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A Dynamical Transition from Confinement to Blow-Up in an Oscillatory Differential Equation

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Abstract In this paper, we investigate the nonlinear ordinary differential equation

$$y' = y^p(\sin y + \varepsilon), \qquad p > 1,$$

with positive initial data. We show that the qualitative behavior of solutions strongly depends on the perturbation parameter ε. For $0 \leq \varepsilon < 1$, the nonlinearity possesses infinitely many zeros, which generate invariant intervals trapping the trajectories. Consequently, all solutions are global and converge to equilibrium points. In contrast, when $\varepsilon > 1$, all zeros disappear, and the equation becomes strictly positive, leading to finite-time blow-up for every positive solution via an Osgood-type argument. The results provide a simple example illustrating how a small structural perturbation may completely change the global dynamics of an oscillatory differential equation.

Keywords Finite-time blow-up · Global existence · Oscillatory nonlinearity · Invariant intervals · Ordinary differential equations · Osgood criterion

1 Introduction

The qualitative study of nonlinear ordinary differential equations has attracted considerable attention because of its importance in nonlinear analysis, mathematical physics, population dynamics, and reaction–diffusion theory. In particular, researchers have extensively investigated questions of global existence, asymptotic behaviour, and finite-time blow-up over the past decades; see, for instance, the classical monographs by Hartman [1], Hale [2], and Friedman [3].

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A fundamental model in this context is the scalar autonomous equation

$$y'(t) = f(y(t)), \tag{1}$$

where the long-time behavior of solutions strongly depends on the growth and sign properties of the nonlinearity f. One of the classical criteria for finite-time blow-up is the Osgood criterion introduced by Osgood [4], which states that blow-up occurs whenever

$$\int^{+\infty} \frac{ds}{f(s)} < +\infty, \tag{2}$$

under suitable positivity assumptions on f. This criterion has played a major role in the qualitative theory of differential equations and has been extended in several directions; see, e.g., [5, 6].

For superlinear nonlinearities such as

$$f(y) = y^p, \qquad p > 1, \tag{3}$$

all positive solutions blow up in finite time. However, the situation changes drastically when the nonlinearity possesses zeros or oscillatory behavior. In such a case, equilibrium points may generate invariant regions trapping the trajectories and preventing blow-up phenomena.

Oscillatory nonlinearities appear naturally in several mathematical models and may lead to unexpected qualitative behaviors. In particular, the equation

$$y'(t) = y(t)^p \sin(y(t)) \tag{4}$$

provides a simple example in which infinitely many equilibria divide the phase space into invariant intervals. Consequently, trajectories remain confined between two consecutive zeros and converge asymptotically toward equilibrium points.

The purpose of the present paper is to investigate how a constant perturbation modifies this confinement mechanism. More precisely, we study the equation

$$y'(t) = y(t)^p\big(\sin(y(t)) + \varepsilon\big), \qquad p > 1, \tag{5}$$

where $\varepsilon \geq 0$ is a real parameter, we show that the qualitative behavior of solutions undergoes a sharp transition depending on the value of ε. When $0 \leq \varepsilon < 1$, the perturbed nonlinearity still possesses infinitely many zeros. These zeros continue to generate invariant intervals, and all solutions remain global. On the other hand, when $\varepsilon > 1$, all equilibria disappear because $\sin y + \varepsilon > 0$ for every $y \in \mathbb{R}$. The equation then behaves like a purely superlinear differential equation, and all positive solutions blow up in finite time by an Osgood-type argument.

Therefore, this work illustrates how a simple perturbation can destroy the invariant-interval structure generated by oscillatory nonlinearities, producing a transition from global existence to finite-time blow-up.

The paper is organized as follows. Section 2 introduces the initial-value problem. Section 3 analyzes the equilibrium structure and the associated invariant intervals. Section 3.1 establishes the global-existence and blow-up alternatives. Section 4 gives an estimate of the blow-up time. Concluding remarks are presented in Section 5.

2 Setting and assumptions

We consider the initial-value problem

$$\begin{cases} y'(t) = y(t)^p\big(\sin(y(t)) + \varepsilon\big), \\ y(0) = y_0 > 0, \end{cases} \tag{6}$$

where $p > 1$ and $\varepsilon \geq 0$. We assume that $y(t)$ is a classical solution defined on its maximal interval of existence.

3 Structure of equilibria

The positive equilibria satisfy

$$\sin y + \varepsilon = 0. \tag{7}$$

Hence:

1. if $0 \leq \varepsilon < 1$, the equation admits infinitely many equilibria;
2. if $\varepsilon = 1$, the equation is in the critical regime and possesses degenerate equilibria;
3. if $\varepsilon > 1$, there is no equilibrium.

When equilibria exist, we denote consecutive equilibrium points by $a_k < b_k$ and the corresponding open intervals by (a_k, b_k).

3.1 Main theorem

Theorem 1 (Global existence versus blow-up transition) *Consider the initial-value problem (6).*

(i) Subcritical case: $0 \leq \varepsilon < 1$. *Every positive solution is global and remains trapped in the interval determined by its initial data. Moreover,*

$$\lim_{t\to+\infty} y(t) = y_*, \tag{8}$$

where y_* is an equilibrium satisfying

$$y_*^p(\sin y_* + \varepsilon) = 0. \tag{9}$$

Since $y_* > 0$, this is equivalent to $\sin y_* + \varepsilon = 0$.

(ii) Critical case: $\varepsilon = 1$. *Every positive solution is global and is either stationary or satisfies*

$$\lim_{t\to+\infty} y(t) = \frac{3\pi}{2} + 2k\pi \tag{10}$$

for an integer k determined by the initial condition.

(iii) Supercritical case: $\varepsilon > 1$. *Every positive solution blows up in finite time.*

3.2 Proof of the theorem

Proof (i) Let $0 \leq \varepsilon < 1$. Fix two consecutive positive equilibrium points $a < b$ such that $y_0 \in (a, b)$. By construction, $\sin y + \varepsilon$ does not change sign on (a, b). Therefore, the right-hand side of (5) has constant sign on this interval, and the solution $y(t)$ is monotone.

We claim that (a, b) is invariant. Indeed, assume by contradiction that the solution reaches one of the endpoints, say a, at some finite time $t_1 > 0$. Since

$$a^p(\sin a + \varepsilon) = 0, \tag{11}$$

the constant function $y(t) \equiv a$ is also a solution. By uniqueness of solutions for ordinary differential equations, a trajectory cannot cross the equilibrium point a, contradicting the assumption. The same argument applies to b. Hence

$$a < y(t) < b \qquad \text{for all } t \geq 0. \tag{12}$$

Since $y(t)$ is monotone and bounded, the finite limit

$$y_* := \lim_{t\to+\infty} y(t) \tag{13}$$

exists. Passing to the limit in the differential equation shows that

$$y_*^p(\sin y_* + \varepsilon) = 0. \tag{14}$$

Because $y_* > 0$, we conclude that $\sin y_* + \varepsilon = 0$. Thus y_* is an equilibrium point, and the solution is global.

Equivalently, the real line is partitioned into invariant intervals by the zeros of $\sin y + \varepsilon$. Inside each interval the right-hand side has constant sign, the solution is monotone, and the trajectory remains bounded between two equilibria.

(ii) Let $\varepsilon = 1$. Then

$$1 + \sin y \geq 0 \tag{15}$$

for all $y \in \mathbb{R}$, with degenerate equilibria at

$$y_k = \frac{3\pi}{2} + 2k\pi, \qquad k \in \mathbb{Z}. \tag{16}$$

If $y_0 = y_k$ for some positive equilibrium y_k, the solution is stationary. Otherwise, the solution is strictly increasing and remains below the next equilibrium. It therefore exists globally and converges to that equilibrium.

(iii) Let $\varepsilon > 1$. For every $y \in \mathbb{R}$,

$$\sin y + \varepsilon \geq \varepsilon - 1 > 0. \tag{17}$$

Consequently,

$$y'(t) \geq (\varepsilon - 1)y(t)^p. \tag{18}$$

Comparison with the solution of

$$z'(t) = (\varepsilon - 1)z(t)^p, \qquad z(0) = y_0, \tag{19}$$

or direct application of the Osgood criterion (2), gives finite-time blow-up. □

4 Blow-up-time estimate

Proposition 1 *If* $\varepsilon > 1$*, the maximal existence time* T *of a positive solution satisfies*

$$T \leq \frac{y_0^{,1-p}}{(p-1)(\varepsilon - 1)}. \tag{20}$$

Proof The explicit solution of the comparison equation (19) is

$$z(t) = \frac{y_0}{\left[1 - (p-1)(\varepsilon - 1)y_0^{p-1}t\right]^{1/(p-1)}}. \tag{21}$$

It blows up at

$$T_z = \frac{y_0^{,1-p}}{(p-1)(\varepsilon - 1)}. \tag{22}$$

Since (18) implies $y(t) \geq z(t)$ while both solutions exist, the solution y must blow up no later than z. This proves (20). □

5 Concluding remarks

Remark 1 The preceding theorem shows that the disappearance of equilibrium points fundamentally changes the equation's qualitative behaviour. In the regime $0 \leq \varepsilon < 1$, the oscillatory structure generates invariant intervals preventing trajectories from escaping to infinity. In contrast, when $\varepsilon > 1$, all equilibria disappear, and the dynamics becomes purely expansive, leading to finite-time blow-up.

Remark 2 The critical value $\varepsilon = 1$ separates two fundamentally different dynamical regimes: a confinement regime characterized by invariant intervals and global existence, and a blow-up regime characterized by monotone growth and finite-time singularity. Hence, $\varepsilon = 1$ may be viewed as a bifurcation threshold for the global dynamics of the equation.

In this paper, we investigated the qualitative behavior of solutions to the oscillatory differential equation (5). Our analysis shows that the structure of the equilibrium points generated by the oscillatory nonlinearity strongly influences the global dynamics. For $0 \leq \varepsilon < 1$, the equation possesses infinitely many equilibria, which partition the phase space into invariant intervals. As a consequence, every solution remains confined inside the interval determined by its initial data and converges asymptotically toward an equilibrium point. In particular, all solutions are global. In contrast, when $\varepsilon > 1$, all equilibrium points disappear. The equation then becomes strictly positive and behaves like a superlinear autonomous equation. Using a comparison argument together with the Osgood criterion, we proved that every positive solution blows up in finite time. The critical value $\varepsilon = 1$ therefore represents a threshold separating two qualitatively different dynamical regimes: confinement with invariant intervals and global existence, and blow-up characterized by monotone growth and finite-time singularity. The present work shows how a simple perturbation can completely change the global behaviour of an oscillatory nonlinear differential equation by destroying its equilibrium structure. Future work may consider several extensions, including reaction–diffusion equations, coupled systems, and nonautonomous perturbations.

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