2n+1}) &= 1 + \nu_3((2n+1)\binom{2n}{n}), \\ &= 1 + \nu_3((2n+1)) + \nu_3(\binom{2n}{n}). \end{aligned} \tag{114}$$

The last equation above proves Eq. (22).

Note that we use notation ν_3 instead of ω_3 because 3 is a prime number.

By using Eq. (14), it follows that:

$$\nu_3(D_{2n}) = \nu_3(\binom{2n}{n}).$$

The last equation above proves Eq. (23).

This completes the proof of Theorem 3.

10 Proofs of Theorem 5 and Corollary 3

Proof of Theorem 5 directly follows from the our proof of Theorem 1. Just replace ab with a, and $a+b$ with b in the proof of Theorem 1.

Also Corollary 3 follows from Theorem 1 and Eq. (3).

11 A Proof of Theorem 6

Proof of Theorem 6 is similar to the proof of Theorem 1.

Note that:

$$M_n(a,b) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{k}\binom{n-k}{k} \cdot \frac{1}{k+1} \cdot a^k \cdot b^{n-2k}. \tag{115}$$

11.1 A Proof of Eq. (30)

By setting $n := 2n$ in Eq. (115), it follows that:

$$M_{2n}(a,b) = \sum_{k=0}^{n} \binom{2n}{k}\binom{2n-k}{k} \cdot \frac{1}{k+1} \cdot a^k \cdot b^{2n-2k}. \tag{116}$$

By using the substitution $k = n - s$, Eq. (116) becomes gradually:

$$\begin{aligned} M_{2n}(a,b) & \\ &= \sum_{s=0}^{n} \binom{2n}{n-s}\binom{n+s}{n-s} \cdot \frac{1}{n+1-s} \cdot b^{2s} \cdot a^{n-s}, \\ &= \sum_{s=0}^{n} \binom{2n}{n+s}\binom{n+s}{2s} \cdot \frac{1}{n+1-s} \cdot b^{2s} \cdot a^{n-s}. \end{aligned} \tag{117}$$

By using Eq. (50), Eq. (117) becomes, as follows:

$$M_{2n}(a,b) = \sum_{s=0}^{n} \left(\frac{\binom{2n}{n}\binom{n}{s}^2}{\binom{2s}{s}}\right) \cdot \frac{1}{n+1-s} \cdot b^{2s} \cdot a^{n-s}. \tag{118}$$

It is readily verified that:

$$\binom{n}{s} \cdot \frac{1}{n+1-s} = \frac{1}{n+1} \cdot \binom{n+1}{s}. \tag{119}$$

By using Eq. (119), Eq. (118) becomes, as follows:

$$M_{2n}(a,b) = \sum_{s=0}^{n} \left(\frac{C_n\binom{n}{s}\binom{n+1}{s}}{\binom{2s}{s}}\right) \cdot b^{2s} \cdot a^{n-s}. \tag{120}$$

Let p be a prime number such that divides b. For $s = 0$, the first summand in the right-side of the Eq. (120) is $C_n \cdot a^n$. Since a and b are relatively prime, it follows that:

$$v_p(C_n \cdot a^n) = v_p(C_n). \tag{121}$$

Let $1 \leq s \leq n$. Then, for the $(s+1)$-th summand in the right-side of Eq. (120), we have:

$$v_p\left(\frac{C_n\binom{n}{s}\binom{n+1}{s}}{\binom{2s}{s}} b^{2s}a^{n-s}\right) \geq v_p(C_n) - v_p\left(\binom{2s}{s}\right) + 2s\,v_p(b). \tag{122}$$

By using Ineq. (53), from Eq. (122), we obtain that:

$$v_p\left(\frac{C_n\binom{n}{s}\binom{n+1}{s}}{\binom{2s}{s}} b^{2s}a^{n-s}\right) \geq v_p(C_n) + s\big(2v_p(b) - 1\big). \tag{123}$$

Obviously, for $1 \leq s \leq n$,

$$v_p(C_n) + s \cdot (2v_p(b) - 1) \geq v_p(C_n) + s \geq v_p(C_n) + 1. \tag{124}$$

By Ineqs. (123) and (124), for $1 \leq s \leq n$, it follows that:

$$v_p\left(\left(\frac{C_n\binom{n}{s}\binom{n+1}{s}}{\binom{2s}{s}}\right) \cdot b^{2s} \cdot a^{n-s}\right) \geq v_p(C_n) + 1. \tag{125}$$

By using Eqns. (121) and (125), we obtain that:

$$v_p(M_{2n}(a,b)) = v_p(C_n), \tag{126}$$

where p is a prime number that divides b. Furthermore, by using Relation (13), it follows that:

$$\begin{aligned}\omega_b\big(M_{2n}(a,b)\big) &= \min_{\substack{p \text{ prime} \\ p \mid b}} \left\lfloor \frac{v_p\big(M_{2n}(a,b)\big)}{v_p(b)} \right\rfloor \\ &= \min_{\substack{p \text{ prime} \\ p \mid b}} \left\lfloor \frac{v_p(C_n)}{v_p(b)} \right\rfloor \\ &= \omega_b(C_n).\end{aligned} \tag{127}$$

The Eq. (127) completes the proof of Eq. (30).

11.2 A Proof of Eq. (31)

By setting $n := 2n+1$ in Eq. (115), we obtain that:

$$\begin{aligned}M_{2n+1}(a,b) &= \sum_{k=0}^{n} \binom{2n+1}{k}\binom{2n+1-k}{k}\frac{1}{k+1} \\ &\quad \times a^k b^{2n+1-2k} \\ &= b\sum_{k=0}^{n} \binom{2n+1}{k}\binom{2n+1-k}{k}\frac{1}{k+1} \\ &\quad \times a^k b^{2n-2k}.\end{aligned} \tag{128}$$

By using the substitution $k = n - s$, Eq. (128) becomes gradually:

$$\begin{aligned}&M_{2n+1}(a,b) \\ &\quad = b\sum_{s=0}^{n} \binom{2n+1}{n-s}\binom{n+s+1}{n-s}\frac{1}{n-s+1} \\ &\quad \times b^{2s}a^{n-s} \\ &\quad = b\sum_{s=0}^{n} \binom{2n+1}{n+s+1}\binom{n+s+1}{2s+1}\frac{1}{n+1-s} \\ &\quad \times b^{2s}a^{n-s}.\end{aligned} \tag{129}$$

By using Eq. (62), Eq. (129) becomes, as follows:

$$\begin{aligned}M_{2n+1}(a,b) &= b\sum_{s=0}^{n}\left[\frac{(2n+1)\binom{2n}{n}\binom{n}{s}^2}{(2s+1)\binom{2s}{s}}\right] \\ &\quad \times \frac{1}{n+1-s} b^{2s}a^{n-s}.\end{aligned} \tag{130}$$

By using Eq. (119), it follows that

$$\begin{aligned}M_{2n+1}(a,b) &= b\sum_{s=0}^{n}\left[\frac{(2n+1)C_n\binom{n}{s}\binom{n+1}{s}}{(2s+1)\binom{2s}{s}}\right] \\ &\quad \times b^{2s}a^{n-s}.\end{aligned} \tag{131}$$

The rest of the proof is similar to the proof of Eq. (15) of Theorem 1

Let p be a prime that divides b. Then, by using Lemma 1, it can be shown that:

$$\begin{aligned}&v_p\left(\sum_{s=0}^{n}\left[\frac{(2n+1)C_n\binom{n}{s}\binom{n+1}{s}}{(2s+1)\binom{2s}{s}}\right] \times b^{2s}a^{n-s}\right) = \\ &\quad v_p\big((2n+1)C_n\big).\end{aligned} \tag{132}$$

This completes the proof of Theorem 6.

12 A Proof of Theorem 4

We use Theorem 6 and Eq. (12). By setting $a := 2$ and $b := 3$ in the Eq. (30), it follows that:

$$v_3(M_{2n}(2,3)) = v_3(C_n). \tag{133}$$

By Eq. (12), it follows that

$$s_{2n+1} = M_{2n}(2,3). \tag{134}$$

By using Eqns. (133) and (134), it follows that:

$$\nu_3(s_{2n+1}) = \nu_3(C_n), \tag{135}$$

where n is a non-negative integer. The Eq. (24) follows from Eq. (135) and the fact that $\nu_3(S_n) = \nu_3(s_n)$.

Similarly, by setting $a := 2$ and $b := 3$ in Eq. (31), it follows that:

$$\nu_3(M_{2n+1}(2,3)) = 1 + \nu_3((2n+1)\cdot C_n). \tag{136}$$

By Eq. (12), it follows that

$$s_{2n+2} = M_{2n+1}(2,3). \tag{137}$$

By using Eqns. (136) and (137), it follows that:

$$\nu_3(s_{2n+2}) = 1 + \nu_3(2n+1) + \nu_3(C_n), \tag{138}$$

where n is a non-negative integer.

Now, Eq. (25) follows from Eq. (138) and the fact that $\nu_3(S_n) = \nu_3(s_n)$.

This completes the proof of Theorem 4.

Remark 3 Theorem 4 can be almost proved by using Theorem 3, Eqns. (5), (6), and Kummer's theorem. See [18].

13 Concluding Remarks

Remark 4 Let us consider the following sum $B(n,m,1,-1) = \sum_{k=0}^{n} \binom{n}{k}^m (-1)^k$, where n and m are natural numbers such that $m \geq 2$. Obviously,

$$B(2n-1,m,1,-1) = 0. \tag{139}$$

By using Eq. (7) and applying Kummer's formula, it follows that

$$B(2n,2,1,-1) = (-1)^n \binom{2n}{n}. \tag{140}$$

By using Eq. (8) and applying Dixon's formula, it follows that

$$B(2n,3,1,-1) = (-1)^n \binom{2n}{n}\binom{3n}{2n}. \tag{141}$$

In 1998, Calkin proved [2, Thm. 1, p. 1] that the alternating binomial sum $B(2n,m,1,-1)$ is divisible by $\binom{2n}{n}$ for all non-negative integers n and all positive integers m. In 2007, Guo, Jouhet, and Zeng proved, among other things, two generalizations of Calkin's result [6, Thm. 1.2, Thm. 1.3, p. 2]. In 2010, Cao and Pan [3] proved an extension of Calkin's result [2, Thm. 1]. In 2020, Calkin's result [2, Thm. 1] was proved by using new class of binomial sums [20, Section 5]. See also [17, Section 8, p. 14].

Let a and b be relatively prime integers such that $a+b \neq 0$. Note that:

$$B(n,m,a,b) \equiv a^n \cdot B(n,m,1,-1) \mod (a+b). \tag{142}$$

By Eqns. (139) and (142), it follows that $B(2n-1,m,a,b)$ is always divisible by $a+b$. By Eqns. (140) and (142), it follows that $B(2n,2,a,b)$ is divisible by $a+b$ if and only if $\binom{2n}{n}$ is divisible by $a+b$. Similarly, by Eqns. (141) and (142), it follows that $B(2n,3,a,b)$ is divisible by $a+b$ if and only if $\binom{2n}{n}\binom{3n}{n}$ is divisible by $a+b$.

Furthermore, by Calkin's result [2, Thm. 1, p. 1], if $\binom{2n}{n}$ is divisible by $a+b$ then $B(2n,m,a,b)$ is divisible by $a+b$ for all natural numbers n and m.

Remark 5 It is known [33, page 2] and [34, Eq. (1.5)] that

$$H_n = M_n(1,3), \tag{143}$$

where H_n are restricted hexagonals numbers [7]. By using Theorem 6, it can be shown that:

$$\begin{aligned} \nu_3(H_{2n}) &= \nu_3(C_n), \\ \nu_3(H_{2n+1}) &= 1 + \nu_3(2n+1) + \nu_3(C_n). \end{aligned} \tag{144}$$

Remark 6 It is known [33, page 2] that

$$C_{n+1} = M_n(1,2), \tag{145}$$

where C_n denotes the nth Catalan number. By using Theorem 6, it can be shown that:

$$\begin{aligned} \nu_2(C_{2n+1}) &= \nu_2(C_n), \\ \nu_2(C_{2n+2}) &= 1 + \nu_2(C_n). \end{aligned} \tag{146}$$

Remark 7 By using Theorem 1 and Kummer's theorem [16, Theorem 1. 2, p. 2], we can efficiently calculate $\omega_{a+b}(B(n,2,a,b))$.

Let us consider the following sum:

$$B(4046,2,37,62) = \sum_{k=0}^{4046} \binom{4046}{k}^2 \times 37^{4046-k} 62^k. \tag{147}$$

Here $a := 37$, $b := 62$, $2n := 4046$, and $a+b := 99$. Obviously, $\gcd(37,62) = 1$.

By Eq. (14) of Theorem 1, it follows that:

$$\omega_{99}\left(B(4046,2,37,62)\right) = \omega_{99}\left(\binom{4046}{2023}\right). \tag{148}$$

Since $99 = 3^2 \cdot 11$, it follows $\nu_3(99) = 2$ and $\nu_{11}(99) = 1$. We need to find $\nu_3\left(\binom{4046}{2023}\right)$ and $\nu_{11}\left(\binom{4046}{2023}\right)$.

The expansion of 2023 in the base 3 is $(2202221)_3$. If we add the number $(2202221)_3$ to itself, there are five 'carry-overs' in total. By Kummer's theorem,

$$\nu_3\left(\binom{4046}{2023}\right) = 5. \tag{149}$$

The expansion of 2023 in the base 11 is $(157A)_{11}$. If we add the number $(157A)_{11}$. to itself, there are three 'carry-overs' in total. By Kummer's theorem,

$$\nu_{11}\left(\binom{4046}{2023}\right) = 3. \tag{150}$$

By Eq. (13), it follows that

$$\begin{aligned}\omega_{99}\left(\binom{4046}{2023}\right) &= \min\left\{\left\lfloor\frac{\nu_{11}\left(\binom{4046}{2023}\right)}{\nu_{11}(99)}\right\rfloor,\right.\\ &\qquad\left.\left\lfloor\frac{\nu_3\left(\binom{4046}{2023}\right)}{\nu_3(99)}\right\rfloor\right\}\\ &= \min\left\{\left\lfloor\frac{3}{1}\right\rfloor, \left\lfloor\frac{5}{2}\right\rfloor\right\}\\ &= \min\{3,2\}\\ &= 2.\end{aligned} \tag{151}$$

By Theorem 1, it follows that

$$\omega_{99}\left(B(4046,2,37,62)\right) = 2. \tag{152}$$

Hence, the sum $B(4046,2,37,62)$ is divisible by 99^2 and not divisible by 99^3. Moreover, by Theorem 2, it follows that the sum $B(4046,m,37,62)$ is always divisible by 99^2 for all natural numbers m such that $m \geq 2$.

Also, let us consider the following sum

$$\begin{aligned}B(4047,2,37,62) = \sum_{k=0}^{4047}\binom{4047}{k}^2\\ \times 37^{4047-k}62^k.\end{aligned} \tag{153}$$

By Eq. (15) of Theorem 1, it follows that:

$$\begin{aligned}&\omega_{99}\left(B(4047,2,37,62)\right)\\ &\quad = 1+\omega_{99}\left(4047\binom{4046}{2023}\right).\end{aligned} \tag{154}$$

We have

$$\begin{aligned}\nu_3\left(4047\binom{4046}{2023}\right) &= \nu_3(4047)+\nu_3\left(\binom{4046}{2023}\right)\\ &= 1+5 = 6.\end{aligned} \tag{155}$$

Furthermore,

$$\begin{aligned}\nu_{11}\left(4047\binom{4046}{2023}\right) &= \nu_{11}(4047)+\nu_{11}\left(\binom{4046}{2023}\right)\\ &= 0+3 = 3.\end{aligned} \tag{156}$$

By Eq. (13), it follows that:

$$\begin{aligned}\omega_{99}\left(4047\binom{4046}{2023}\right) &= \min\left\{\left\lfloor\frac{\nu_{11}\left(4047\binom{4046}{2023}\right)}{\nu_{11}(99)}\right\rfloor,\right.\\ &\qquad\left.\left\lfloor\frac{\nu_3\left(4047\binom{4046}{2023}\right)}{\nu_3(99)}\right\rfloor\right\}\\ &= \min\left\{\left\lfloor\frac{3}{1}\right\rfloor, \left\lfloor\frac{6}{2}\right\rfloor\right\}\\ &= \min\{3,3\} = 3.\end{aligned} \tag{157}$$

By Eq. (15) of Theorem 1, it follows that:

$$\begin{aligned}\omega_{99}\left(B(4047,2,37,62)\right) &= 1+\omega_{99}\left(4047\binom{4046}{2023}\right)\\ &= 1+3 = 4.\end{aligned} \tag{158}$$

It follows that:

$$\omega_{99}\left(B(4047,2,37,62)\right) = 4. \tag{159}$$

Hence, the sum $B(4047,2,37,62)$ is divisible by 99^4 and not divisible by 99^5.

Moreover, by Theorem 2, it follows that the sum $B(4047,m,37,62)$ is always divisible by 99^4 for all natural numbers m such that $m \geq 2$.

Remark 8 By using Eqns. (35) and (37), we can derive a slight generalization of MacMahon's formula (8). It can be shown that:

$$\begin{aligned}M_B(n,j,2;a,b) = \binom{n}{j}\sum_{l=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n}{2l}\binom{2l}{l}\\ \times\binom{n+l-j}{n}(ab)^l(a+b)^{n-2l}.\end{aligned} \tag{160}$$

By setting $j := 0$ in Eq. (160), Eq. (8) follows.

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