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Quantitative Non-Unique Ćirić Fixed-Point Theory under Residual-Sensitive C-Class Controls

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Abstract Building on the non-unique fixed-point theory of Čirić and its known extensions by c-comparison functions, we introduce a residual-sensitive three-variable C-class condition designed according to an orbit-extraction principle. The global condition involves the mutual distance and the two pointwise residuals, but on a Picard pair (x, Tx) it collapses to the scalar recurrence $d(Tx, T^2x) \le \phi\big(d(x, Tx)\big)$, which governs summable decay of successive Picard displacements. This structural hypothesis and its diagonal reduction, rather than the use of a nonlinear comparison function by itself, constitute the principal contribution. For an orbitally continuous self-map on a T-orbitally complete metric space, the recurrence makes the map weakly Picard and yields both an a priori tail estimate and a genuine a posteriori estimate in terms of the current residual $d(T^n x, T^{n+1}x)$. These bounds induce an explicit retraction-displacement modulus. Consequently, we obtain set-wise generalized Ulam stability, well-posedness with respect to the fixed-point set, and a Hausdorff data-dependence estimate for the entire fixed-point set under perturbations of the operator. A nonlinear example with two fixed points exhibits non-geometric residual decay and cannot satisfy a corresponding constant-coefficient inequality with $q < 1$. A second example shows that the linear Hausdorff perturbation bound is sharp. The results place residual-sensitive C-class conditions within the quantitative theory of weakly Picard operators without imposing uniqueness.

Keywords: non-unique fixed point; Čirić mapping; c-comparison function; weakly Picard operator; C-class control; a posteriori error estimate; Ulam stability; data dependence.

1 Introduction

Banach's contraction principle remains the basic model for constructive fixed-point theory [3], but its uniform coefficient forces uniqueness and excludes many nonuniform iterative mechanisms. Classical alternatives due to Kannan, Chatterjea, Rakotch and Čirić substantially broadened the admissible contractive structure. Of particular relevance here is Čirić's non-unique framework, in which a minimum-distance inequality is compatible with several fixed points and nevertheless forces Picard convergence [6]. Non-unique Čirić theory has since been extended through rational and hybrid inequalities, interpolation, generalized metric spaces and nonlinear comparison functions; see, for example, [1, 7–11]. An important point for the present paper is that this literature already provides a replacement of a constant factor $q < 1$ with a nonlinear comparison function. In particular, the b-metric result of Alsulami, Karapınar and Rakočević [1] contains, in the metric case, the estimate

$$m_T(x, y) - c_T(x, y) \le \phi\big(d(x, y)\big). \tag{1}$$

with a c-comparison function ϕ. Thus, nonlinear control alone is not the contribution sought here.

A second relevant line of development is the theory of weakly Picard operators. A map is weakly Picard when every Picard orbit converges to some fixed point, not necessarily the same one. This point of view naturally accommodates non-unique fixed points and leads to retraction-displacement estimates, well-posedness, Ulam-type stability and data dependence [4, 12–14]. These quantitative consequences are particularly useful when the fixed-point set is not a singleton.

C-class functions provide another flexible language for writing generalized contractive inequalities. They originated as two-variable controls and continue to appear in recent fixed-point results in various settings [2, 15]. In a non-unique

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Čirić argument, however, an unrestricted multivariable C-class function does not by itself give the summable recurrence needed to control the Picard tail. The decisive information is its behavior on the configuration generated by two consecutive iterates.

The motivation for the residual-sensitive formulation comes from a simple distinction between *global verification* and *orbital convergence*. As regards global, a generalized contractive condition may naturally depend not only on $d(x,y)$, but also on the residuals $d(x,Tx)$ and $d(y,Ty)$, which measure the local displacement of each point under the operator. These quantities can reflect nonuniform behavior that a single constant coefficient cannot represent. Along a Picard orbit, however, convergence is governed by the successive residuals

$$r_n = d\big(T^n x, T^{n+1} x\big). \tag{2}$$

The useful design principle is therefore to allow residual dependence in the global inequality while requiring that, on the Picard pair (x,Tx), the auxiliary three-variable control reduces to a one-dimensional comparison rule for r_{n+1} in terms of r_n. This principle explains the diagonal requirement introduced below. When $y = Tx$, the first two inputs of the auxiliary function coincide, because $d(x,y) = d(x,Tx)$, and the crossing term in the Čirić inequality vanishes. The condition $F(t,t,s) \le \phi(t)$ then extracts the recurrence

$$d\big(Tx, T^2 x\big) \le \phi\big(d(x,Tx)\big). \tag{3}$$

Thus, the three-variable hypothesis is not introduced merely as another generalized contraction. It is a globally verifiable structural condition deliberately shaped so that its restriction to consecutive iterates yields the orbital recurrence required for quantitative convergence. The paper's logical structure follows this motivation. We first isolate the orbital summability principle and derive a priori and a posteriori Picard estimates from the scalar residual recurrence. We then prove that the residual-sensitive C-class condition reduces to that recurrence. The principal novelty therefore lies in the structural hypothesis together with this reduction mechanism. The retraction-displacement estimate, generalized Ulam stability, set-wise well-posedness, and Hausdorff data dependence follow from the resulting quantitative weakly Picard framework. The linear specialization is shown to be sharp. We also include a nonlinear two-fixed-point example for which the effective residual ratio tends to one, so no uniform coefficient $q < 1$ is possible. We do not claim that this example lies outside all previously known nonlinear comparison-function conditions; it illustrates why a residual-sensitive global formulation and a current-residual error estimate are natural in a non-geometric regime.

2 Preliminaries and residual-sensitive controls

Throughout, (X,d) is a metric space and $T : X \to X$ is a self-map. We write

$$\begin{aligned} &\mathrm{Fix}(T) = \{x \in X : Tx = x\},\\ &O(x,T) = \{T^n x : n \in \mathbb{N} \cup \{0\}\}. \end{aligned} \tag{4}$$

For an initial point $x_0 \in X$, let $x_n = T^n x_0$ and define the *Picard residual sequence*

$$r_n = d(x_n, x_{n+1}) = d\big(T^n x_0, T^{n+1} x_0\big), \qquad n \ge 0. \tag{5}$$

The term *residual-sensitive* refers to a contractive condition that records pointwise displacements such as $d(x,Tx)$ and $d(y,Ty)$ and is designed so that, when $y = Tx$, it yields a recurrence controlling r_{n+1} by r_n.

Definition 1 The metric space (X,d) is called T-orbitally complete if every Cauchy sequence contained in an orbit $O(x,T)$ converges to a point of X.

Definition 2 The map T is called orbitally continuous if, whenever $T^{n_j} x \to z$ for some $x \in X$ and subsequence $\{n_j\}$, one has $T^{n_j+1} x \to Tz$.

Ordinary completeness implies orbital completeness, and ordinary continuity implies orbital continuity.

Definition 3 A map $\phi : [0,\infty) \to [0,\infty)$ is called a c-comparison function if ϕ is nondecreasing, $\phi(0) = 0$, and

$$\Theta_\phi(t) := \sum_{k=0}^{\infty} \phi^k(t) < \infty \qquad (t > 0), \tag{6}$$

where ϕ^k denotes the k-fold iterate.

This formulation is equivalent to the summability property commonly associated with c-comparison functions in metric fixed-point theory; see [1, 5, 12]. In particular,

$$\phi(t) < t \qquad (t > 0), \tag{7}$$

and Θ_ϕ is nondecreasing and continuous at zero. The linear choice $\phi(t) = qt$, $0 \le q < 1$, gives

$$\Theta_\phi(t) = \frac{t}{1-q}. \tag{8}$$

Definition 4 A continuous function $F : [0,\infty)^3 \to [0,\infty)$ is called a three-variable modified C-class function if

$$F(a,b,c) \le a \qquad (a,b,c \ge 0), \tag{9}$$

and

$$F(a,b,c) = a \quad \Longrightarrow \quad a = 0 \text{ or } b = 0 \text{ or } c = 0. \tag{10}$$

Given a c-comparison function ϕ, the function F is called ϕ-diagonally admissible if

$$F(t,t,s) \le \phi(t) \qquad (t, s \ge 0). \tag{11}$$

The terminology in (9)–(10) is motivated by the C-class principle of Ansari [2]. The additional condition (11) is the residual-sensitive requirement: it prescribes exactly what the auxiliary function must do on the diagonal configuration generated by a Picard pair.

Proposition 1 *Let ϕ be a c-comparison function. Then*

$$F_\phi(a,b,c)=\phi(a)+\big(a-\phi(a)\big)\frac{|a-b|}{1+|a-b|}\frac{c}{1+c} \tag{12}$$

is a ϕ-diagonally admissible three-variable modified C-class function.

Proof Continuity and nonnegativity are immediate. Since both fractions in (12) belong to $[0,1)$,

$$\phi(a)\le F_\phi(a,b,c)\le a. \tag{13}$$

If $a>0$, then $\phi(a)<a$ and the product of the two fractions is strictly smaller than one, so $F_\phi(a,b,c)<a$. Hence equality with a can occur only when $a=0$, which is stronger than (10). Finally, $b=a$ gives $F_\phi(a,a,c)=\phi(a)$.

For $x,y\in X$, define the Čirić minimum terms

$$m_T(x,y)=\min\{d(Tx,Ty),d(x,Tx),d(y,Ty)\}, \tag{14}$$
$$c_T(x,y)=\min\{d(x,Ty),d(y,Tx)\}. \tag{15}$$

Definition 5 Let ϕ be a c-comparison function and let F be ϕ-diagonally admissible. We call T an (F,ϕ) residual-sensitive non-unique Čirić map if

$$m_T(x,y)\le F\big(d(x,y),d(x,Tx),d(y,Ty)\big)+c_T(x,y) \tag{16}$$

for all $x,y\in X$.

Condition (16) embodies the orbit-extraction principle described in the introduction. Its three inputs retain information about the separation of x and y and their individual residuals, so the inequality can be checked globally without requiring a uniform contraction ratio. At the same time, the diagonal admissibility of F is tuned to the single configuration needed in the Picard argument. If $u,v\in \mathrm{Fix}(T)$, then $m_T(u,v)=0$, so (16) does not constrain the distance between distinct fixed points. This is why uniqueness is not built into the condition.

3 Orbital summability and Picard convergence

We begin with the mechanism that actually drives the iterative argument.

Theorem 1 **(Orbital summability principle)** *Let $T:X\to X$ be orbitally continuous and let (X,d) be T-orbitally complete. Assume that there exists a c-comparison function ϕ such that*

$$d(Tx,T^2x)\le\phi\big(d(x,Tx)\big)\qquad (x\in X). \tag{17}$$

Then T is a weakly Picard operator: for every $x_0\in X$, the Picard sequence $x_{n+1}=Tx_n$ converges to a point $P_Tx_0\in$ Fix(T).

If $r_n=d(x_n,x_{n+1})$, then

$$r_n\le\phi^n(r_0),\qquad n\ge 0, \tag{18}$$

and the following estimates hold:

$$d(x_n,P_Tx_0)\le\sum_{j=n}^{\infty}\phi^j(r_0)=\Theta_\phi\big(\phi^n(r_0)\big), \tag{19}$$
$$d(x_n,P_Tx_0)\le\Theta_\phi(r_n). \tag{20}$$

Equation (19) is an a priori estimate, while (20) is an a posteriori estimate based on the current Picard residual.

Proof If $r_n=0$ for some n, then $x_n\in\mathrm{Fix}(T)$ and the orbit is constant thereafter. Otherwise, (17) applied at x_n gives

$$r_{n+1}\le\phi(r_n). \tag{21}$$

Monotonicity of ϕ yields (18) by induction. For $m>n$, the triangle inequality gives

$$d(x_n,x_m)\le\sum_{j=n}^{m-1}r_j\le\sum_{j=n}^{m-1}\phi^j(r_0). \tag{22}$$

The series in (6) has a convergent tail, hence $\{x_n\}$ is Cauchy. Orbital completeness gives a limit $p\in X$. Orbital continuity implies

$$x_{n+1}=Tx_n\longrightarrow Tp. \tag{23}$$

while $x_{n+1}\to p$, so $Tp=p$. Set $P_Tx_0=p$.

Letting $m\to\infty$ in the preceding estimate gives (19). Starting instead at the observed residual r_n, repeated use of (17) yields

$$r_{n+k}\le\phi^k(r_n),\qquad k\ge 0. \tag{24}$$

so

$$d(x_n,P_Tx_0)\le\sum_{k=0}^{\infty}r_{n+k}\le\sum_{k=0}^{\infty}\phi^k(r_n)=\Theta_\phi(r_n). \tag{25}$$

which is (20).

The next proposition is the structural bridge between the global three-variable inequality and the scalar recurrence that controls the orbit.

Proposition 2 **(Diagonal reduction to the residual recurrence)** *Let T satisfy (16). Then*

$$d(Tx,T^2x)\le\phi\big(d(x,Tx)\big)\qquad (x\in X). \tag{26}$$

Proof Fix $x \in X$ and put

$$r = d(x, Tx), \qquad s = d\big(Tx, T^2x\big). \tag{27}$$

If $r = 0$, then $Tx = x$ and $s = 0$. Suppose $r > 0$. Apply (16) to the pair (x, Tx). Since

$$c_T(x, Tx) = \min\big\{d(x, T^2x), d(Tx, Tx)\big\} = 0. \tag{28}$$

and

$$m_T(x, Tx) = \min\{s, r, s\} = \min\{r, s\}. \tag{29}$$

we obtain

$$\min\{r, s\} \le F(r, r, s) \le \phi(r). \tag{30}$$

If $s \ge r$, then $r \le \phi(r) < r$, a contradiction. Hence $s < r$, and therefore $s \le \phi(r)$.

Proposition 2 identifies the mathematical core of the framework. The global inequality (16) is allowed to contain three variables and residual information, but the Picard substitution $y = Tx$ forces the crossing term to vanish and the first two arguments of F to coincide. Diagonal admissibility then removes the auxiliary structure and leaves exactly the scalar recurrence (17). The subsequent quantitative conclusions depend on this recurrence through Theorem 1.

Theorem 2 (Quantitative non-unique Čirić theorem) *Let $T : X \to X$ be orbitally continuous on a T-orbitally complete metric space. If T satisfies (16) for a c-comparison function ϕ and a ϕ-diagonally admissible modified C-class function F, then all conclusions of Theorem 1 hold.*

Proof Proposition 2 gives (17), so Theorem 1 applies.

Corollary 1 (Classical scalar comparison form) *Under the orbital assumptions above, suppose*

$$m_T(x, y) - c_T(x, y) \le \phi\big(d(x, y)\big) \tag{31}$$

for all $x, y \in X$, where ϕ is a c-comparison function. Then T is weakly Picard and satisfies (18)–(20).

Proof Take $F(a, b, c) = \phi(a)$. Then (16) is exactly (31), and F is ϕ-diagonally admissible.

Remark 1 Corollary 1 is an explicitly quantitative metric-space form of the nonlinear non-unique Čirić mechanism already present in [1]. The role of Theorem 2 is not merely to replace a constant coefficient by a comparison function. Its new element is the residual-sensitive structural condition and the proof that this globally stated condition reduces, on Picard pairs, to the recurrence that activates the quantitative weakly Picard theory developed below.

Corollary 2 (Constant coefficient) *If $\phi(t) = qt$, $0 \le q < 1$, then*

$$d(T^n x, P_T x) \le \frac{q^n}{1-q} d(x, Tx) \tag{32}$$

and

$$d(T^n x, P_T x) \le \frac{d(T^n x, T^{n+1}x)}{1-q}. \tag{33}$$

Corollary 3 (Absence of nontrivial periodic points) *Under the assumptions of Theorem 2, every periodic point of T is a fixed point.*

Proof If x has prime period $k > 1$, then all residuals along the cycle are positive and satisfy $r_{n+1} < r_n$. After one full period this gives

$$r_0 > r_1 > \cdots > r_k = r_0. \tag{34}$$

which is impossible.

4 Retraction-displacement, stability and data dependence

The results in this section are unified by one estimate. The a posteriori bound in Theorem 2 first gives

$$d(x, P_T x) \le \Theta_\phi\big(d(x, Tx)\big). \tag{35}$$

where $P_T x$ is the Picard limit. Since $P_T x \in \mathrm{Fix}(T)$, the same inequality bounds the distance from any approximate fixed point to the entire fixed-point set. Generalized Ulam stability and well-posedness follow by applying this estimate to points with small residual, while Hausdorff data dependence follows by viewing a fixed point of one operator as an approximate fixed point of the perturbed operator. Thus, Proposition 3 and Theorems 3–4 are different consequences of a single residual-to-solution-set modulus.

Proposition 3 (Picard limit map and retraction-displacement) *Under the assumptions of Theorem 2, define*

$$P_T : X \to \mathrm{Fix}(T), \qquad P_T x = \lim_{n\to\infty} T^n x. \tag{36}$$

Then

$$P_T \circ T = P_T, \qquad P_T|_{\mathrm{Fix}(T)} = I_{\mathrm{Fix}(T)}, \qquad P_T^2 = P_T, \tag{37}$$

and

$$d(x, P_T x) \le \Theta_\phi\big(d(x, Tx)\big) \qquad (x \in X). \tag{38}$$

Thus, P_T is a set retraction onto $\mathrm{Fix}(T)$ and satisfies an explicit retraction-displacement estimate.

Proof The identities in (37) follow from the fact that the orbits of x and Tx differ only in their first term and that fixed points generate constant orbits. Estimate (38) is (20) with $n = 0$.

The wording "set retraction" is used because continuity of P_T is not assumed; this agrees with the weakly Picard framework in [13, 14].

Definition 6 For a nonempty subset $A \subseteq X$, let

$$\mathrm{dist}(x, A) = \inf_{a \in A} d(x, a). \tag{39}$$

We say that the fixed-point problem for T is set-wise well posed if every sequence $\{z_n\}$ satisfying $d(z_n, Tz_n) \to 0$ also satisfies

$$\mathrm{dist}\big(z_n, \mathrm{Fix}(T)\big) \longrightarrow 0. \tag{40}$$

We say that it is generalized Ulam stable with modulus Ω if

$$d(x, Tx) \le \varepsilon \quad \Longrightarrow \quad \mathrm{dist}\big(x, \mathrm{Fix}(T)\big) \le \Omega(\varepsilon). \tag{41}$$

Theorem 3 (Generalized Ulam stability and well-posedness) *Under the assumptions of Theorem 2, the fixed-point problem is generalized Ulam stable with modulus Θ_ϕ:*

$$d(x, Tx) \le \varepsilon \quad \Longrightarrow \quad \mathrm{dist}(x, \mathrm{Fix}(T)) \le \Theta_\phi(\varepsilon). \tag{42}$$

It is also set-wise well posed.

Proof Since $P_T x \in \mathrm{Fix}(T)$, Proposition 3 gives

$$\mathrm{dist}\big(x, \mathrm{Fix}(T)\big) \le d(x, P_T x) \le \Theta_\phi\big(d(x, Tx)\big). \tag{43}$$

Monotonicity yields (42). If $d(z_n, Tz_n) \to 0$, then

$$\mathrm{dist}\big(z_n, \mathrm{Fix}(T)\big) \le \Theta_\phi\big(d(z_n, Tz_n)\big) \longrightarrow 0. \tag{44}$$

because Θ_ϕ is continuous at zero.

For perturbations, define the excess and Hausdorff distance between nonempty sets $A, B \subseteq X$ by

$$\begin{aligned} \mathrm{exc}(A, B) &= \sup_{a \in A} \mathrm{dist}(a, B), \\ H(A, B) &= \max\big\{\mathrm{exc}(A, B), \mathrm{exc}(B, A)\big\}. \end{aligned} \tag{45}$$

Theorem 4 (Hausdorff data dependence of the fixed-point set) *Let $T, S : X \to X$ each satisfy the assumptions of Theorem 2, with c-comparison functions ϕ_T and ϕ_S, respectively. Suppose*

$$\delta := \sup_{x \in X} d(Tx, Sx) < \infty. \tag{46}$$

Then

$$\mathrm{exc}(\mathrm{Fix}(T), \mathrm{Fix}(S)) \le \Theta_{\phi_S}(\delta), \tag{47}$$

$$\mathrm{exc}(\mathrm{Fix}(S), \mathrm{Fix}(T)) \le \Theta_{\phi_T}(\delta), \tag{48}$$

and consequently

$$H(\mathrm{Fix}(T), \mathrm{Fix}(S)) \le \max\{\Theta_{\phi_T}(\delta), \Theta_{\phi_S}(\delta)\}. \tag{49}$$

If both maps have the same control ϕ, then

$$H(\mathrm{Fix}(T), \mathrm{Fix}(S)) \le \Theta_\phi(\delta). \tag{50}$$

Proof Let $u \in \mathrm{Fix}(T)$. Then

$$d(u, Su) = d(Tu, Su) \le \delta. \tag{51}$$

Applying Theorem 3 to the map S gives

$$\mathrm{dist}\big(u, \mathrm{Fix}(S)\big) \le \Theta_{\phi_S}(\delta). \tag{52}$$

Taking the supremum over $u \in \mathrm{Fix}(T)$ proves (47). Interchanging T and S gives (48), and (49)–(50) follow from (45).

Corollary 4 (Linear perturbation estimate) *If both maps in Theorem 4 have the common control $\phi(t) = qt$, $0 \le q < 1$, then*

$$H(\mathrm{Fix}(T), \mathrm{Fix}(S)) \le \frac{\delta}{1 - q}. \tag{53}$$

The abstract retraction-displacement, Ulam and perturbation viewpoints are established parts of weakly Picard theory [13, 14]. Theorems 3 and 4 identify the explicit modulus produced when the residual-sensitive Čirić condition is reduced to the orbital recurrence.

5 Examples

The two examples have complementary purposes. Example 1 uses the genuine three-variable function F_ϕ and shows how residual sensitivity accommodates non-geometric local decay while still extracting a summable Picard recurrence. Example 2 uses the linear special case to isolate the fixed-set perturbation mechanism and demonstrate the sharpness of the resulting Hausdorff bound.

5.1 A nonlinear non-geometric example with two fixed points

Consider

$$\phi(t) = \frac{t}{1 + \sqrt{t}}, \qquad t \ge 0. \tag{54}$$

Lemma 1 *The function ϕ in (54) is a c-comparison function. It is increasing, concave and subadditive.*

Proof Direct differentiation shows that ϕ is increasing and concave on $(0,\infty)$, with $\phi(0)=0$ and $\phi(t)<t$ for $t>0$. A nonnegative concave function vanishing at zero is subadditive.

It remains to prove summability. Let $t_{n+1}=\phi(t_n)$, $t_0>0$, and put $s_n=t_n^{-1/2}$. Then

$$s_{n+1}=\sqrt{s_n(s_n+1)}. \tag{55}$$

so

$$s_{n+1}-s_n=\frac{1}{\sqrt{1+1/s_n}+1}>0. \tag{56}$$

The sequence $\{s_n\}$ cannot have a finite positive limit, because such a limit s would satisfy $s=\sqrt{s(s+1)}$. Hence $s_n\to\infty$. Choose N such that $s_N\geq 1$. For $n\geq N$,

$$s_{n+1}-s_n\geq\frac{1}{\sqrt{2}+1}=\sqrt{2}-1. \tag{57}$$

Therefore

$$t_n\leq\frac{1}{\left[s_N+(n-N)(\sqrt{2}-1)\right]^2},\qquad n\geq N. \tag{58}$$

and $\sum_{n=0}^{\infty}t_n<\infty$.

Example 1 Let

$$X=[0,1]\cup\{2\}. \tag{59}$$

with the Euclidean metric, and define

$$T(x)=\phi(x)=\frac{x}{1+\sqrt{x}}\quad(x\in[0,1]),\qquad T(2)=2. \tag{60}$$

Then X is complete, T is continuous, and

$$\mathrm{Fix}(T)=\{0,2\}. \tag{61}$$

Let F_ϕ be the genuine three-variable control in (12). For $x,y\in[0,1]$, subadditivity and monotonicity give

$$|\phi(x)-\phi(y)|\leq\phi(|x-y|). \tag{62}$$

Hence

$$\begin{aligned} m_T(x,y)&\leq d(Tx,Ty)\\ &\leq\phi(d(x,y))\\ &\leq F_\phi(d(x,y),d(x,Tx),d(y,Ty)). \end{aligned} \tag{63}$$

If one of the points is 2, then $d(2,T2)=0$, so $m_T(x,y)=0$. Thus (16) holds on all of X.

The residual-sensitive formulation is particularly natural here. For arbitrary pairs, F_ϕ may depend on the mismatch between the mutual distance $d(x,y)$ and the residual $d(x,Tx)$, as well as on the second residual $d(y,Ty)$. This gives a flexible global sufficient condition. On a Picard pair (x,Tx), however, the first two arguments are equal, so the correction term in (12) vanishes and

$$F_\phi(r,r,s)=\phi(r). \tag{64}$$

The global three-variable inequality therefore retains off-orbit flexibility without weakening the exact recurrence needed to control successive residuals.

The example does not admit any corresponding constant coefficient $q<1$. For $x\in(0,1]$, put

$$r(x)=d(x,Tx)=x-\phi(x)=\frac{x^{3/2}}{1+\sqrt{x}}. \tag{65}$$

For the pair (x,Tx), first note that

$$\frac{r(Tx)}{r(x)}=\frac{1}{\sqrt{1+\sqrt{x}}+\sqrt{x}}<1, \tag{66}$$

so $m_T(x,Tx)=r(Tx)$ and $c_T(x,Tx)=0$. If a constant-coefficient inequality

$$m_T(x,y)-c_T(x,y)\leq qd(x,y). \tag{67}$$

held for some $q<1$, then (66) would give $r(Tx)/r(x)\leq q$ for every $x>0$. But

$$\lim_{x\downarrow 0}\frac{r(Tx)}{r(x)}=1. \tag{68}$$

a contradiction.

Thus the Picard decay is genuinely non-geometric. Nevertheless, Theorem 2 gives the computable residual estimate

$$d(T^nx,P_Tx)\leq\Theta_\phi(d(T^nx,T^{n+1}x)). \tag{69}$$

The current displacement is therefore not only part of the structural hypothesis; it is also an observable stopping quantity that bounds the remaining distance to the orbit's limiting fixed point. Every orbit starting in $[0,1]$ converges to 0, while the orbit starting at 2 remains at 2.

5.2 Sharpness of the linear data-dependence bound

Example 2 Let

$$X=[0,1]\cup[2,3]. \tag{70}$$

with the Euclidean metric and fix $\varepsilon\in(0,1)$. Define

$$T(x)=\begin{cases}x/2, & x\in[0,1],\\ 2+(x-2)/2, & x\in[2,3],\end{cases} \tag{71}$$

and

$$S_\varepsilon(x)=\begin{cases}\varepsilon+(x-\varepsilon)/2, & x\in[0,1],\\ 2+\varepsilon+\left(x-(2+\varepsilon)\right)/2, & x\in[2,3].\end{cases} \tag{72}$$

Both maps are continuous self-maps of X. Their fixed-point sets are

$$\mathrm{Fix}(T)=\{0,2\},\qquad\mathrm{Fix}(S_\varepsilon)=\{\varepsilon,2+\varepsilon\}. \tag{73}$$

For points in the same connected component,

$$d(Tx, Ty) = \frac{1}{2} d(x, y),$$
$$d(S_\varepsilon x, S_\varepsilon y) = \frac{1}{2} d(x, y). \tag{74}$$

For points in different components, the crossing term c_T (respectively c_{S_ε}) is at least 1, while m_T (respectively m_{S_ε}) is at most $1/2$. Hence both maps satisfy the classical non-unique Ćirić inequality with $q = 1/2$.

Moreover,

$$\delta = \sup_{x \in X} d(Tx, S_\varepsilon x) = \frac{\varepsilon}{2}, \tag{75}$$

whereas

$$H\big(\mathrm{Fix}(T), \mathrm{Fix}(S_\varepsilon)\big) = \varepsilon. \tag{76}$$

Since $q = 1/2$, Corollary 4 gives

$$\frac{\delta}{1-q} = \frac{\varepsilon/2}{1/2} = \varepsilon, \tag{77}$$

which equals (76). Therefore the coefficient in the linear Hausdorff data-dependence bound is sharp.

This example is the residual-insensitive boundary case $F(a, b, c) = qa$ of the proposed framework. Its role is to show that, once the structural reduction has produced the residual-to-fixed-set modulus, the same modulus converts the measurable operator discrepancy δ into a best-possible bound for the displacement of the full fixed-point sets.

6 Relation to existing theory

Table 1 summarizes the positioning of the present result. The comparison is intentionally conservative: the existence part of the scalar c-comparison theory is acknowledged as known. The main novelty is the residual-sensitive three-variable structural hypothesis, its diagonal reduction to the Picard residual recurrence, and the explicit quantitative consequences derived from that recurrence.

Table 1 Position of the present framework relative to representative results.

Framework	Main mechanism	Relation to the present paper
Ćirić [6]	Minimum-distance inequality with constant $q < 1$; non-unique fixed points allowed.	Recovered by $F(a, b, c) = qa$ and $\phi(t) = qt$. The present paper additionally derives current-residual error and fixed-set perturbation estimates.
Alsulami–Karapınar–Rakočević [1]	Non-unique Ćirić inequality with a c/b-comparison function.	Establishes that nonlinear summable comparison control is already known. Corollary 1 is a quantitative metric specialization, not a novelty claim.
Karapınar–Agarwal [7] and Karapınar [8]	Non-unique Ćirić extensions in generalized settings and through interpolation.	Complementary in setting and contractive form.
C-class methods [2, 15]	Functional auxiliary controls of C-class type.	The three-variable control is used as a residual-sensitive sufficient structural condition. Its diagonal restriction is what produces the Picard recurrence.
Weakly Picard theory [13, 14]	Retraction-displacement, well-posedness, Ulam stability and data dependence.	Provides the conceptual framework. Here the modulus Θ_ϕ is derived explicitly from the residual recurrence generated by the proposed structural condition.
Almost weakly Picard operators [4]	Recent weak forms of Picard convergence and comparison-function controls.	Shows the continuing relevance of non-unique iterative convergence; the present focus is the global-to-orbital reduction, quantitative residual control and fixed-set stability.

Three points deserve emphasis. First, the diagonal requirement (11) is not presented as a replacement for established c-comparison theory. It is a practical structural requirement ensuring that a flexible three-variable inequality collapses, on a Picard pair (x, Tx), to the summable residual recurrence (17). Second, the novelty is not the formal appearance of residual terms alone; it is the deliberate coupling of a globally verifiable residual-sensitive condition with a precise reduction mechanism and an explicit quantitative weakly Picard modulus. Third, the conclusions concern the whole fixed-point set. This is essential in the non-unique setting: one cannot in general estimate the displacement between two selected limits of perturbed maps, but Theorem 4 controls the Hausdorff distance between the fixed-point sets.

7 Conclusion

A non-unique Ćirić theorem with nonlinear summable comparison control should not be justified merely by replacing a constant q with a nonlinear function, because such comparison-function extensions are already available. The contribution developed here is structural and quantitative. The essential orbital assumption was isolated as a recurrence for the Picard residual sequence. A residual-sensitive three-variable C-class inequality was then introduced as a globally stated sufficient condition whose diagonal restriction, on (x, Tx), produces precisely that recurrence.

This reduction bridges to the quantitative theory. It yields a priori information from the initial residual and a genuine a posteriori estimate from the current Picard displacement. The latter gives an explicit retraction-displacement modulus and, through the same estimate, set-wise generalized Ulam stability and well-posedness. For operator perturbations, the modulus controls the Hausdorff distance between the complete fixed-point sets; the constant-coefficient version of this estimate is sharp.

The framework also has clear limitations. Condition (16) is sufficient rather than necessary: some maps may satisfy the orbital recurrence directly without admitting a convenient global verification through a chosen three-variable function F. The conclusions require orbital completeness and orbital continuity, and the Picard limit retraction P_T need not be continuous. Because uniqueness is deliberately not imposed, the theory controls distance to the fixed-point set rather than selection of a particular fixed point. The Hausdorff perturbation result further assumes a finite uniform discrepancy $\sup_{x\in X} d(Tx, Sx)$, which may be restrictive on unbounded domains.

The present formulation is restricted to standard metric spaces to keep the global-to-orbital mechanism transparent. Extensions to b-metric spaces would require the weighted comparison series familiar from b-comparison functions. Multivalued weakly Picard mappings and ordered or graph-based settings are further natural directions, provided that an explicit summable residual estimate and an appropriate fixed-set distance can be preserved.

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