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Generalized coherent states with shifted (displaced) arguments[1]

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Abstract In the paper we developed a procedure for constructing generalized coherent states with shifted argument, as a result of the action of the generalized analogue of the displacement operator. This was based on the successive action of a pair of nonlinear ladder operators, which generate nonlinear coherent states. To examine the properties of coherent states with shifted argument, the rules of the diagonal operator ordering technique (DOOT) were used. The results obtained were verified for a series of particular cases, obtaining expressions consistent with those in the literature. The expressions obtained will be, among others, useful in quantum optics, e.g. for calculating the Wigner operator in the representation of coherent states.

keywords: coherent states, displacement operator, hypergeometric function.

1 Introduction

It is well known that, in many branches of physics, the so-called displacement operator is used. In essence, this operator translates the variable x, characteristic of a given physical quantity, by an amount Δx. It transforms a function of x into the same function evaluated at the shifted argument $x + \Delta x$.

In general, the unitary translation or displacement operator is the exponential operator

$$\hat{D}(\Delta x) = \exp\left(\Delta x \frac{\partial}{\partial x}\right),$$

which transfers an initial state or function $\Phi(x)$ into a state or function with shifted argument:

$$\exp\left(\Delta x \frac{\partial}{\partial x}\right) \Phi(x) = \Phi(x + \Delta x). \tag{1}$$

In quantum mechanics, displacement operators are used to change the position q and/or the momentum p in phase space. The associated generators $\hat{Q}$ and $\hat{P}$ satisfy

$$[\hat{Q}, \hat{P}] = i, \qquad \hbar = 1.$$

The displacement operator is then defined by

$$\hat{D}(q,p) = \exp\left[i\left(p\hat{Q} - q\hat{P}\right)\right]. \tag{2}$$

The next step toward quantum optics is the introduction of the complex variable z, together with the canonical creation operator $\hat{a}^\dagger$ and annihilation operator $\hat{a}$:

$$\begin{pmatrix} z \\ z^* \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} q + ip \\ q - ip \end{pmatrix}, \qquad \begin{pmatrix} \hat{a} \\ \hat{a}^\dagger \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \hat{Q} + i\hat{P} \\ \hat{Q} - i\hat{P} \end{pmatrix}. \tag{3}$$

The inverse relations are

$$q = \frac{1}{\sqrt{2}}\left(z^* + z\right), \qquad p = \frac{i}{\sqrt{2}}\left(z^* - z\right), \qquad \hat{Q} = \frac{1}{\sqrt{2}}\left(\hat{a}^\dagger + \hat{a}\right), \qquad \hat{P} = \frac{i}{\sqrt{2}}\left(\hat{a}^\dagger - \hat{a}\right). \tag{4}$$

Since the complex variable z enters the formalism of coherent states (CSs), the preceding relations give the following expression for the displacement operator in quantum optics:

$$\hat{D}(z) = \exp\left(z\hat{a}^\dagger - z^*\hat{a}\right). \tag{5}$$

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[1]The article is intended to be a tribute to Emeritus Professor Hong-Yi Fan, from the University of Science and Technology of China (USTC), China, for his contribution to the creation and promotion of the IWOP calculation technique in Quantum Mechanics.

In the physical world, however, some phenomena require operators that are more complicated than the canonical operators associated with the linear one-dimensional harmonic oscillator. Whereas canonical operators generate linear coherent states (LCSs), these more complicated operators correspond to nonlinear coherent states (NCSs). It is therefore natural to assume that the formalism of nonlinear or generalized coherent states involves a generalized analogue of the displacement operator.

The formalism of LCSs underwent considerable development following the introduction of the technique of integration within an ordered product (IWOP). This significant computational advance, which makes full use of Dirac's symbolic representation, was developed by Hong-Yi Fan and his collaborators beginning in 1982; see Refs. [1–4] and the references therein. It made calculations within the formalism of coherent states more transparent and easier to interpret. Nevertheless, applications of the IWOP technique have largely remained restricted to canonical coherent states, without being extended to nonlinear coherent states.

To generalize the IWOP technique to nonlinear systems beyond the one-dimensional harmonic oscillator, we developed a related method called the *diagonal ordering operation technique* (DOOT) [5]. We subsequently applied this technique to several nonlinear oscillators characterized by different classes of coherent states [6–10].

Consequently, the construction of the generalized analogue of the displacement operator $\hat{D}(z)$ is based on the *DOOT–IWOP correspondence principle.* This principle belongs to a broader framework relating the characteristic quantities of nonlinear systems to those of linear systems. It states that every characteristic quantity $X_{\rm DOOT}$ in the DOOT formalism corresponds to a quantity $X_{\rm IWOP}$ in the IWOP formalism through the following limit:

$$\begin{aligned} &\lim_{\substack{p=q\\ \widetilde{a}=\widetilde{b}}} X_{\rm DOOT} = X_{\rm IWOP}, \\ &\lim_{\substack{p=q\\ \widetilde{a}=\widetilde{b}}} \hat{D}_{\rm DOOT}(z) = \hat{D}_{\rm IWOP}(z). \end{aligned} \tag{6}$$

Here, as will be shown below, the integers p and q, together with the sets of real parameters $\widetilde{a}$ and $\widetilde{b}$, characterize the DOOT formalism.

In this article, we apply the rules of the DOOT technique to examine the expressions and properties of coherent states with shifted or displaced arguments. All the intermediate results are verified, by taking the preceding limit, in the case of the canonical coherent states associated with the one-dimensional harmonic oscillator (HO-1D).

In this sense, the article may be regarded as a tribute to Professor Hong-Yi Fan, one of the initiators and leading proponents of the IWOP calculation technique for canonical coherent states. The application of this technique has not only clarified several questions involving quantum operators and coherent states, but has also produced new formulas useful in the theory of complex functions.

2 The case of canonical (linear) coherent states

In this section, we recall several properties of canonical coherent states and displacement operators that will be useful below. For the one-dimensional harmonic oscillator, the normalized canonical coherent states have the same expression in the Barut–Girardello, Klauder–Perelomov, and Gazeau–Klauder definitions [11].

These states can be expanded in the orthonormal Fock basis $\{|n\rangle\}_{n=0}^{\infty}$, whose elements are eigenstates of the number operator:

$$\hat{N}|n\rangle = n|n\rangle, \qquad \langle n|m\rangle = \delta_{nm}. \tag{7}$$

The canonical coherent states are then given by

$$|z\rangle = e^{-\frac{1}{2}|z|^2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}}\,|n\rangle. \tag{8}$$

They are labeled by the complex variable

$$z = |z|e^{i\phi}, \qquad 0 \le |z| < \infty, \qquad 0 \le \phi < 2\pi. \tag{9}$$

The same expression can be obtained using the displacement operator:

$$\begin{aligned} \hat{D}(z) &= \exp\big(z\hat{a}^\dagger - z^*\hat{a}\big) \\ &= e^{-\frac{1}{2}|z|^2} e^{z\hat{a}^\dagger} e^{-z^*\hat{a}}. \end{aligned} \tag{10}$$

This normally ordered form follows from the Baker–Campbell–Hausdorff formula

$$\begin{aligned} \exp\Big(\hat{X} + \hat{Y}\Big) &= \exp(\hat{X})\,\exp(\hat{Y}) \\ &\quad\times \exp\left(-\frac{1}{2}[\hat{X},\hat{Y}]\right). \end{aligned} \tag{11}$$

provided that $[\hat{X},\hat{Y}]$ commutes with both $\hat{X}$ and $\hat{Y}$.

The canonical creation and annihilation operators, $\hat{a}^\dagger$ and $\hat{a}$, satisfy

$$[\hat{a},\hat{a}^\dagger] = 1. \tag{12}$$

Their actions on the Fock states and the corresponding bra states are

$$
\begin{aligned}
&\hat{a}|n\rangle = \sqrt{n}\,|n-1\rangle,\\
&\langle n|\hat{a}^\dagger = \sqrt{n}\,\langle n-1|,\\
&\hat{a}^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle,\\
&\langle n|\hat{a} = \sqrt{n+1}\,\langle n+1|,\\
&\left(\hat{a}^\dagger\right)^n|0\rangle = \sqrt{n!}\,|n\rangle,\\
&\langle 0|\hat{a}^n = \sqrt{n!}\,\langle n|,\\
&\langle n|\hat{a}^\dagger\hat{a}|n\rangle = n,\\
&\langle n|\hat{a}\hat{a}^\dagger|n\rangle = n+1.
\end{aligned} \tag{13}
$$

To these operators, we may add the particle-number operator, usually called simply the number operator, $\hat{N} = \hat{a}^\dagger\hat{a}$. This operator will be useful below. It is self-adjoint:

$$
\hat{N}^\dagger = \hat{N} = \hat{a}^\dagger\hat{a}. \tag{14}
$$

The coherent states can be obtained by applying the displacement operator $\hat{D}(z)$ to the vacuum state $|0\rangle$. Using

$$
\hat{a}|0\rangle = 0, \qquad e^{-z^*\hat{a}}|0\rangle = |0\rangle. \tag{15}
$$

we obtain

$$
\begin{aligned}
|z\rangle &= \hat{D}(z)|0\rangle\\
&= e^{-\frac{1}{2}|z|^2}e^{z\hat{a}^\dagger}e^{-z^*\hat{a}}|0\rangle\\
&= e^{-\frac{1}{2}|z|^2}e^{z\hat{a}^\dagger}|0\rangle.
\end{aligned} \tag{16}
$$

The Hermitian-conjugate relation is

$$
\begin{aligned}
\langle z| &= \langle 0|\hat{D}^\dagger(z) = \langle 0|\hat{D}(-z)\\
&= \langle 0|e^{z^*\hat{a}-z\hat{a}^\dagger}\\
&= \langle 0|e^{-\frac{1}{2}|z|^2}e^{z^*\hat{a}}e^{-z\hat{a}^\dagger}.
\end{aligned} \tag{17}
$$

Since coherent states are normalized to unity,

$$
\begin{aligned}
\langle z|z\rangle &= \langle 0|\hat{D}^\dagger(z)\hat{D}(z)|0\rangle\\
&= 1.
\end{aligned} \tag{18}
$$

It follows that the displacement operator is unitary:

$$
\hat{D}^\dagger(z)\hat{D}(z) = \hat{D}(z)\hat{D}^\dagger(z) = \hat{I}. \tag{19}
$$

We now apply a second displacement operator, depending on the complex variable σ, to the previously obtained coherent state. Since $\hat{a}|z\rangle = z|z\rangle$, we successively obtain

$$
\begin{aligned}
\hat{D}(\sigma)|z\rangle &= e^{-\frac{1}{2}|\sigma|^2}e^{\sigma\hat{a}^\dagger}e^{-\sigma^*\hat{a}}|z\rangle\\
&= e^{-\frac{1}{2}|\sigma|^2-\sigma^* z}e^{\sigma\hat{a}^\dagger}|z\rangle\\
&= e^{-\frac{1}{2}|\sigma|^2-\frac{1}{2}|z|^2-\sigma^* z}\sum_{l=0}^{\infty}\frac{z^l}{\sqrt{l!}}e^{\sigma\hat{a}^\dagger}|l\rangle.
\end{aligned} \tag{20}
$$

The action of the exponential operator on a Fock state is

$$
e^{\sigma\hat{a}^\dagger}|l\rangle = \sum_{m=0}^{\infty}\frac{\sigma^m}{m!}\sqrt{\frac{(l+m)!}{l!}}\,|l+m\rangle. \tag{21}
$$

Substituting this expression gives

$$
\begin{aligned}
\hat{D}(\sigma)|z\rangle = e^{-\frac{1}{2}|\sigma|^2-\frac{1}{2}|z|^2-\sigma^* z}&\\
\times\sum_{m=0}^{\infty}\sum_{l=0}^{\infty}\frac{(l+m)!}{\sqrt{(l+m)!\,l!\,m!}}&\\
\times z^l\sigma^m\,|l+m\rangle.&
\end{aligned} \tag{22}
$$

Introducing the new summation index

$$
n = l+m, \qquad l = n-m \geq 0, \qquad 0 \leq m \leq n. \tag{23}
$$

we find

$$
\begin{aligned}
\hat{D}(\sigma)|z\rangle &= e^{-\frac{1}{2}|\sigma|^2-\frac{1}{2}|z|^2-\sigma^* z}\\
&\quad\times\sum_{n=0}^{\infty}\frac{1}{\sqrt{n!}}\sum_{m=0}^{n}\frac{n!}{(n-m)!\,m!}\\
&\quad\times z^{n-m}\sigma^m\,|n\rangle\\
&= e^{-\frac{1}{2}|\sigma|^2-\frac{1}{2}|z|^2-\sigma^* z}\\
&\quad\times\sum_{n=0}^{\infty}\frac{1}{\sqrt{n!}}\left[\sum_{m=0}^{n}\binom{n}{m}z^{n-m}\sigma^m\right]|n\rangle\\
&= e^{-\frac{1}{2}|\sigma|^2-\frac{1}{2}|z|^2-\sigma^* z}\\
&\quad\times\sum_{n=0}^{\infty}\frac{(z+\sigma)^n}{\sqrt{n!}}|n\rangle\\
&= e^{\frac{1}{2}(\sigma z^*-\sigma^* z)}|z+\sigma\rangle.
\end{aligned} \tag{24}
$$

Consequently, the coherent state with shifted argument satisfies

$$
\begin{aligned}
|z+\sigma\rangle &= e^{-\frac{1}{2}(\sigma z^*-\sigma^* z)}\hat{D}(\sigma)\hat{D}(z)|0\rangle\\
&= \hat{D}(z+\sigma)|0\rangle.
\end{aligned} \tag{25}
$$

which means that the successive displacements are equivalent to a single displacement up to an phase factor. The above relation can be written in an equivalent manner,

$$
|z+\sigma\rangle e^{-\ \frac{1}{2}\ \ |z+\sigma|^2}\sum_{n=0}^{\infty}\frac{(z+\sigma)^n}{\sqrt{n\ !}}|n> \tag{2.14}
$$

This expression of CSs with shifted argument could also be deduced directly, if we consider that the sum of two complex numbers z and σ is also a complex number$z+\sigma$. It is observed that the expression on the right-hand side is symmetric in the variables z andσ, which thus have "equal rights".

3 Generalized coherent states

For several decades since the introduction, in 1926, by Schrödinger, of the notion of coherent state [12], the scientific world was only interested in the canonical CSs, connected with the one-dimensional harmonic oscillator. Over time, however, other types of CSs began to appear and were named nonlinear CSs (NCSs) or, more recently, generalized CSs (GCSs). As a counterpart, the canonical CSs of HO-1D were named linear CSs.

Mainly, any nonlinear or generalized CSs have the following structure:

$$|z\rangle = \frac{1}{\sqrt{N(|z|^2)}} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{\rho(n)}} |n\rangle, \tag{26}$$

where $\{\rho(n)\}_{n=0}^{\infty}$ is the sequence of strictly positive numbers, called *structure constants*, because they determine the structure and character of the normalization function. This must be an analytical function:

$$N(|z|^2) = \sum_{n=0}^{\infty} \frac{1}{\rho(n)} (|z|^2)^n. \tag{27}$$

This function must be strictly positive and convergent within a disc, a circle or domain of the complex z-plane. This condition assures that the probabilities are well-defined. Since the numbers are strictly positive, it follows automatically that the terms of the series are positive.

In essence, generalized CSs are generated by a conjugate (adjoint) pair of ladder operators: the creation $\hat{A}_+$ and the annihilation $\hat{A}_-$, where $(\hat{A}_+)^\dagger = \hat{A}_-$ and $(\hat{A}_-)^\dagger = \hat{A}_+$. These operators are connected with the canonical operators $\hat{a}^\dagger$ and $\hat{a}$ through a function depending on the particle number operator $\hat{N} = \hat{a}^\dagger \hat{a}$. This real function $f(\hat{N})$ is called the nonlinearity function.

Nonlinear operators have the following structure:

$$\hat{A}_- = \hat{a}\, f(\hat{N}), \qquad \hat{A}_+ = f(\hat{N})\, \hat{a}^\dagger. \tag{28}$$

The nonlinearity function $f(\hat{N})$ can be, up to a point, arbitrary, as long as it ensures that the normalization function is an analytic function (non-analytic ones having no physical meaning). Moreover, the nonlinearity function $f(\hat{N})$ satisfies the following general relation:

$$f(\hat{N} + m)\, |n + p\rangle = f(n + p + m)\, |n + p\rangle. \tag{29}$$

At the limit the pair of newly introduced operators reduces to the canonical operators:

$$\lim_{\substack{p=q \\ a=b}} \begin{pmatrix} \hat{A}_- \\ \hat{A}_+ \end{pmatrix} = \lim_{f(\hat{N})\to 1} \begin{pmatrix} \hat{A}_- \\ \hat{A}_+ \end{pmatrix} = \begin{pmatrix} \hat{a} \\ \hat{a}^\dagger \end{pmatrix}. \tag{30}$$

We will choose the nonlinearity function in the following manner:

$$f(\hat{N}) = \sqrt{\frac{\prod_{j=1}^{q} (b_j - 1 + \hat{N})}{\prod_{i=1}^{p} (a_i - 1 + \hat{N})}}. \tag{31}$$

Its eigenvalue equation is

$$f(\hat{N})\, |n\rangle = \sqrt{\frac{\prod_{j=1}^{q} (b_j - 1 + n)}{\prod_{i=1}^{p} (a_i - 1 + n)}}\, |n\rangle. \tag{32}$$

The actions of these operators and their Hermitian conjugates on the Fock vectors are

$$\begin{aligned} \hat{A}_- |n\rangle &= \sqrt{e(n)}\, |n - 1\rangle, \\ \langle n|\, \hat{A}_+ &= \sqrt{e(n)}\, \langle n - 1|, \\ \hat{A}_+ |n\rangle &= \sqrt{e(n + 1)}\, |n + 1\rangle, \\ \langle n|\, \hat{A}_- &= \sqrt{e(n + 1)}\, \langle n + 1|, \\ \langle n|\, \hat{A}_+ \hat{A}_- |n\rangle &= e(n). \end{aligned} \tag{33}$$

Here we used the notation:

$$\begin{aligned} e(n) &\equiv n\ \ (f(n))^2 = n\ \ g(n) \quad , \\ g(n) &\equiv (f(n))^2 = \frac{\prod_{j=1}^{q} (b_j - 1 + n)}{\prod_{i=1}^{p} (a_i - 1 + n)} \end{aligned} \tag{3.9}$$

The repeated action of the creation operator on the vacuum ket vector $|0>$ (and similarly for their hermitian conjugate $0|$) is summarized as follows:

$$\left(\hat{A}_+\right)^n |0\rangle = \sqrt{\prod_{s=1}^{n} e(s)}\, |n\rangle = \sqrt{\rho(n)}\, |n\rangle, \tag{34}$$

and

$$\langle 0| \left(\hat{A}_-\right)^n = \langle n| \sqrt{\prod_{s=1}^{n} e(s)} = \langle n| \sqrt{\rho(n)}. \tag{35}$$

where we introduced a new notation

$$\rho(n) = [e(n)]! = \prod_{s=1}^{n} e(s) = n!\, [g(n)]!, \tag{36}$$

with

$$\lim_{f(n)\to 1} \rho(n) = n!, \qquad \rho(0) = 1. \tag{37}$$

Here we used the *functional factorial* defined as

$$[x(n)]! \equiv x(1) x(2) \cdots x(n), \qquad [x(0)]! = 1, \tag{38}$$

as well as the Pochhammer symbol

$$(x)_n = \prod_{s=1}^{n} (x + s - 1) = \frac{\Gamma(x + n)}{\Gamma(x)}. \tag{39}$$

Consequently, the structure function can be written as

$$\rho(n) = n!\,\frac{\prod_{j=1}^{q}(b_j)_n}{\prod_{i=1}^{p}(a_i)_n} \equiv n!\,[g(n)]!, \tag{40}$$

where

$$[g(n)]! \equiv \frac{\prod_{j=1}^{q}(b_j)_n}{\prod_{i=1}^{p}(a_i)_n}, \qquad [g(0)]! = 1. \tag{41}$$

This entity, in which a function that depends on a number appears instead of a number, can be called *functional factorial*.

In the following we will also use the short notations:

$$\binom{\rho(l)}{\rho(n)} = \frac{\rho(l)}{\rho(n)\,\rho(l-n)}, \tag{42}$$

and

$$\binom{g(l)}{g(n)} \equiv \frac{[g(l)]!}{[g(n)]!\,[g(l-n)]!}, \tag{43}$$

so that

$$\binom{\rho(l)}{\rho(n)} = \binom{l}{n}\binom{g(l)}{g(n)}. \tag{44}$$

The entity $\binom{\rho(l)}{\rho(n)}$ can be interpreted as the *functional binomial coefficients*, because at the limit, this turns into the usual binomial coefficient:

$$\lim_{\substack{p=q\\ \widetilde{a}=\widetilde{b}=1}} \binom{\rho(l)}{\rho(n)} = \binom{l}{n}. \tag{45}$$

Generally, the ladder operators $\hat{A}_+$ and $\hat{A}_-$ are not commutable, but, as we will see below, they obey the rules of the *Diagonal Operator Ordering Technique* (DOOT), which we introduced in a previous paper [5]. The main rules of the DOOT are:

a) Inside the symbol $\#\cdots\#$ the order of operators $\hat{A}_+$ and $\hat{A}_-$ can be permuted, $\hat{A}_-\hat{A}_+ = \hat{A}_+\hat{A}_-$, so that finally we obtain a function or an expression that is normally ordered (operator $\hat{A}_+$ is located on the left side and operator $\hat{A}_-$ is on its right side).

b) Inside the symbol $:\cdots:$ we can perform all algebraic operations (derivations, integrations), according to the usual rules.

Consequently, all operator functions involved in the DOOT technique must have a formal series expansion of a specific form in order to yield non-zero states:

$$f(\hat{A}_+, \hat{A}_-) = \sum_{n=0}^{\infty} C_n\,(\hat{A}_+\hat{A}_-)^n. \tag{46}$$

c) The ladder operators $\hat{A}_+$ and $\hat{A}_-$ can be treated as simple c-numbers;

d) the vacuum state projector $|0\rangle\langle 0|$, in the frame of DOOT, has the following normal ordered form:

$$\frac{1}{{}_pF_q(\hat{A}_+\hat{A}_-)} = \sum_{n=0}^{\infty} \frac{(\hat{A}_+\hat{A}_-)^n}{\rho(n)},$$

where the abbreviated notation for the coefficients are: $\widetilde{a} \equiv \{a_1, a_2, \ldots, a_p\} \equiv \{a_i\}_1^p$ and ${}_pF_q(\widetilde{a}; \widetilde{b}; x)$ for the generalized hypergeometric function.

It follows from the above rules of the DOOT technique that creation and annihilation operators commute only if they are located within the ## sign. Outside this sign, these operators do not commute:

$$\hat{A}_-\hat{A}_+ \neq \hat{A}_+\hat{A}_-, \tag{47}$$

but within the ## sign they commute:

$$\#\hat{A}_-\hat{A}_+\# = \#\hat{A}_+\hat{A}_-\#. \tag{48}$$

Furthermore, to obtain a non-zero result, the creation and annihilation operators within the ordered product must have the same power.

$$(\hat{A}_-)^m(\hat{A}_+)^n = (\hat{A}_+)^n(\hat{A}_-)^m = \delta_{nm}(\hat{A}_+\hat{A}_-)^n. \tag{49}$$

To simplify writing formulas, everywhere possible, without causing confusion, in the following we will write the hypergeometric functions without mentioning the set of numbers $\widetilde{a}$ and $\widetilde{b}$ and indicating only their argument, e.g. ${}_pF_q(\widetilde{a}; \widetilde{b}; x) \equiv {}_pF_q(x)$. Instead, we will use the same $|z\rangle$ notation for both generalized CSs and canonical CSs. We will explicitly specify the sets of numbers $\widetilde{a}$ and $\widetilde{b}$ only where strictly necessary, to avoid confusion.

With this notation, the generalized hypergeometric function will be written as

$$\begin{aligned} {}_pF_q(\widetilde{a}; \widetilde{b}; x) &\equiv {}_pF_q(x) \\ &= \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{p}(a_i)_n}{\prod_{j=1}^{q}(b_j)_n}\frac{x^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{x^n}{\rho(n)}. \end{aligned} \tag{50}$$

One of the important advantages of using the DOOT technique (as well as the IWOP technique) is that it allows us to express various entities in terms of creation/annihilation operators $\hat{A}_+$ and $\hat{A}_-$. Including the vacuum projector $|0\rangle\langle 0|$ which is expressed as a function that depends on the normal ordered product of creation and annihilation operators $\hat{A}_+\hat{A}_-$. Thus we end up working with functions that have operators as arguments, and these are viewed as simple c-numbers.

Let us emphasize here that the DOOT technique is a generalization of the technique of *Integration Within an Ordered Product* (IWOP), introduced by Hong-yi Fan (see, e.g. [3] and references therein). But, the difference between the two techniques consists in the fact that the IWOP is applicable only for Bose operators, referring to the CSs of the HO-1D, while DOOT applies to any pair of creation/annihilation operators, adopting the above rules. In this sense, the IWOP appears as a particular case of DOOT.

The choice of ladder operators $\hat{A}_+$ and $\hat{A}_-$, having the above properties and which obey the rules of the DOOT technique was not accidental. This choice leads to the fact that the normalization function of CSs will be the generalized hypergeometric function itself, ${}_pF_q(x)$.

Consequently, let us define the CSs in the Barut-Girardello manner, i.e. as the eigenvalues of the annihilation operator

$$\hat{A}_-|z\rangle = z|z\rangle, \qquad \langle z|\hat{A}_+ = z^*\langle z|. \tag{51}$$

The second relation can be obtained either by complex conjugation of the first relation, or by using the property $\langle n|\hat{A}_+ = \sqrt{e(n)}\,\langle n-1|$, see Eq. (3.8).

Using this definition, as well as the rules of the DOOT technique, it is obtained that the expression of generalized CSs is

$$\begin{aligned}|z\rangle &= \frac{1}{\sqrt{{}_pF_q(|z|^2)}}\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{\rho(n)}}|n\rangle \\ &= \frac{1}{\sqrt{{}_pF_q(|z|^2)}}\,{}_pF_q(z\hat{A}_+)|0\rangle,\end{aligned} \tag{52}$$

and their complex conjugate

$$\begin{aligned}\langle z| &= \frac{1}{\sqrt{{}_pF_q(|z|^2)}}\sum_{n=0}^{\infty}\frac{(z^*)^n}{\sqrt{\rho(n)}}\langle n| \\ &= \frac{1}{\sqrt{{}_pF_q(|z|^2)}}\,\langle 0|\,{}_pF_q(z^*\hat{A}_-).\end{aligned} \tag{53}$$

Using the DOOT rules, the projector, in the CSs representation can be written as

$$\begin{aligned}|z\rangle\langle z| &= \frac{{}_pF_q(z\hat{A}_+)|0\rangle\langle 0|\,{}_pF_q(z^*\hat{A}_-)}{{}_pF_q(|z|^2)} \\ &= \frac{1}{{}_pF_q(|z|^2)}\frac{{}_pF_q(z\hat{A}_+)\,{}_pF_q(z^*\hat{A}_-)}{{}_pF_q(\hat{A}_+\hat{A}_-)}.\end{aligned} \tag{54}$$

Because the normalization function must be a strictly positive convergent function, their convergence radius R_c of the series must fulfill the inequality:

$$\begin{aligned}0 < |z| < R_c &= \sqrt{\lim_{n\to\infty}\frac{\rho(n+1)}{\rho(n)}} \\ &= \sqrt{\lim_{n\to\infty}(n+1)\frac{\prod_{j=1}^{q}(b_j+n)}{\prod_{i=1}^{p}(a_i+n)}}.\end{aligned} \tag{55}$$

Particularly, for the case of canonical coherent states, $\rho(n) = n!$, and we obtain the correct result, $R_c = \infty$.

From the theory of hypergeometric series, it follows that the function ${}_pF_q(\widetilde{a};\widetilde{b};x) \equiv {}_pF_q(\{a_i\}_1^p;\{b_j\}_1^q;x)$ exhibits the following behavior:

- if $p \le q$ and $|z| < \infty$, the series converges over the entire complex plane.

- if $p = q+1$ and $|z| < 1$, is convergent, and for $|z| > 1$ is divergent.

- if $p > q+1$ and $|z| \ne 0$, is divergent, if the series does not terminate and becomes a polynomial.

In addition to these, a series of conditions are imposed on the parameters which appear in the expression of generalized hypergeometric functions [13]. For example, for the case $p = q+1$ convergence will be achieved if:

- for $|z| = 1$ if $\eta = 1$;

- for $|z| = 1$ and $|z| \ne 1$, if $0 \le \eta \le 1$, where

$$\eta = \mathrm{Re}\left(\sum_{i=1}^{p}a_i - \sum_{j=1}^{q}b_j\right). \tag{56}$$

From the perspective of the coherent state formalism, only the cases where the series converges are of interest.

In 1963 Klauder established that all generalized CSs (therefore, linear and non-linear) must satisfy the following minimum conditions, which were later called "the Klauder's prescriptions" [14]: (I). NCSs are normalized but non-orthogonal:

$$\langle z|z'\rangle = \frac{{}_pF_q(z^*z')}{\sqrt{{}_pF_q(|z|^2)}\sqrt{{}_pF_q(|z'|^2)}}, \tag{57}$$

and thus the generalized CSs form an overcomplete set.

(II). Continuity in the label variable z of the NCSs:

$$\lim_{z\to z'}\|\,|z\rangle - |z'\rangle\| = 0. \tag{58}$$

(III). NCSs must satisfy the resolution of the identity, or the closure relation which can be expressed by the resolution of the identity operator $\hat{I}$ in the vector space of quantum states, where $d\mu(z)$ is suitable integration measure.

$$\int d\mu(z)\,|z\rangle\langle z| = \hat{I} = \sum_{n=0}^{\infty}|n\rangle\langle n|. \tag{59}$$

As a consequence of this relation, the property of CSs to form an overcomplete set translates into the fact that any state can be decomposed on the set of CSs:

$$|z'\rangle = \int d\mu(z)\,|z\rangle\langle z|z'\rangle. \tag{60}$$

It can be verified that the integration measure is [5]:

$$d\mu(z) = \Gamma(\widetilde{a}/\widetilde{b})\,\frac{d\phi}{2\pi}\,d(|z|^2)\,{}_pF_q(|z|^2) \times G^{q+1,0}_{p,q+1}\left(|z|^2 \,\middle|\, \begin{matrix}\widetilde{a}-1\\ 0,\ \widetilde{b}-1\end{matrix}\right), \tag{61}$$

where we used the notation

$$\Gamma(\widetilde{a}/\widetilde{b}) \equiv \frac{\prod_{i=1}^{p}\Gamma(a_i)}{\prod_{j=1}^{q}\Gamma(b_j)}. \tag{62}$$

Here $G^{q+1,0}_{p,q+1}(|z|^2|\cdots)$ is the Meijer G-function which can be connected with the hypergeometric function ${}_pF_q(|z|^2)$ as follows [15]:

$${}_pF_q(|z|^2) = \Gamma(\widetilde{b}/\widetilde{a})\,G^{1,p}_{p,q+1}\left(-|z|^2 \,\middle|\, \begin{matrix}1-\widetilde{a}\\ 0,\ 1-\widetilde{b}\end{matrix}\right), \tag{63}$$

where $1-\widetilde{a} \equiv \{1-a_1, 1-a_2, \ldots, 1-a_p\}$ and so on.

With the above considerations, the relation of decomposition of unity operator can be also written as

$$\int d\mu(z)\,|z\rangle\langle z| = \Gamma(\widetilde{a}/\widetilde{b}) \sum_{n,m=0}^{\infty} \frac{|n\rangle\langle m|}{\sqrt{\rho(n)\rho(m)}} \times \int d(|z|^2)\,G^{q+1,0}_{p,q+1}\left(|z|^2 \,\middle|\, \begin{matrix}\widetilde{a}-1\\ 0,\ \widetilde{b}-1\end{matrix}\right) \times \int_0^{2\pi}\frac{d\phi_z}{2\pi} z^n (z^*)^m. \tag{64}$$

Using the orthogonality of the angular integral, $\int_0^{2\pi}\frac{d\phi}{2\pi} z^n (z^*)^m = \delta_{nm}|z|^{2n}$, we get

$$\int d\mu(z)\,|z\rangle\langle z| = \Gamma(\widetilde{a}/\widetilde{b}) \times \sum_{n=0}^{\infty}\frac{|n\rangle\langle n|}{\rho(n)} \int_0^{\infty} d(|z|^2)\,(|z|^2)^n \times G^{q+1,0}_{p,q+1}\left(|z|^2 \,\middle|\, \begin{matrix}\widetilde{a}-1\\ 0,\ \widetilde{b}-1\end{matrix}\right). \tag{65}$$

Using the properties of the Meijer G-function, the integral equals $\rho(n)\Gamma(\widetilde{b}/\widetilde{a})$, so that the final result is

$$\int d\mu(z)\,|z\rangle\langle z| = \sum_{n=0}^{\infty}|n\rangle\langle n| = \hat{I}. \tag{66}$$

The proof is simple and can be found by consulting our preprint [16]. Let us also focus our attention on an integral that results from the decomposition relations of the unit operator and which will be useful to us further:

$$\int \frac{d^2 z}{\pi}\,G^{q+1,0}_{p,q+1}\left(|z|^2 \,\middle|\, \begin{matrix}\widetilde{a}-1\\ 0,\ \widetilde{b}-1\end{matrix}\right)(|z|^2)^n = \Gamma(\widetilde{b}/\widetilde{a})\,\rho(n). \tag{67}$$

By analogy with the displacement operator defined by the canonical operators $\hat{a}^\dagger$ and $\hat{a}$, the generalized exponential displacement operator, corresponding to the ladder operators $\hat{A}_+$ and $\hat{A}_-$, can also be defined as:

$$\hat{D}(z) = \exp\left(z\hat{A}_+ - z^*\hat{A}_-\right). \tag{68}$$

At the end of this section, let us point out that, besides Barut-Girardello type CSs, other type of CSs can also be defined in the so called Klauder-Perelomov manner, as a result of the action of the generalized analogue of the displacement operator on the vacuum state:

$$|z\rangle_{\rm KP} = \frac{1}{\sqrt{N_{\rm KP}(|z|^2)}}\,\hat{D}(z)\,|0\rangle = \frac{1}{\sqrt{N_{\rm KP}(|z|^2)}}\,\exp\left(z\hat{A}_+\right)\,|0\rangle, \tag{69}$$

where we took into account that $\hat{A}_-|0\rangle = 0$. Expanding the exponential in a power series and taking into account the result of the action of the operator $\hat{A}_+$ on the vacuum state, we obtain that the set of CSs of Klauder-Perelomov type has the expression

$$|z\rangle_{\rm KP} = \frac{1}{\sqrt{{}_qF_p(\widetilde{b};\widetilde{a};|z|^2)}}\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{\rho_{\rm KP}(n)}}|n\rangle = \frac{1}{\sqrt{{}_qF_p(\widetilde{b};\widetilde{a};|z|^2)}}\,{}_qF_p(\widetilde{b};\widetilde{a};z\hat{A}_+)|0\rangle. \tag{70}$$

The notation used here is

$$\rho_{\rm KP}(n) = \frac{(n!)^2}{\rho(n)}. \tag{71}$$

It can be seen that between the sets of CSs of the Barut-Girardello type, on the one hand, and the Klauder-Perelomov, on the other hand, there is a dualism. This dualism consists mainly of the interchange of the indices p and q, as well as of the sets of numbers $\{a_i\}$ and $\{b_j\}$, that is, instead of the hypergeometric function ${}_pF_q(\widetilde{a};\widetilde{b};x)$, the function ${}_qF_p(\widetilde{b};\widetilde{a};x)$ appears, as well as instead of the structure function $\rho(n)$, the function $\rho_{\rm KP}(n)$ appears [17].

4 Generalized analogue of the displacement operator

Previously we have seen that the generalized CSs can also be obtained by applying the generalized hypergeometric function ${}_pF_q(z\hat{A}_+)$ on the vacuum ket vector $|0\rangle$. This idea appeared for the first time, for a particular case of pseudoharmonic oscillator, in Mojaveri and Dehghani's paper [16].

On the other hand, any function that has as its argument the annihilation operator, therefore also ${}_pF_q(-z^*\hat{A}_-)$, applied to the vector $|0\rangle$ leaves it unchanged, i.e.

$$ {}_pF_q(-z^*\hat{A}_-)\,|0\rangle = |0\rangle. \tag{72}$$

It must be explicitly emphasized that a relation of the form $f(\hat{A}_-)|0\rangle = |0\rangle$, containing the annihilation operator $\hat{A}_-$ in its argument, is valid only if the function $f(\hat{A}_-)$ acts on the vacuum state vector $|0\rangle$. If this operator function acts upon a "non-vacuum" coherent state, then the result of this action is different from $|0\rangle$. This procedure will also be evident in the calculations that follow.

So, it is obvious that a generalized analogue of the displacement operator can be constructed, similar to the "usual" one $\hat{D}(z) = e^{-\frac{1}{2}|z|^2} e^{z\hat{a}^\dagger - z^*\hat{a}}$.

Consequently, by analogy, we can define the *generalized analogue of the displacement operator* as follows:

$$\hat{D}(z) \equiv \frac{1}{\sqrt{{}_pF_q(|z|^2)}}\,{}_pF_q(\;z\hat{A}_+)\;{}_pF_q(\;-z^*\hat{A}_-) \tag{4.2}$$

Correspondingly, its adjoint operator (Hermitian conjugate) is

$$\begin{aligned}\hat{D}^\dagger(z) &= \frac{1}{\sqrt{{}_pF_q(|z|^2)}}\,{}_pF_q(z^*\hat{A}_-)\,{}_pF_q(-z\hat{A}_+)\\ &= \frac{1}{\sqrt{{}_pF_q(|z|^2)}}\,{}_pF_q(-z\hat{A}_+)\,{}_pF_q(z^*\hat{A}_-)\\ &= \hat{D}(-z).\end{aligned} \tag{73}$$

However, the inequalities hold:

$$\hat{D}^{-1}(z) \neq \hat{D}(-z), \qquad \hat{D}^\dagger(z)\hat{D}(z) \neq 1, \tag{74}$$

therefore, the operator $\hat{D}(z)$ is not unitary.

In the complete definition of the displacement operator, the minus sign in ${}_pF_q(-z^*\hat{A}_-)$ is retained so that, in the harmonic limit, it coincides with the displacement operator of canonical coherent states.

As a consequence, the following relationship will have to be valid at the limit:

$$\begin{aligned}\lim_{\substack{p=q\\a=b}} \hat{D}(z) &= \lim_{\substack{p=q\\a=b}} \frac{1}{\sqrt{{}_pF_q(|z|^2)}}\\ &\quad\times {}_pF_q(z\hat{A}_+)\,{}_pF_q(-z^*\hat{A}_-)\\ &= e^{-\frac{1}{2}|z|^2} e^{z\hat{a}^\dagger - z^*\hat{a}} = \hat{D}(z).\end{aligned} \tag{75}$$

In the calculations, but only when the function ${}_pF_q(-z^*\hat{A}_-)$ acts on the vacuum state $|0\rangle$, below we can drop this function, so we will consider that the generalized analogue of the displacement operator has the expression (which agrees with [18]):

$$\hat{D}(z) \equiv \frac{1}{\sqrt{{}_pF_q(|z|^2)}}\,{}_pF_q(z\hat{A}_+). \tag{76}$$

Using the definition of the CSs (and their complex conjugate counterpart), let's highlight another property of the generalized analogue of the displacement operator, which can be considered as their eigenvalue equation

$$\begin{aligned}{}_pF_q(-z^*\hat{A}_-)|\sigma\rangle &= \sum_{n=0}^{\infty} \frac{(z^*)^n}{\rho(n)}(\hat{A}_-)^n|\sigma\rangle\\ &= \sum_{n=0}^{\infty} \frac{(z^*)^n}{\rho(n)}\sigma^n|\sigma\rangle\\ &= {}_pF_q(-z^*\sigma)|\sigma\rangle,\end{aligned} \tag{77}$$

and similarly

$$\langle\sigma|\,{}_pF_q(z\hat{A}_+) = \langle\sigma|\,{}_pF_q(z\sigma^*). \tag{78}$$

It means that the function ${}_pF_q(-z^*\sigma)$ is the eigenvalue of the operator ${}_pF_q(-z^*\hat{A}_-)$ attached to the eigenfunction $|\sigma\rangle$ and similar for their complex conjugate.

The diagonal elements of the generalized analogue of the displacement operator in CSs representation are

$$\begin{aligned}\langle\sigma|\hat{D}(z)|\sigma\rangle &= \frac{\langle\sigma|\,{}_pF_q(z\hat{A}_+)\,{}_pF_q(z^*\hat{A}_-)|\sigma\rangle}{\sqrt{{}_pF_q(|z|^2)}}\\ &= \frac{1}{\sqrt{{}_pF_q(|z|^2)}}\frac{{}_pF_q(z\sigma^*)\,{}_pF_q(z^*\sigma)}{{}_pF_q(|\sigma|^2)}.\end{aligned} \tag{79}$$

On the other hand, using the DOOT rules we have

$$\begin{aligned}|l\rangle\langle l| &= \frac{1}{\rho(l)}(\hat{A}_+)^l|0\rangle\langle 0|(\hat{A}_-)^l\\ &= \frac{1}{{}_pF_q(\hat{A}_+\hat{A}_-)}\frac{(\hat{A}_+\hat{A}_-)^l}{\rho(l)},\end{aligned} \tag{80}$$

and it can be easily verified that this expression fulfills the completeness relation

$$\sum_{l=0}^{\infty} |l\rangle\langle l| = \hat{I}. \tag{81}$$

Now, we will be able to write, which means that the displacement operator applied to the vacuum ket vector $|0\rangle$ transforms it into a vector corresponding to the generalized CSs $|z\rangle$.

$$|z\rangle = \hat{D}(z)\,|0\rangle. \tag{82}$$

For the moment, for reasons of clarity of exposition, we will return to the initial notation for generalized hypergeometric functions, ${}_pF_q(\widetilde{a};\widetilde{b};x)$.

Previously we saw that CSs satisfy the decomposition relation of the unit operator. Let us see how generalized states of displacement are involved in this relation, also taking into account the DOOT rules.

$$\begin{aligned}\hat{I} &= \int d\mu(z)\,|z\rangle\langle z| \\ &= \int d\mu(z)\,\hat{D}(z)|0\rangle\langle 0|\hat{D}^{\dagger}(z) \\ &= \frac{1}{{}_pF_q(\widetilde{a};\widetilde{b};\hat{A}_+\hat{A}_-)}\int d\mu(z)\,\hat{D}(z)\hat{D}^{\dagger}(z).\end{aligned} \tag{83}$$

From this follows a new property of the generalized analogue of the displacement operator:

$$\int d\mu(z)\,\hat{D}(z)\hat{D}^{\dagger}(z) = {}_pF_q(\widetilde{a};\widetilde{b};\hat{A}_+\hat{A}_-). \tag{84}$$

or, more explicitly, using the expression of integration measure

$$\begin{aligned}&\int \frac{d^2z}{\pi}\,G^{q+1,0}_{p,q+1}\left(|z|^2\,\middle|\,\begin{matrix}\widetilde{a}-1\\ 0,\ \widetilde{b}-1\end{matrix}\right)\times {}_pF_q(\widetilde{a};\widetilde{b};|z|^2)\\ &\qquad\qquad\times\hat{D}(z)\hat{D}^{\dagger}(z)\\ &\qquad\qquad= \Gamma(\widetilde{b}/\widetilde{a})_pF_q(\widetilde{a};\widetilde{b};\hat{A}_+\hat{A}_-).\end{aligned} \tag{85}$$

Moreover, the CSs can also be defined as follows:

$$|z\rangle = \hat{D}(z)|0\rangle = \frac{1}{\sqrt{{}_pF_q(\widetilde{a};\widetilde{b};|z|^2)}}{}_pF_q(\widetilde{a};\widetilde{b};z\hat{A}_+)|0\rangle. \tag{86}$$

$$\langle z| = \langle 0|\hat{D}^{\dagger}(z) = \frac{\langle 0|{}_pF_q(\widetilde{a};\widetilde{b};z^*\hat{A}_-)}{\sqrt{{}_pF_q(\widetilde{a};\widetilde{b};|z|^2)}}. \tag{87}$$

Substituting into the decomposition relation the unit operator, as well as with the DOOT rules, we obtain

$$\begin{aligned}\hat{I} &= \int d\mu(z)\,|z\rangle\langle z| \\ &= \int d\mu(z)\,\frac{{}_pF_q(\widetilde{a};\widetilde{b};z\hat{A}_+){}_pF_q(\widetilde{a};\widetilde{b};z^*\hat{A}_-)}{{}_pF_q(\widetilde{a};\widetilde{b};|z|^2){}_pF_q(\widetilde{a};\widetilde{b};\hat{A}_+\hat{A}_-)}.\end{aligned} \tag{88}$$

After substituting the expression of the integration measure, we will obtain an important equality:

$$\begin{aligned}&\int \frac{d^2z}{\pi}\,G^{q+1,0}_{p,q+1}\left(|z|^2\,\middle|\,\begin{matrix}\widetilde{a}-1\\ 0,\ \widetilde{b}-1\end{matrix}\right)\\ &\quad\times \#{}_pF_q(\widetilde{a};\widetilde{b};z\hat{A}_+){}_pF_q(\widetilde{a};\widetilde{b};z^*\hat{A}_-)\#\\ &= \Gamma(\widetilde{b}/\widetilde{a})_pF_q(\widetilde{a};\widetilde{b};\hat{A}_+\hat{A}_-).\end{aligned} \tag{89}$$

With the help of this integral, a series of similar integrals can be calculated, for example, an integral whose integrand also contains the product $z^n(z^*)^m$ (for calculation details, see [19]). The following will also be useful for this purpose:

- The differentiation rule for hypergeometric functions [15]:

$$\begin{aligned}&\left(\frac{\partial}{\partial\hat{A}_+}\right)^n {}_pF_q(\widetilde{a};\widetilde{b};z\hat{A}_+)\\ &\quad= z^n\frac{\prod_{i=1}^{p}(a_i)_n}{\prod_{j=1}^{q}(b_j)_n}{}_pF_q(\widetilde{a}+n;\widetilde{b}+n;z\hat{A}_+).\end{aligned} \tag{90}$$

Consequently, we have

$$\begin{aligned}&z^n{}_pF_q(\widetilde{a};\widetilde{b};z\hat{A}_+)\\ &\quad= \frac{\prod_{j=1}^{q}(1-b_j)_n}{\prod_{i=1}^{p}(1-a_i)_n}\\ &\qquad\times\left(\frac{\partial}{\partial\hat{A}_+}\right)^n{}_pF_q(\widetilde{a}-n;\widetilde{b}-n;z\hat{A}_+).\end{aligned} \tag{91}$$

as well as their complex conjugate. Here we used the equality between the Pochhammer symbols [20]:

$$(a-n)_n = (-1)^n(1-a)_n. \tag{92}$$

- The angular integral has the following expression:

$$\begin{aligned}&\int_0^{2\pi}\frac{d\phi}{2\pi}\,{}_pF_q(\widetilde{a};\widetilde{b};z\hat{A}_+){}_pF_q(\widetilde{a};\widetilde{b};z^*\hat{A}_-)\\ &\qquad\qquad= \sum_{n=0}^{\infty}\frac{(\hat{A}_+\hat{A}_-)^n}{[\rho(n)]^2}\,(|z|^2)^n.\end{aligned} \tag{93}$$

- The result of the integral after $|z|^2$ is

$$\begin{aligned}&\int_0^{\infty} d(|z|^2)\,G^{q+1,0}_{p,q+1}\left(|z|^2\,\middle|\,\begin{matrix}\widetilde{a}-1\\ 0,\ \widetilde{b}-1\end{matrix}\right)(|z|^2)^n\\ &\qquad\qquad= \Gamma(\widetilde{b}/\widetilde{a})\,\rho(n).\end{aligned} \tag{94}$$

Consequently, the generalized analogue of the displacement operators satisfy the relation

$$\int d\mu(z)\,\hat{D}(z)\hat{D}^{\dagger}(z) = {}_pF_q(\widetilde{a};\widetilde{b};\hat{A}_+\hat{A}_-). \tag{95}$$

Considering the DOOT rules, according to which the creation and annihilation operators commute, so that in the end we have an ordered product of the operators, we can easily verify the validity of the properties below.

The generalized analogue of the displacement operators are not unitary operators, in other words, $\hat{D}(z)\hat{D}^{-1}(z) \neq 1$, because it is evident that FF equals 1. This can be shown.

By considering the scalar product of self-consistent states, which proves that the coherent states are normalized to unity, we obtain:

$$\langle z|z\rangle = \langle 0|\hat{D}^{\dagger}(z)\hat{D}(z)|0\rangle = 1. \tag{96}$$

This result might lead to the false conclusion that generalized shift operators are unitary operators.

In order to demonstrate this assertion, we will make use of the relations that generalize the actions of ladder operators on Fock vectors:

$$\begin{aligned}(\hat{A}_+)^s|n\rangle &= \sqrt{\frac{[e(n+s)]!}{[e(n)]!}}\,|n+s\rangle,\\(\hat{A}_-)^s|n\rangle &= \sqrt{\frac{[e(n)]!}{[e(n-s)]!}}\,|n-s\rangle,\\\langle n|(\hat{A}_+)^s &= \sqrt{\frac{[e(n)]!}{[e(n-s)]!}}\,\langle n-s|,\\\langle n|(\hat{A}_-)^s &= \sqrt{\frac{[e(n+s)]!}{[e(n)]!}}\,\langle n+s|.\end{aligned} \tag{97}$$

Consequently, successively we have

$$\begin{aligned}\langle z|z\rangle &= \langle 0|\hat{D}^\dagger(z)\hat{D}(z)|0\rangle\\&= \frac{1}{{}_pF_q(|z|^2)}\langle 0|{}_pF_q(z^*\hat{A}_-)\,{}_pF_q(z\hat{A}_+)|0\rangle\\&= \frac{1}{{}_pF_q(|z|^2)}\sum_{n,m=0}^{\infty}\frac{(z^*)^m}{\rho(m)}\frac{z^n}{\rho(n)}\\&\quad\times\langle 0|(\hat{A}_-)^m(\hat{A}_+)^n|0\rangle.\end{aligned} \tag{98}$$

Now,

$$\begin{aligned}\langle 0|(\hat{A}_-)^m(\hat{A}_+)^n|0\rangle &= \sqrt{[e(n)]![e(m)]!}\,\langle n|m\rangle\\&= [e(n)]!\,\delta_{nm}\\&= \rho(n)\,\delta_{nm},\end{aligned} \tag{99}$$

due to the fact that $\rho(n) = [e(n)]!$.

It is evident that this relation generally is not equal to unity, only in the case $n = m$.

Therefore, it is incorrect to state that the generalized analogue of the displacement operator is a unitary operator.

Regarding the name itself, however, we prefer to retain the term "generalized analogue of the displacement operator"—as it appears in the paper by Mojaveri and Dehghani [18]—which they introduced and used.

5 Construction of the displaced coherent states

Let's see what is the result of the successive application of two displacement operators, which depend on different arguments, respectively $\hat{D}(\varepsilon\ \sigma)$ and $\hat{D}(\lambda\ z)$ on the vacuum ket vector $|0>$. The goal is to obtain the CSs that have as argument a linear combination of complex variables $z = |z|\ \exp(i\ \phi_z)$ and $\sigma = |\sigma|\ \exp(i\ \phi_\sigma)$, that is $Z = \varepsilon\ z + \lambda\ \sigma$, where ε and λ are real numbers.

The complex number Z can be viewed from two perspectives:

Perspective 1: as a complex number resulting from the linear combination of two complex numbers, z and σ, with phases ϕ_z respectively ϕ_σ.

$$\begin{aligned}&Z = \varepsilon z + \lambda\sigma = \varepsilon|z|e^{i\phi_z} + \lambda|\sigma|e^{i\phi_\sigma},\\&|Z|^2 = \varepsilon^2|z|^2 + \lambda^2|\sigma|^2 + 2\varepsilon\lambda|z|\,|\sigma|\cos(\phi_z - \phi_\sigma).\end{aligned} \tag{100}$$

Perspective 2, as a single phasor (rotating vector) in the complex space $\mathbb{C}$, having magnitude $|Z|$ and phase ϕ_Z.

$$\begin{aligned}&Z = |Z|\exp(i\phi_Z) = \mathrm{Re}\,Z + i\,\mathrm{Im}\,Z,\\&\phi_Z = \arctan\left(\frac{\varepsilon|z|\sin\phi_z + \lambda|\sigma|\sin\phi_\sigma}{\varepsilon|z|\cos\phi_z + \lambda|\sigma|\cos\phi_\sigma}\right).\end{aligned} \tag{101}$$

The expressions obtained will be, among others, useful for calculating the Wigner operator in the representation of CSs [21]. Consequently, CSs having as argument the linear combination of complex numbers is defined as follows:

$$|Z\rangle = |\varepsilon z + \lambda\sigma\rangle = \hat{D}(\lambda\sigma)\hat{D}(\varepsilon z)|0\rangle. \tag{102}$$

Let us see what is the expression of the development of these CSs in terms of the Fock vectors $|n\rangle$.

We will start from the definition relation of CSs with the argument εz

$$\hat{D}(\varepsilon z)|0\rangle = |\varepsilon z\rangle, \tag{103}$$

and we will act on it with the displacement operator $\hat{D}(\lambda\sigma)$. We obtain, successively

$$\begin{aligned}\hat{D}(\lambda\sigma)|\varepsilon z\rangle &= \frac{{}_pF_q(\lambda\sigma\hat{A}_+)\,{}_pF_q(-\lambda\sigma^*\hat{A}_-)}{\sqrt{{}_pF_q(|\lambda\sigma|^2)}}\,|\varepsilon z\rangle\\&= \frac{{}_pF_q(\lambda\sigma\hat{A}_+)\,{}_pF_q(-\lambda\sigma^*\hat{A}_-)}{\sqrt{{}_pF_q(|\lambda\sigma|^2)}}\\&\quad\times\frac{{}_pF_q(\varepsilon z\hat{A}_+)}{\sqrt{{}_pF_q(|\varepsilon z|^2)}}|0\rangle.\end{aligned} \tag{104}$$

Now,

$$\begin{aligned}{}_pF_q(\lambda\sigma\hat{A}_+)|\varepsilon z\rangle &= \frac{1}{\sqrt{{}_pF_q(|\varepsilon z|^2)}}\\&\quad\times\sum_{m=0}^{\infty}\frac{(\lambda\sigma)^m}{\rho(m)}(\hat{A}_+)^m|\varepsilon z\rangle.\end{aligned} \tag{105}$$

Using

$$(\hat{A}_+)^m|n\rangle = \sqrt{\frac{\rho(n+m)}{\rho(n)}}|n+m\rangle, \tag{106}$$

and substituting above, using a new summation index $l = n + m$ and proceeding similarly to the case of Eqs. (2.11) and (2.12), we obtain

$$
\begin{aligned}
{}_pF_q(\lambda\sigma\hat{A}_+)|\varepsilon z\rangle &= \frac{1}{\sqrt{{}_pF_q(|\varepsilon z|^2)}} \\
&\times \sum_{l=0}^{\infty} \frac{1}{\sqrt{\rho(l)}} \left[\sum_{m=0}^{l} \binom{\rho(l)}{\rho(m)} (\varepsilon z)^{l-m} (\lambda\sigma)^m \right] |l\rangle,
\end{aligned} \tag{107}
$$

where we multiplied the numerator and denominator by $\sqrt{\rho(l)}$ and swapped the summations over m and l. Due to $\rho(l-m)$, we must have $m \le l$, so the summation over m must stop at $m = l$.

In addition, since $m \le l$ and the sum over the index l no longer depends on the index m, we can take $m = 0$.

The variable Z, defined previously, is ultimately also a complex number. Therefore, the shift operator corresponding to this variable is

$$
\hat{D}(Z) \equiv \frac{1}{\sqrt{{}_pF_q(|Z|^2)}} {}_pF_q(Z\hat{A}_+)\, {}_pF_q(-Z^*\hat{A}_-). \tag{108}
$$

Consequently, the coherent state in the variable Z is

$$
|Z\rangle = \hat{D}(Z)|0\rangle = \frac{1}{\sqrt{{}_pF_q(|Z|^2)}} {}_pF_q(Z\hat{A}_+)|0\rangle. \tag{109}
$$

Now,

$$
\begin{aligned}
{}_pF_q(Z\hat{A}_+)|0\rangle &= \sum_{n=0}^{\infty} \frac{Z^n}{\rho(n)} (\hat{A}_+)^n |0\rangle \\
&= \sum_{n=0}^{\infty} \frac{Z^n}{\sqrt{\rho(n)}} |n\rangle,
\end{aligned} \tag{110}
$$

so that the final expression is

$$
|Z\rangle = \frac{1}{\sqrt{{}_pF_q(|Z|^2)}} \sum_{n=0}^{\infty} \frac{Z^n}{\sqrt{\rho(n)}} |n\rangle. \tag{111}
$$

The quantity between curly brackets $\{\}$ represents precisely the *generalized expression of Newton's binomial*. For this quantity we will use curly brackets:

$$
\{Z\}^l \equiv \{\varepsilon z + \lambda\sigma\}^l = \sum_{m=0}^{l} \binom{\rho(l)}{\rho(m)} (\varepsilon z)^{l-m} (\lambda\sigma)^m. \tag{112}
$$

So, a sum between curly brackets raised to a power represents the generalized expression of the Newton's binomial:

$$
\begin{aligned}
\{x + y\}^l &\equiv \sum_{m=0}^{l} \binom{\rho(l)}{\rho(m)} x^{l-m} y^m \\
&= \sum_{m=0}^{l} \binom{l}{m} \binom{g(l)}{g(m)} x^{l-m} y^m.
\end{aligned} \tag{113}
$$

Note: We will use curly brackets $\{\}$ to avoid potential confusion with the generalized factorial $[\,]!$.

With this notation, we will obtain precisely the expression of the coherent state with a shifted argument.

$$
\begin{aligned}
{}_pF_q(\lambda\sigma\hat{A}_+)|\varepsilon z\rangle &= \frac{1}{\sqrt{{}_pF_q(|\varepsilon z|^2)}} \\
&\times \sum_{l=0}^{\infty} \frac{\{\varepsilon z + \lambda\sigma\}^l}{\sqrt{\rho(l)}} |l\rangle \\
&= |\varepsilon z + \lambda\sigma\rangle.
\end{aligned} \tag{114}
$$

It should be noted that the generalized Newton binomial, defined above, appears in the context of calculations involving coherent states (where, incidentally, the structure constant also appears). In other cases, the generalized Newton binomial obviously differs from the standard binomial coefficient expression.

$$
\{x + y\}^l \neq (x + y)^l. \tag{115}
$$

Therefore, the two notations should not be confused—that is,

$$
\begin{aligned}
\{x + y\}^l &= \sum_{k=0}^{l} \binom{\rho(l)}{\rho(k)} x^{l-k} y^k \\
&= \sum_{k=0}^{l} \binom{l}{k} \binom{g(l)}{g(k)} x^{l-k} y^k, \\
(x + y)^l &= \sum_{k=0}^{l} \binom{l}{k} x^{l-k} y^k.
\end{aligned} \tag{116}
$$

The connection between the two expressions is, see Eqs. (3.14)–(3.16)

$$
\lim_{\substack{g(l) \to 1 \\ g(l-k) \to 1}} \{x + y\}^l = (x + y)^l. \tag{117}
$$

Finally, after substitutions, we will get

$$
\hat{D}(\lambda\sigma)|\varepsilon z\rangle = \frac{\sqrt{{}_pF_q\left(|\{\varepsilon z + \lambda\sigma\}|^2\right)}}{\sqrt{{}_pF_q(|\lambda\sigma|^2)\, {}_pF_q(|\varepsilon z|^2)}} |Z\rangle, \tag{118}
$$

where the generalized hypergeometric function depending on the displaced variable is

$$
{}_pF_q\left(|\{Z\}|^2\right) \equiv {}_pF_q\left(|\{\varepsilon z + \lambda\sigma\}|^2\right). \tag{119}
$$

$$
\begin{aligned}
{}_pF_q(|Z|^2) &= \sum_{n=0}^{\infty} \frac{1}{\rho(n)} |\{\epsilon z + \lambda\sigma\}|^{2n} \\
&= \sum_{n=0}^{\infty} \frac{1}{\rho(n)} \sum_{s=0}^{n} \sum_{t=0}^{n} \binom{n}{s} \binom{n}{t} \\
&\quad \times (\epsilon z)^{n-s} (\lambda\sigma)^s (\epsilon z^*)^{n-t} (\lambda\sigma^*)^t.
\end{aligned} \tag{120}
$$

The final expression for the shifted CSs, defined by the generalized analogue of the displacement operators is then

$$|\varepsilon z + \lambda\sigma\rangle = \sqrt{\frac{{}_pF_q(|\lambda\sigma|^2)\,{}_pF_q(|\varepsilon z|^2)}{{}_pF_q\left(|\{\varepsilon z + \lambda\sigma\}|^2\right)}} \times \hat{D}(\lambda\sigma)\hat{D}(\varepsilon z)|0\rangle, \tag{121}$$

or, in equivalent manner, as their expansion with respect to the Fock vectors

$$|\varepsilon z + \lambda\sigma\rangle = \frac{1}{\sqrt{{}_pF_q\left(|\{\varepsilon z + \lambda\sigma\}|^2\right)}} \times \sum_{n=0}^{\infty} \frac{\{\varepsilon z + \lambda\sigma\}^n}{\sqrt{\rho(n)}}|n\rangle. \tag{122}$$

Finally, this is the expression for *generalized CSs with displaced (shifted) argument.*

This can also be written in the following form:

$$|\varepsilon z + \lambda\sigma\rangle = \frac{{}_pF_q\left(\{\varepsilon z + \lambda\sigma\}\hat{A}_+\right)}{\sqrt{{}_pF_q\left(|\{\varepsilon z + \lambda\sigma\}|^2\right)}}|0\rangle, \tag{123}$$

and similar for their complex conjugate counterpart.

Let's find the expression of the integration measure $d\mu(\varepsilon z + \lambda\sigma) = d\mu(Z)$ from the decomposition relation of the unit operator.

In essence, the variable Z is a linear combination of two independent complex variables, z and σ; therefore, ultimately, it is also a complex variable. In the complex plane $\mathbb{C}$, generally the Lebesgue integration measure is

$$\begin{aligned} d\mu(Z) &= \frac{1}{\pi} d(\mathrm{Re}\, Z)\, d(\mathrm{Im}\, Z)\, h(|Z|^2) \\ &= d(|Z|^2)\, h(|Z|^2)\, \frac{d\phi_Z}{2\pi}. \end{aligned} \tag{124}$$

where $h\ \left(|Z|^2\right)$ is a weighting function that must be determined specifically at a later stage.

If we consider the linear combination of two independent complex variables with equal "rights", $z, \sigma \in \mathbb{C}$, then the integration measure in polar coordinates will be symmetric in both variables.

$$\begin{aligned} d\mu(z,\sigma) &= d\mu(z)\, d\mu(\sigma)\, {}_pF_q\left(|\{\varepsilon z + \lambda\sigma\}|^2\right) \\ &= d(|z|^2)\, h(|z|^2) \frac{d\phi_z}{2\pi}\, d(|\sigma|^2)\, h(|\sigma|^2) \frac{d\phi_\sigma}{2\pi} \\ &\quad \times {}_pF_q\left(|\{\varepsilon z + \lambda\sigma\}|^2\right). \end{aligned} \tag{125}$$

Consequently, the integral over the entire space $\mathbb{C}^2$, in polar coordinates, is a quadruple integral:

$$\begin{aligned} &\int_{\mathbb{C}^2} f(z,\sigma)\, d\mu(z,\sigma) \\ &\quad = \int_0^\infty d(|z|^2)\, h(|z|^2) \int_0^{2\pi} \frac{d\phi_z}{2\pi} \\ &\quad \times \int_0^\infty d(|\sigma|^2)\, h(|\sigma|^2) \int_0^{2\pi} \frac{d\phi_\sigma}{2\pi} \\ &\quad \times f\left(|z| e^{i\phi_z}, |\sigma| e^{i\phi_\sigma}\right). \end{aligned} \tag{126}$$

The weight function of the one-variable integration measure is, see Eq. (3.31).

Given that the linear combination of the two complex variables, z and σ, is also a complex number $Z = \varepsilon z + \lambda\sigma$, expressing the formulas in terms of z and σ is equivalent to expressing them in terms of Z. Therefore, for the sake of simplicity, we will use the variable $\{Z\}$ where appropriate.

For simplification, we will use the variable $Z = \varepsilon z + \lambda\sigma$, so that the CSs will be

$$\begin{aligned} |Z\rangle &= \frac{1}{\sqrt{{}_pF_q\left(|\{Z\}|^2\right)}} \sum_{n=0}^{\infty} \frac{\{Z\}^n}{\sqrt{\rho(n)}}|n\rangle \\ &= \frac{1}{\sqrt{{}_pF_q\left(|\{Z\}|^2\right)}}\, {}_pF_q\Big(\{Z\}\hat{A}_+\Big)\, |0\rangle. \end{aligned} \tag{127}$$

$$\begin{aligned} \langle Z| &= \frac{1}{\sqrt{{}_pF_q\left(|\{Z\}|^2\right)}} \sum_{n=0}^{\infty} \frac{\{Z^*\}^n}{\sqrt{\rho(n)}}\langle n| \\ &= \frac{1}{\sqrt{{}_pF_q\left(|\{Z\}|^2\right)}} \langle 0|\, {}_pF_q\Big(\{Z^*\}\hat{A}_+\Big). \end{aligned} \tag{128}$$

and the decomposition relation of the unit operator will be written as

$$\int d\mu(Z)\, |Z\rangle\langle Z| = \hat{I}, \tag{129}$$

where the integration measure is

$$\begin{aligned} d\mu(Z) &= \frac{d^2 Z}{\pi}\, h\big(|\{Z\}|^2\big)\, {}_pF_q\big(|\{Z\}|^2\big) \\ &= \frac{d\phi_Z}{2\pi}\, d\big(|\{Z\}|^2\big)\, h\big(|\{Z\}|^2\big)\, {}_pF_q\big(|\{Z\}|^2\big), \end{aligned} \tag{130}$$

and we need to find the weighting function $h\big(|\{Z\}|^2\big)$.

By substituting the corresponding expressions, we will use the expressions of the hypergeometric functions written as power series of the respective variables. Consequently, we will first have to solve the angular integral

$$\int_0^{2\pi} \frac{d\phi_Z}{2\pi} \{Z\}^l \{Z^*\}^s = \left(|\{Z\}|^2\right)^l \delta_{ls}. \tag{131}$$

The angular integral is non-zero only if we have equal exponents: $s = l$, which is in accordance with the condition of satisfying the closure relation of the Fock vectors.

Therefore, we will have:

$$\sum_{l=0}^{\infty} \frac{|l\rangle\langle l|}{\rho(l)} \times \int_0^{\infty} d(|Z|^2)\, h(|Z|^2)\, {}_pF_q(|Z|^2)\, (|Z|^2)^l = \hat{I}. \tag{132}$$

It is obvious that the above integral must have the value

$$\int_0^{\infty} d\big(|\{Z\}|^2\big)\, h\big(|\{Z\}|^2\big)\, {}_pF_q\big(|\{Z\}|^2\big)\, \big(|\{Z\}|^2\big)^l = \rho(l) = \Gamma(\widetilde{a}/\widetilde{b})\, \Gamma(l+1)\, \frac{\prod_{j=1}^{q} \Gamma(b_j + l)}{\prod_{i=1}^{p} \Gamma(a_i + l)}. \tag{133}$$

This is an integral of the type of Stieltjes moments problem and their solution imply the Meijer G-function:

$$h\big(|\{Z\}|^2\big) = \Gamma(\widetilde{a}/\widetilde{b})\, {}_pF_q\big(|\{Z\}|^2\big) \times G_{p,q+1}^{q+1,0}\left(|\{Z\}|^2 \,\middle|\, \begin{matrix} \{a_i - 1\}_1^p \\ 0,\ \{b_j - 1\}_1^q \end{matrix}\right). \tag{134}$$

Consequently, the integration measure will have the final expression

$$d\mu(Z) = \Gamma(\widetilde{a}/\widetilde{b})\, d\big(|\{Z\}|^2\big)\, \frac{d\phi_Z}{2\pi}\, {}_pF_q\big(|\{Z\}|^2\big) \times G_{p,q+1}^{q+1,0}\left(|\{Z\}|^2 \,\middle|\, \begin{matrix} \{a_i - 1\}_1^p \\ 0,\ \{b_j - 1\}_1^q \end{matrix}\right). \tag{135}$$

In the notation of old variables, we will have

$$|\varepsilon z + \lambda\sigma\rangle = \frac{1}{\sqrt{{}_pF_q(|\{\varepsilon z + \lambda\sigma\}|^2)}} \times \sum_{n=0}^{\infty} \frac{\{\varepsilon z + \lambda\sigma\}^n}{\sqrt{\rho(n)}} |n\rangle. \tag{136}$$

It is not difficult to verify that the coherent states with displaced (shifted) argument are normalized to unity.

$$\langle \varepsilon z + \lambda\sigma | \varepsilon z + \lambda\sigma\rangle = \frac{1}{{}_pF_q(|\{\varepsilon z + \lambda\sigma\}|^2)} \times \sum_{n=0}^{\infty} \frac{\big(|\{\varepsilon z + \lambda\sigma\}|^2\big)^n}{\rho(n)} = 1. \tag{137}$$

It is well known from quantum mechanics or quantum optics literature that the choice of the integration measure, implicitly of their weight function, using a Hilbert space basis, is made such that the completeness relation for this basis is satisfied.

In the complex $\mathbb{C}^2$ space the complex variables z and σ are independent and equivalent, then it is logical that the global integration measure can be written as a product of integration measures corresponding to a single complex variable, in a perfect symmetric manner.

$$d\mu(\varepsilon z + \lambda\sigma) = \frac{d\mu(z)}{{}_pF_q(|\varepsilon z|^2)} \times \frac{d\mu(\sigma)}{{}_pF_q(|\lambda\sigma|^2)} \times {}_pF_q\big(|\{\varepsilon z + \lambda\sigma\}|^2\big), \tag{138}$$

where the integration measure for one variable is given by Eq. (3.28), see also [5]:

$$d\mu(z) = d(|z|^2)\, h(|z|^2) \frac{d\phi_z}{2\pi} = \Gamma(\widetilde{a}/\widetilde{b})\, d(|\varepsilon z|^2) \frac{d\phi_z}{2\pi}\, {}_pF_q(|\varepsilon z|^2) \times G_{p,q+1}^{q+1,0}\left(|\varepsilon z|^2 \,\middle|\, \begin{matrix} \{a_i - 1\}_1^p \\ 0,\ \{b_j - 1\}_1^q \end{matrix}\right), \tag{139}$$

and similar for the variable σ.

It can be demonstrated (for details, see Appendix 2) that the coherent states with displaced (shifted) argument satisfy the completeness relation (or decomposition of the identity operator):

$$\int d\mu(\varepsilon z + \lambda\sigma)\, |\varepsilon z + \lambda\sigma\rangle\langle \varepsilon z + \lambda\sigma| = \hat{I}. \tag{140}$$

The proof is presented in the Appendix.

The product $\{Z\}^n\{Z^*\}^m$ appears in many calculations. Let us evaluate it.

$$\{Z\}^n\{Z^*\}^m = \{\varepsilon z + \lambda\sigma\}^n\{\varepsilon z^* + \lambda\sigma^*\}^m = \sum_{s=0}^{n} \binom{\rho(n)}{\rho(s)} (\varepsilon z)^{n-s}(\lambda\sigma)^s \times \sum_{k=0}^{m} \binom{\rho(m)}{\rho(k)} (\varepsilon z^*)^{m-k}(\lambda\sigma^*)^k. \tag{141}$$

The expression of the integration measure can be deduced much more simply if the DOOT rules are used.

After suitable substitutions, the decomposition relation of the unit operator will be (see, Appendix 2)

$$\int \frac{d\mu(Z)}{{}_pF_q(|\{Z\}|^2)} \frac{{}_pF_q(\{Z\}\hat{A}_+)\, {}_pF_q(\{Z^*\}\hat{A}_-)}{{}_pF_q(\hat{A}_+\hat{A}_-)} = \frac{1}{{}_pF_q(\hat{A}_+\hat{A}_-)} \int_0^{\infty} d(|Z|^2)\, h(|Z|^2) \times \int_0^{2\pi} \frac{d\phi_Z}{2\pi}\, {}_pF_q(\{Z\}\hat{A}_+)\, {}_pF_q(\{Z^*\}\hat{A}_-) = 1. \tag{142}$$

The angular integral is

$$\int_0^{2\pi} \frac{d\phi_Z}{2\pi} {}_pF_q(\{Z\}\hat{A}_+)\,{}_pF_q(\{Z^*\}\hat{A}_-)$$
$$= \# \sum_{n=0}^{\infty} \frac{(|\{Z\}|\hat{A}_+)^n}{\rho(n)} \sum_{m=0}^{\infty} \frac{(|\{Z^*\}|\hat{A}_-)^m}{\rho(m)} \#$$
$$\times \int_0^{2\pi} \frac{d\phi_Z}{2\pi} \exp\{i(n-m)\phi_Z\}$$
$$= \sum_{n=0}^{\infty} \frac{(\hat{A}_+\hat{A}_-)^n}{[\rho(n)]^2} (|\{Z\}|^2)^n\, \delta_{nm}. \tag{143}$$

The last angular integral has the value $(|\{Z\}|^2)^l \delta_{ls}$, so that the previous relationship becomes

$$\frac{1}{{}_pF_q(\hat{A}_+\hat{A}_-)} \sum_{l=0}^{\infty} \frac{(\hat{A}_+\hat{A}_-)^l}{[\rho(l)]^2}$$
$$\times \int_0^{\infty} d(|Z|^2)\, h(|Z|)\, (|Z|^2)^l = 1, \tag{144}$$

and we obtain the same expression as above and finally, the integration measure.

The decomposition relation of the unit operator will now be written as

$$\int d\mu(Z)\, |Z\rangle\langle Z| = \frac{1}{{}_pF_q(\hat{A}_+\hat{A}_-)}$$
$$\times \int d\mu(Z) \frac{1}{{}_pF_q(|\{Z\}|^2)}$$
$$\times {}_pF_q(\{Z\}\hat{A}_+)\,{}_pF_q(\{Z^*\}\hat{A}_-)$$
$$= 1. \tag{145}$$

In the old variables, this becomes

$$\int \frac{d^2(\epsilon z + \lambda\sigma)}{\pi}$$
$$G^{q+1,0}_{p,q+1}\left(|\{\epsilon z + \lambda\sigma\}|^2 \,\middle|\, \begin{matrix} \{a_i - 1\}_1^p \\ 0,\ \{b_j - 1\}_1^q \end{matrix} \right)$$
$$\times {}_pF_q(\{\epsilon z + \lambda\sigma\}\hat{A}_+)$$
$$\times {}_pF_q(\{\epsilon z^* + \lambda\sigma^*\}\hat{A}_-)$$
$$= {}_pF_q(\hat{A}_+\hat{A}_-). \tag{146}$$

In the context of what is indicated above, let us define the generalized analogue of the displacement operator corresponding to the variable Z, as follows:

$$\hat{D}(Z) = \frac{1}{\sqrt{{}_pF_q(|\{Z\}|^2)}} {}_pF_q(\{Z\}\hat{A}_+),$$
$$\hat{D}(\varepsilon z + \lambda\sigma) = \frac{{}_pF_q(\{\varepsilon z + \lambda\sigma\}\hat{A}_+)}{\sqrt{{}_pF_q(|\{\varepsilon z + \lambda\sigma\}|^2)}}. \tag{147}$$

so CSs with shifted argument can be defined as follows

$$|Z\rangle = \hat{D}(Z)|0\rangle$$
$$= \frac{1}{\sqrt{{}_pF_q(|\{Z\}|^2)}} {}_pF_q(\{Z\}\hat{A}_+)|0\rangle,$$
$$|\varepsilon z + \lambda\sigma\rangle = \hat{D}(\varepsilon z + \lambda\sigma)|0\rangle$$
$$= \frac{1}{\sqrt{{}_pF_q(|\{\varepsilon z + \lambda\sigma\}|^2)}}$$
$$\times {}_pF_q(\{\varepsilon z + \lambda\sigma\}\hat{A}_+)|0\rangle. \tag{148}$$

or, in an equivalent manner

$$|Z\rangle = \frac{1}{\sqrt{{}_pF_q(|\{Z\}|^2)}} \times \sum_{n=0}^{\infty} \frac{\{Z\}^n}{\sqrt{\rho(n)}}|n\rangle, \tag{149}$$

respectively, in the old variables

$$|\varepsilon z + \lambda\sigma\rangle = \frac{1}{\sqrt{{}_pF_q(|\{\varepsilon z + \lambda\sigma\}|^2)}}$$
$$\times \sum_{n=0}^{\infty} \frac{\{\varepsilon z + \lambda\sigma\}^n}{\sqrt{\rho(n)}}|n\rangle. \tag{150}$$

In conclusion, bringing together the two ways of defining CSs with displaced argument (i.e. the *Perspective 1* and *2*), we will be able to write

$$|\varepsilon z + \lambda\sigma\rangle = \hat{D}(\varepsilon z + \lambda\sigma)|0\rangle = \hat{D}(\varepsilon z)\hat{D}(\lambda\sigma)|0\rangle. \tag{151}$$

$$|\varepsilon z + \lambda\sigma\rangle = \frac{{}_pF_q\Big(\{\varepsilon z + \lambda\sigma\}\hat{A}_+\Big)}{\sqrt{{}_pF_q(|\{\varepsilon z + \lambda\sigma\}|^2)}}|0\rangle$$
$$= \frac{{}_pF_q(\lambda z\hat{A}_+)}{\sqrt{{}_pF_q(|\varepsilon z|^2)}} \frac{{}_pF_q(\varepsilon\sigma\hat{A}_+)}{\sqrt{{}_pF_q(|\lambda\sigma|^2)}}|0\rangle. \tag{152}$$

From here the operational equalities for generalized analogue of the displacement operators follow finally, as a conclusion:

$$\hat{D}(\epsilon z + \lambda\sigma)\,|0\rangle = \hat{D}(\epsilon z)\,\hat{D}(\lambda\sigma)\,|0\rangle. \tag{153}$$

Remark: This relation holds only when the operators act on the vacuum state. As an operator equality, $\hat{D}(\epsilon z + \lambda\sigma) \neq \hat{D}(\epsilon z)\hat{D}(\lambda\sigma)$ in general, because the generalized displacement operators are not unitary and do not form a group representation.

$${}_pF_q\Big(\{\varepsilon z + \lambda\sigma\}\hat{A}_+\Big)$$
$$= \sqrt{\frac{{}_pF_q(|\{\varepsilon z + \lambda\sigma\}|^2)}{{}_pF_q(|\varepsilon z|^2)\,{}_pF_q(|\lambda\sigma|^2)}}$$
$$\times {}_pF_q(\varepsilon z\hat{A}_+)\,{}_pF_q(\lambda\sigma\hat{A}_+). \tag{154}$$

In general, a function of two independent variables can be expanded into a power series of the form

$$
\begin{aligned}
&{}_pF_q(\{\epsilon z+\lambda\sigma\}\hat{A}_+) \\
&\quad=\sum_{n=0}^{\infty}\frac{1}{\rho(n)}\left[\sum_{k=0}^{n}\binom{\rho(n)}{\rho(k)}(\epsilon z)^{n-k}(\lambda\sigma)^k\right](\hat{A}_+)^n.
\end{aligned}
\tag{155}
$$

Let us consider an operator-valued function depending on creation and annihilation operators, ordered according to the DOOT technique.

$$
f(\hat{A}_+,\hat{A}_-)=\sum_{k,s=0}^{\infty}C_{ks}\,(\hat{A}_+)^k(\hat{A}_-)^s. \tag{156}
$$

We aim to calculate the expected value of the operator $\hat{D}^\dagger(z)f(\hat{A}_+,\hat{A}_-)\hat{D}(z)$ in the coherent state $|\sigma\rangle$, that is, practically the expected value of the operator $f(\hat{A}_+,\hat{A}_-)$ in the displaced coherent state $|z+\sigma\rangle$. We will have, successively

$$
\begin{aligned}
&\langle\sigma|\hat{D}^\dagger(z)f(\hat{A}_+,\hat{A}_-)\hat{D}(z)|\sigma\rangle \\
&\quad=\langle z+\sigma|f(\hat{A}_+,\hat{A}_-)|z+\sigma\rangle \\
&\quad=\sum_{k,s=0}^{\infty}C_{ks}\langle z+\sigma|(\hat{A}_+)^k(\hat{A}_-)^s|z+\sigma\rangle.
\end{aligned}
\tag{157}
$$

Their expectation value in the coherent states representation is

$$
\begin{aligned}
&\langle\varepsilon z+\lambda\sigma|f(\hat{A}_+,\hat{A}_-)|\varepsilon z+\lambda\sigma\rangle \\
&=\sum_{k,s=0}^{\infty}C_{ks}\langle\varepsilon z+\lambda\sigma|(\hat{A}_+)^k(\hat{A}_-)^s|\varepsilon z+\lambda\sigma\rangle \\
&=\sum_{k,s=0}^{\infty}C_{ks}\{\varepsilon z^*+\lambda\sigma^*\}^k\{\varepsilon z+\lambda\sigma\}^s \\
&=f(\varepsilon z^*+\lambda\sigma^*,\varepsilon z+\lambda\sigma),
\end{aligned}
\tag{158}
$$

because

$$
(\hat{A}_-)^s|\varepsilon z+\lambda\sigma\rangle=\{\varepsilon z+\lambda\sigma\}^s|\varepsilon z+\lambda\sigma\rangle, \tag{159}
$$

$$
\langle\varepsilon z+\lambda\sigma|(\hat{A}_+)^k=\{\varepsilon z^*+\lambda\sigma^*\}^k\langle\varepsilon z+\lambda\sigma|, \tag{160}
$$

and

$$
\begin{aligned}
&\{\varepsilon z^*+\lambda\sigma^*\}^k\{\varepsilon z+\lambda\sigma\}^s \\
&=\sum_{m=0}^{k}\binom{\rho(k)}{\rho(m)}(\varepsilon z^*)^{k-m}(\lambda\sigma^*)^m \\
&\quad\times\sum_{j=0}^{s}\binom{\rho(s)}{\rho(j)}(\varepsilon z)^{s-j}(\lambda\sigma)^j.
\end{aligned}
\tag{161}
$$

Finally, we obtain

$$
\begin{aligned}
&\langle\sigma|\hat{D}^\dagger(z)f(\hat{A}_+,\hat{A}_-)\hat{D}(z)|\sigma\rangle \\
&=\sum_{k,s=0}^{\infty}C_{ks}\sum_{m=0}^{k}\binom{\rho(k)}{\rho(m)}(\varepsilon z^*)^{k-m}(\lambda\sigma^*)^m \\
&\quad\times\sum_{j=0}^{s}\binom{\rho(s)}{\rho(j)}(\varepsilon z)^{s-j}(\lambda\sigma)^j.
\end{aligned}
\tag{162}
$$

or, in a more compact form:

$$
\begin{aligned}
&\langle\sigma|\hat{D}^\dagger(z)f(\hat{A}_+,\hat{A}_-)\hat{D}(z)|\sigma\rangle \\
&\quad=\sum_{k,s=0}^{\infty}C_{ks}\{\varepsilon z^*+\lambda\sigma^*\}^k\{\varepsilon z+\lambda\sigma\}^s \\
&\quad=f(\varepsilon z^*+\lambda\sigma^*,\varepsilon z+\lambda\sigma).
\end{aligned}
\tag{163}
$$

Of particular interest are the following two cases, where the function $f(\hat{A}_+,\hat{A}_-)$ is equal to one of the operators, $\hat{A}_+$ or $\hat{A}_-$ and $\varepsilon=\lambda=1$, $C_{00}=0$.

Case 1 $f(\hat{A}_+)=\hat{A}_+$, $k=1$, $s=0$, $C_{10}=1$

$$
\langle\sigma|\hat{D}^\dagger(z)\hat{A}_+\hat{D}(z)|\sigma\rangle=z^*+\sigma^*. \tag{164}
$$

Case 2 $f(\hat{A}_-)=\hat{A}_-$, $k=0$, $s=1$, $C_{01}=1$

$$
\langle\sigma|\hat{D}^\dagger(z)\hat{A}_-\hat{D}(z)|\sigma\rangle=z+\sigma. \tag{165}
$$

The following formula, satisfied by the canonical displacement operators, is well known from quantum optics textbooks:

$$
\hat{D}^\dagger(z)\hat{a}\hat{D}(z)=\hat{a}+z. \tag{166}
$$

so that we obtain a result similar to the previous one:

$$
\langle\sigma|\hat{D}^\dagger(z)\hat{a}\hat{D}(z)|\sigma\rangle=z+\sigma. \tag{167}
$$

and similar for $\hat{a}^\dagger$.

These results also justify retaining the name the generalized analogue of the displacement operator for $\hat{D}(z)$, despite the fact that this operator is not unitary.

In line with the above, considering an analytical function of operators that takes the normal-ordered product—in the DOOT sense—of generalized creation and annihilation operators as its argument,

$$
f(\hat{A}_+\hat{A}_-)=\sum_{k=0}^{\infty}C_{kk}(\hat{A}_+\hat{A}_-)^k, \tag{168}
$$

we obtain the following relation, the proof being similar:

$$
\begin{aligned}
&\langle\sigma|\hat{D}^\dagger(z)f(\hat{A}_+\hat{A}_-)\hat{D}(z)|\sigma\rangle \\
&\quad=\sum_{k=0}^{\infty}C_{kk}\left(|\{\varepsilon z+\lambda\sigma\}|^2\right)^k \\
&\quad=f\left(|\{\varepsilon z+\lambda\sigma\}|^2\right).
\end{aligned}
\tag{169}
$$

In the particular case, for $\varepsilon = \lambda = 1$, $p = q = 0$, $\widetilde{a} = \widetilde{b}$ and considering σ as a constant, the following equality holds.

$$\begin{aligned}\int \frac{d^2z}{\pi} G^{1,0}_{0,1}\big(|\{z+\sigma\}|^2\big|0\big) \\ \times {}_0F_0\Big(\{z+\sigma\}\hat{A}_+\Big)\, {}_0F_0\Big(\{z^*+\sigma^*\}\hat{A}_-\Big) \\ = {}_0F_0(\hat{A}_+\hat{A}_-) \\ = \exp(\hat{A}_+\hat{A}_-).\end{aligned} \tag{170}$$

and, because $\{z+\sigma\} = z+\sigma$, and $\rho(1) = 1$, as well as of the following limits

$$\lim_{\substack{p=q=0\\ a=b}} {}_pF_q(\widetilde{a};\widetilde{b};x) = {}_0F_0(;;x) = e^x. \tag{171}$$

$$\lim_{\substack{p=q=0\\ a=b}} G^{q+1,0}_{p,q+1}\left(x\middle|\begin{matrix}\widetilde{a}-1\\ 0,\widetilde{b}-1\end{matrix}\right) = G^{1,0}_{0,1}(x|0) = e^{-x}. \tag{172}$$

we obtain the same results as in [4].

$$\int \frac{d^2z}{\pi} e^{-|z+\sigma|^2} e^{(z+\sigma)\hat{A}_+} e^{(z^*+\sigma^*)\hat{A}_-} = e^{\hat{A}_+\hat{A}_-}. \tag{173}$$

This integral can be transformed as follows

$$\begin{aligned}\int \frac{d^2z}{\pi} \exp\big\{-|z|^2 \\ + (\hat{A}_+ - \sigma^*)z + (\hat{A}_- - \sigma)z^*\big\} \\ = \exp\big\{(\hat{A}_+ - \sigma^*)(\hat{A}_- - \sigma)\big\}.\end{aligned} \tag{174}$$

6 Particular cases

Let us specify the values of the real constants ε and λ.

a) Case $\varepsilon = 1$ ***and*** $\lambda = 0$ ***(generalized CSs with non-displaced argument).*** This is the usual case of obtaining expressions for generalized CSs. Since the variable is no longer a binomial, we can give up the curly brace $\{\cdots\}$, so $\{z\} = z$, and the expansion of the generalized CSs is then

$$\begin{aligned}|z\rangle &= \frac{1}{\sqrt{{}_pF_q(|z|^2)}} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{\rho(n)}}|n\rangle \\ &= \frac{1}{\sqrt{{}_pF_q(|z|^2)}} {}_pF_q(z\hat{A}_+)|0\rangle.\end{aligned} \tag{175}$$

with the generalized analogue of the displacement operator

$$\hat{D}(z) = \frac{1}{\sqrt{{}_pF_q(|z|^2)}} {}_pF_q(z\hat{A}_+)\, {}_pF_q(z^*\hat{A}_-). \tag{176}$$

The corresponding expression for the integration measure is

$$\begin{aligned}d\mu(z) = \varGamma(\widetilde{a}/\widetilde{b})\, d(|z|^2)\frac{d\phi_z}{2\pi} {}_pF_q(|z|^2) \\ \times G^{q+1,0}_{p,q+1}\left(|z|^2 \,\middle|\, \begin{matrix}\{a_i-1\}_1^p\\ 0,\ \{b_j-1\}_1^q\end{matrix}\right).\end{aligned} \tag{177}$$

guarantees the validity of the decomposition relation of the unit operator, so that the corresponding integral of the Stieltjes moments is

$$\begin{aligned}\int_0^\infty d(|z|^2) G^{q+1,0}_{p,q+1}\left(|z|^2 \,\middle|\, \begin{matrix}\{a_i-1\}_1^p\\ 0,\ \{b_j-1\}_1^q\end{matrix}\right)(|z|^2)^l \\ = \frac{1}{\varGamma(\widetilde{a}/\widetilde{b})}\rho(l) = \frac{1}{\varGamma(\widetilde{a}/\widetilde{b})} l! \frac{\prod_{j=1}^q \varGamma(b_j+l)}{\prod_{i=1}^p \varGamma(a_i+l)}.\end{aligned} \tag{178}$$

By customizing the indices $p$and q, as well as the sets of coefficients $\widetilde{a}$ and $\widetilde{b}$, numerous examples can be obtained, as well as applications, for the various quantum systems. We continue to reproduce only the case of the pseudoharmonic oscillator (PHO). Let's highlight the following two subcases, which can also be considered as sequels to the previous case.

a1) ***Subcase*** $\varepsilon = 1$, $\lambda = 0$, $p = q$, and $\widetilde{a} = \widetilde{b}$ (*canonical CSs with non-displaced argument*). The canonical CSs are

$$\begin{aligned}|z\rangle &= e^{-\frac{1}{2}|z|^2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}}|n\rangle \\ &= e^{-\frac{1}{2}|z|^2} e^{z\hat{a}^\dagger}|0\rangle.\end{aligned} \tag{179}$$

the integration measure $d\mu(z) = \frac{d^2z}{2\pi} = d(|z|^2)\frac{d\phi_z}{2\pi}$, and the displacement operator is the usual one

$$\begin{aligned}\hat{D}(z) &= \frac{1}{\sqrt{{}_0F_0(|z|^2)}} {}_0F_0(z\hat{a}^\dagger)\, {}_0F_0(-z^*\hat{a}) \\ &= e^{-\frac{1}{2}|z|^2} e^{z\hat{a}^\dagger - z^*\hat{a}}.\end{aligned} \tag{180}$$

a2) ***Subcase*** $\varepsilon = 1$, $\lambda = 0$, $p = 0$, $q = 1$, $\widetilde{a} = \varnothing$, $\widetilde{b} = \varsigma + 1/2$ (*corresponding to the pseudoharmonic oscillator, PHO*)

The pseudoharmonic oscillator (PHO) is an exactly solvable quantum model, proposed by Goldman and Kryvchenkov [19] and their radial part of potential in radial coordinate r is

$$V(r) = \frac{1}{2}m\omega^2 r^2 + \frac{\hbar^2}{2m}\varsigma(\varsigma-1)\frac{1}{r^2}. \tag{181}$$

where m, ω and ς represent respectively: the mass of particle (or the reduced mass if this potential describe the nuclear motion of diatomic molecule), the frequency, and the strength of the external field.

The corresponding creation / annihilation operators for the PHO are

$$\begin{aligned} \hat{J}_+|n;\varsigma\rangle &= \sqrt{(n+1)\left(n+\varsigma+\frac{1}{2}\right)}\,|n+1;\varsigma\rangle, \\ \hat{J}_-|n;\varsigma\rangle &= \sqrt{n\left(n+\varsigma-\frac{1}{2}\right)}\,|n-1;\varsigma\rangle. \end{aligned} \tag{182}$$

The generalized CSs are defined as

$$\hat{J}_-|z;\varsigma\rangle = z|z;\varsigma\rangle, \tag{183}$$

and are expanded in the Fock-vectors basis as follows:

$$\begin{aligned} |z;\varsigma\rangle &= \frac{1}{\sqrt{{}_0F_1(;\varsigma+\frac{1}{2};|z|^2)}} \\ &\quad \times \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!\,(\varsigma+\frac{1}{2})_n}}|n;\varsigma\rangle \\ &= \frac{1}{\sqrt{{}_0F_1(;\varsigma+\frac{1}{2};|z|^2)}} \\ &\quad \times {}_0F_1(;\varsigma+\frac{1}{2};z\hat{J}_+)|0;\varsigma\rangle. \end{aligned} \tag{184}$$

The normalization hypergeometric function is connected with the modified Bessel function of the first kind

$${}_0F_1(;\varsigma+\tfrac{1}{2};|z|^2) = \Gamma(\varsigma+\tfrac{1}{2})\,|z|^{\frac{1}{2}-\varsigma}\,I_{\varsigma-\frac{1}{2}}(2|z|), \tag{185}$$

so that the CSs becomes [16]

$$\begin{aligned} |z;\varsigma\rangle &= \sqrt{\frac{|z|^{\varsigma-\frac{1}{2}}}{I_{\varsigma-\frac{1}{2}}(2|z|)}} \\ &\quad \times \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!\,\Gamma(\varsigma+\frac{1}{2}+n)}}|n;\varsigma\rangle. \end{aligned} \tag{186}$$

Then, the corresponding generalized analogue of the displaced operator acting on the vacuum ket state of the PHO, $|0;\varsigma\rangle$ is

$$\hat{D}(z) \equiv \frac{{}_0F_1(;\varsigma+\frac{1}{2};z\hat{J}_+)\,{}_0F_1(;\varsigma+\frac{1}{2};-z^*\hat{J}_-)}{\sqrt{{}_0F_1(;\varsigma+\frac{1}{2};|z|^2)}}. \tag{187}$$

b) Case $\varepsilon = 1$ and $\lambda = 1$ (*CSs with argument sum of two complex numbers*). The expression for generalized CSs with shifted argument is

$$\begin{aligned} |z+\sigma\rangle &= \frac{1}{\sqrt{{}_pF_q(|\{z+\sigma\}|^2)}} \sum_{n=0}^{\infty} \frac{\{z+\sigma\}^n}{\sqrt{\rho(n)}}|n\rangle \\ &= \frac{1}{\sqrt{{}_pF_q(|\{z+\sigma\}|^2)}}\,{}_pF_q(\{z+\sigma\}\hat{A}_+)|0\rangle. \end{aligned} \tag{188}$$

and similar their conjugate counterpart.

As we stated previously, the expression in the curly brackets {} can be interpreted as a *generalized binomial sum.*

$$\{z+\sigma\}^l \equiv \sum_{n=0}^{l} \binom{\rho(l)}{\rho(n)} z^{l-n}\sigma^n, \tag{189}$$

and similarly their complex conjugate.

It is not difficult to prove that $\langle z+\sigma|z+\sigma\rangle = 1$, and also that the fundamental property of CSs, i.e. the decomposition relation of the unity operator is also satisfied. Since the variables z and σ have "equal rights", the expression must be symmetrical:

$$\iint d\mu(z,\sigma)\,|z+\sigma\rangle\langle z+\sigma| = \hat{I}. \tag{190}$$

Therefore, we will look for the integration measure in the form

$$\begin{aligned} d\mu(z,\sigma) = &\frac{d^2z}{\pi}\,\frac{d^2\sigma}{\pi} \\ &\times h_z(|z|)\,h_\sigma(|\sigma|) \\ &\times {}_pF_q(|\{z+\sigma\}|^2), \end{aligned} \tag{191}$$

and we need to find the weighting functions $h_z(|z|)$ and $h_\sigma(|\sigma|)$. So, we will have

$$\begin{aligned} &\int \frac{d^2z}{\pi}h_z(|z|) \int \frac{d^2\sigma}{\pi}h_\sigma(|\sigma|) \\ &\quad \times \sum_{l=0}^{\infty} \frac{\{z+\sigma\}^l}{\sqrt{\rho(l)}}|l\rangle \sum_{s=0}^{\infty} \frac{\{z^*+\sigma^*\}^s}{\sqrt{\rho(s)}}\langle s| = \hat{I}. \end{aligned} \tag{192}$$

Let's break down the product below

$$\begin{aligned} \{z+\sigma\}^l\,\{z^*+\sigma^*\}^s = &\sum_{n=0}^{l} \binom{\rho(l)}{\rho(n)} z^{l-n}\sigma^n \\ &\times \sum_{m=0}^{s} \binom{\rho(s)}{\rho(m)} (z^*)^{s-m}(\sigma^*)^m. \end{aligned} \tag{193}$$

In order to be able to satisfy the closing relation of the Fock vectors

$$\sum_{l=0}^{\infty} |l\rangle\langle l| = \hat{I}, \tag{194}$$

it is obvious, from now on, that we must have equality between the summation indices $s = l$.

Consequently, we will write

$$\begin{aligned} &\sum_{l=0}^{\infty} \frac{|l\rangle\langle l|}{\rho(l)} \sum_{n=0}^{l} \binom{\rho(l)}{\rho(n)} \int \frac{d^2z}{\pi}h_z(|z|)\,z^{l-n}(z^*)^{l-m} \\ &\quad \times \sum_{m=0}^{l} \binom{\rho(s)}{\rho(m)} \int \frac{d^2\sigma}{\pi}h_\sigma(|\sigma|)\,\sigma^n(\sigma^*)^m = 1. \end{aligned} \tag{195}$$

The angular integrals above will be different from zero only if the powers of the complex variable and the corresponding complex conjugate are equal. For example:

$$\begin{aligned}\int \frac{d^2\sigma}{\pi} h_\sigma(|\sigma|)\,\sigma^n(\sigma^*)^m &= \int_0^\infty d(|\sigma|^2) h_\sigma(|\sigma|) \\ &\times \int_0^{2\pi} \frac{d\phi_\sigma}{2\pi}\sigma^n(\sigma^*)^m \\ &= \int_0^\infty d(|\sigma|) h_\sigma(|\sigma|)\,(|\sigma|^2)^n\,\delta_{nm} \\ &= \Gamma(\widetilde{a}/\widetilde{b})\,\rho(n)\,\delta_{nm}.\end{aligned} \tag{196}$$

Consequently, the relationship becomes

$$\begin{aligned}1 = \sum_{l=0}^{\infty} |l\rangle\langle l|\,[g(l)]! \times \sum_{n=0}^{l}\binom{l}{n} \\ \times \left\{\frac{1}{(l-n)!\,([g(l-n)]!)^2}\,\mathrm{Int}_z\right\} \\ \times \left\{\frac{1}{n!\,([g(n)]!)^2}\,\mathrm{Int}_\sigma\right\},\end{aligned} \tag{197}$$

where by Int_x we denoted the abbreviated integrals above. For reasons of symmetry, we require that they have the following values:

$$\begin{aligned}\mathrm{Int}_{|z|} &\equiv \frac{1}{\sqrt{2^l}}\int_0^\infty d(|z|^2)\,h_z(|z|)\,(|z|^2)^{l-n} \\ &= \frac{1}{\sqrt{2^l\,[g(l)]!}}\,(l-n)!\,([g(l-n)]!)^2,\end{aligned} \tag{198}$$

$$\begin{aligned}\mathrm{Int}_{|\sigma|} &\equiv \frac{1}{\sqrt{2^l}}\int_0^\infty d(|\sigma|^2)\,h_\sigma(|\sigma|)\,(|\sigma|^2)^{n} \\ &= \frac{1}{\sqrt{2^l\,[g(l)]!}}\,n!\,([g(n)]!)^2.\end{aligned} \tag{199}$$

This kind of integral is solved by first changing the exponent of the integration variable, so that it will be $s-1$, with s a complex number:

$$\begin{aligned}\mathrm{Int}_{|z|} &\equiv \int_0^\infty d(|z|^2)\,h_z(|z|)\,(|z|^2)^{s-1}\Big|_{n=l+1-s} \\ &= \frac{1}{\sqrt{2^l\,[g(l)]!}}\,[\Gamma(\widetilde{a}/\widetilde{b})]^2 \\ &\times \frac{\Gamma(s)\prod_{j=1}^q[\Gamma(b_j-1+s)]^2}{\prod_{i=1}^p[\Gamma(a_i-1+s)]^2},\end{aligned} \tag{200}$$

$$\begin{aligned}\mathrm{Int}_{|z|} &\equiv \int_0^\infty d(|z|^2)\,h_z(|z|)\,(|z|^2)^{s-1}\Big|_{n=l+1-s} \\ &= \frac{1}{\sqrt{2^l\,[g(l)]!}}\,[\Gamma(\widetilde{a}/\widetilde{b})]^2 \\ &\times \frac{\Gamma(s)\prod_{j=1}^q[\Gamma(b_j-1+s)]^2}{\prod_{i=1}^p[\Gamma(a_i-1+s)]^2},\end{aligned} \tag{201}$$

$$\begin{aligned}\mathrm{Int}_{|\sigma|} &\equiv \int_0^\infty d(|\sigma|^2)\,h_\sigma(|\sigma|)\,(|\sigma|^2)^{s-1}\Big|_{n=s-1} \\ &= \frac{1}{\sqrt{2^l\,[g(l)]!}}\,[\Gamma(\widetilde{a}/\widetilde{b})]^2 \\ &\times \frac{\Gamma(s)\prod_{j=1}^q[\Gamma(b_j-1+s)]^2}{\prod_{i=1}^p[\Gamma(a_i-1+s)]^2}.\end{aligned} \tag{202}$$

The above indexes were changed in order to be able to use the classical integral of Meijer's G functions [15]:

$$\begin{aligned}&\int_0^\infty dx\,x^{s-1}\,G_{p,q}^{m,n}\left(\beta x \,\middle|\, \begin{matrix}\{a_i\}_1^n & \{a_i\}_{n+1}^p \\ \{b_j\}_1^m & \{b_j\}_{m+1}^q\end{matrix}\right) \\ &= \frac{1}{\beta^s}\,\frac{\prod_{j=1}^m\Gamma(b_j+s)\prod_{i=1}^n\Gamma(1-a_i-s)}{\prod_{j=m+1}^q\Gamma(1-b_j-s)\prod_{i=n+1}^p\Gamma(a_i+s)}.\end{aligned} \tag{203}$$

As a consequence, the following expressions of the weighting functions are obtained

$$\begin{aligned}h_z(|z|) &= \frac{1}{\sqrt{2^l\,[g(l)]!}}\,[\Gamma(\widetilde{a}/\widetilde{b})]^2 \\ &\times G_{2p,\,2q+1}^{2q+1,\,0}\left(|z|^2 \,\middle|\, \begin{matrix}\{a_i-1\}_1^p & \{a_i-1\}_1^p \\ 0,\ \{b_j-1\}_1^q & \{b_j-1\}_1^q\end{matrix}\right),\end{aligned} \tag{204}$$

with the notations

$$\begin{aligned}&\{A\}_1^{2p} \equiv \{a_i-1\}_1^p,\ \{a_i-1\}_1^p, \\ &\{B\}_1^{2q+1} \equiv 0,\ \{b_j-1\}_1^q,\ \{b_j-1\}_1^q.\end{aligned} \tag{205}$$

and similar for $h_\sigma\,(|z|)$.

A beautiful symmetry is observed relative to the variables z and σ, which was to be expected.

Substituting these results in the decomposition relation of the unit operator and taking into account the equality

$$\sum_{m=0}^{l}\binom{l}{m} = 2^l, \tag{206}$$

we will find that the closure relation of the Fock vectors is satisfied.

Let us now examine the particular case when $p = q = 0$, $\{a_i\}_1^p = \{b_j\}_1^q$ and, consequently, $\rho(x) = x!$, $[g(0)]! = 1$, which case corresponds to the canonical CSs (characteristics of the HO-1D one-dimensional harmonic oscillator).

These are, keeping in mind that for this case we have ${}_pF_q(\widetilde{a};\widetilde{b};x) = {}_0F_0(;;x) = e^x$, as well as $\{z+\sigma\}^l = (z+\sigma)^l$:

$$
\begin{aligned}
|z+\sigma\rangle &= e^{-\frac{1}{2}|z+\sigma|^2} \sum_{l=0}^{\infty} \frac{(z+\sigma)^l}{\sqrt{l!}} |l\rangle \\
&= e^{-\frac{1}{2}|z+\sigma|^2 + (z+\sigma)\hat{A}_+} |0\rangle.
\end{aligned}
\tag{207}
$$

The integration measure is

$$
d\mu(z,\sigma) = \frac{d^2 z}{\pi} \frac{d^2\sigma}{\pi} h_z(|z|)\, h_\sigma(|\sigma|)\, e^{|z+\sigma|^2}, \tag{208}
$$

and the weighting functions are, respectively

$$
\begin{aligned}
h_z(|z|) &= G^{1,0}_{0,1}(|z|^2|0) = e^{-|z|^2}, \\
h_\sigma(|\sigma|) &= G^{1,0}_{0,1}(|\sigma|^2|0) = e^{-|\sigma|^2}.
\end{aligned}
\tag{209}
$$

The decomposition relation of the unity operator is written, therefore

$$
\begin{aligned}
\hat{I} &= \iint d\mu(z,\sigma)\, |z+\sigma\rangle\langle z+\sigma| \\
&= \sum_{l=0}^{\infty} |l\rangle\langle l| \sum_{n=0}^{l} \binom{l}{n} \\
&\quad \times \left\{ \frac{1}{(l-n)!} \mathrm{Int}_z \right\} \left\{ \frac{1}{n!} \mathrm{Int}_\sigma \right\},
\end{aligned}
\tag{210}
$$

where

$$
\mathrm{Int}_{|z|} = \frac{1}{\sqrt{2^l}} (l-n)!, \qquad \mathrm{Int}_{|\sigma|} = \frac{1}{\sqrt{2^l}} n!, \tag{211}
$$

from which it follows that the equality is satisfied.

7 Concluding remarks

In the paper we obtain the expression of CSs with shifted or displaced argument, whose argument is a linear combination of two different complex variables $Z = \varepsilon z + \lambda\sigma$, where ε and λ are real numbers. In this sense, we used a pair of generalized ladder operators (annihilation and creation) $\hat{A}_- = \hat{a} f(\hat{N})$ and $\hat{A}_+ = f(\hat{N})\hat{a}^\dagger$, where $f(\hat{N})$ is the nonlinearity function. These operators generate different sets of nonlinear CSs which were obtained in two equivalent ways: 1.) the classical method, also used in the case of canonical operators, i.e. by successively applying two displacement operators, each depending on only one complex variable; 2.) the generalized method, by using the generalized analogue of the displacement operators, which were defined by Mojaveri and Dehghani [18], that is, as normalized generalized hypergeometric function, which had $z\hat{A}_+$ as its argument. In both methods, CSs are obtained by applying the normalized displacement operators on the vacuum state ket $|0\rangle$. The general results obtained were verified for several particular cases: the case of generalized CSs with non-displaced argument, with subcases - canonical CSs with non-displaced argument and CSs corresponding to the pseudoharmonic oscillator, PHO, as well as for the case of CSs whose argument was a sum of two complex numbers. This last case is useful for obtaining the expression for the Wigner operator in quantum optics [21]. In all calculations we used the diagonal operator ordering technique (DOOT) which we introduced in [5], and which is a generalization of the integration of an ordered product of operators (IWOP) technique developed by Hong-Yi Fan and applied only to canonical operators [3]. The DOOT technique applies to any pair of non-linear generalized operators $\hat{A}_- = \hat{a} f(\hat{N})$ and $\hat{A}_+ = f(\hat{N})\hat{a}^\dagger$. In essence, the DOOT technique stipulates that the generalized operators $\hat{A}_-$ and $\hat{A}_+$ appearing in expressions within the DOOT brackets $\#\cdots\#$ are commutative and can be treated as simple c-numbers.

Appendix 1

For the sake of brevity, let us denote the completeness relation as follows:

$$
\mathrm{Int}_{(z,\sigma)} \equiv \int d\mu(\varepsilon z + \lambda\sigma)\, |\varepsilon z + \lambda\sigma\rangle\langle \varepsilon z + \lambda\sigma|. \tag{A.1}
$$

After the appropriate substitutions, it becomes

$$
\begin{aligned}
\mathrm{Int}_{(z,\sigma)} &= \int \frac{d\mu(z)}{{}_pF_q(|\epsilon z|^2)} \frac{d\mu(\sigma)}{{}_pF_q(|\lambda\sigma|^2)} \\
&\quad \times {}_pF_q(\{\epsilon z + \lambda\sigma\}\hat{A}_+) \\
&\quad \times {}_pF_q(\{\epsilon z^* + \lambda\sigma^*\}\hat{A}_-) \\
&\quad \times \frac{1}{{}_pF_q(|\{\epsilon z + \lambda\sigma\}|^2)}.
\end{aligned}
\tag{A.2}
$$

$$
\begin{aligned}
\mathrm{Int}_{(z,\sigma)} &= \sum_{n,m=0}^{\infty} \frac{|n\rangle\langle m|}{\sqrt{\rho(n)\rho(m)}} \\
&\quad \times \int \frac{d\mu(z)}{{}_pF_q(|\epsilon z|^2)} \frac{d\mu(\sigma)}{{}_pF_q(|\lambda\sigma|^2)} \\
&\quad \times \{\epsilon z + \lambda\sigma\}^n \{\epsilon z^* + \lambda\sigma^*\}^m.
\end{aligned}
\tag{A.3}
$$

Now,

$$
\begin{aligned}
&\{\varepsilon z + \lambda\sigma\}^n \{\varepsilon z^* + \lambda\sigma^*\}^m \\
&= \sum_{s=0}^{n} \binom{\rho(n)}{\rho(s)} (\varepsilon z)^{n-s} (\lambda\sigma)^s \\
&\quad \times \sum_{k=0}^{m} \binom{\rho(m)}{\rho(k)} (\varepsilon z^*)^{m-k} (\lambda\sigma^*)^k.
\end{aligned}
\tag{A.4}
$$

Substituting (A.4) into (A.3), we get

$$\begin{aligned}\mathrm{Int}_{(z,\sigma)} = &\sum_{n,m=0}^{\infty} \frac{|n\rangle\langle m|}{\sqrt{\rho(n)\rho(m)}}\\ &\times \sum_{s=0}^{n} \binom{\rho(n)}{\rho(s)} \sum_{k=0}^{m} \binom{\rho(m)}{\rho(k)}\\ &\times \mathrm{Int}_z\, \mathrm{Int}_\sigma,\end{aligned} \tag{A.5}$$

where

$$\mathrm{Int}_z \equiv \int \frac{d\mu(z)}{{}_pF_q(|\varepsilon z|^2)} (\varepsilon z)^{n-s} (\varepsilon z^*)^{m-k}, \tag{A.6}$$

and

$$\mathrm{Int}_\sigma \equiv \int \frac{d\mu(\sigma)}{{}_pF_q(|\lambda\sigma|^2)} (\lambda\sigma)^{s} (\lambda\sigma^*)^{k}. \tag{A.7}$$

with the following notations, as well as using the expression for one-variable integration measure.

$$\begin{aligned}d\mu(z) = &\Gamma(\widetilde{a}/\widetilde{b}) d(|z|^2) \frac{d\phi_z}{2\pi}\, {}_pF_q(|z|^2)\\ &\times G^{q+1,0}_{p,q+1}\left(|z|^2 \,\middle|\, \begin{matrix}\{a_i-1\}_1^p\\ 0,\ \{b_j-1\}_1^q\end{matrix}\right).\end{aligned} \tag{A.8}$$

$$\begin{aligned}\mathrm{Int}_{(z)} \equiv &\frac{1}{\sqrt{2^n}} \int \frac{d\mu(z)}{{}_pF_q(|\varepsilon z|^2)} (\varepsilon z)^{n-s} (\varepsilon z^*)^{m-k}\\ = &\frac{1}{\sqrt{2^n}} \Gamma(\widetilde{a}/\widetilde{b})\\ &\times \int_0^\infty d(|\varepsilon z|^2) G^{q+1,0}_{p,q+1}\left(|\varepsilon z|^2 \,\middle|\, \begin{matrix}\widetilde{a}-1\\ 0,\ \widetilde{b}-1\end{matrix}\right)\\ &\times \int_0^{2\pi} \frac{d\phi_z}{2\pi} (\varepsilon z)^{n-s} (\varepsilon z^*)^{m-k}.\end{aligned} \tag{A.9}$$

The angular integral is

$$\begin{aligned}&\int_0^{2\pi} \frac{d\phi_z}{2\pi} (\varepsilon z)^{n-s} (\varepsilon z^*)^{m-k}\\ &\qquad = (|\varepsilon z|^2)^{n-k}\, \delta_{n-s,\,m-k}.\end{aligned} \tag{A.10}$$

and, the radial integral becomes

$$\begin{aligned}&\int_0^\infty d(|\varepsilon z|^2) G^{q+1,0}_{p,q+1}\left(|\varepsilon z|^2 \,\middle|\, \begin{matrix}\widetilde{a}-1\\ 0,\ \widetilde{b}-1\end{matrix}\right)\\ &\quad \times (|\varepsilon z|^2)^{n-k}\\ &= \Gamma(\widetilde{b}/\widetilde{a})\, \rho(n-k).\end{aligned} \tag{A.11}$$

Finally, we have

$$\mathrm{Int}_{(z)} = \frac{1}{\sqrt{2^n}} \rho(n-k). \tag{A.12}$$

Analogously, the integral over σ will be

$$\begin{aligned}\mathrm{Int}_{(\sigma)} \equiv &\frac{1}{\sqrt{2^m}} \int \frac{d\mu(\sigma)}{{}_pF_q(|\lambda\sigma|^2)} (\lambda\sigma)^{s} (\lambda\sigma^*)^{k}\\ = &\frac{1}{\sqrt{2^m}} \rho(k)\, \delta_{sk}.\end{aligned} \tag{A.13}$$

Consequently, we have $s = k$, $n-s = m-k$ and finally, $m = n$.

$$\begin{aligned}\mathrm{Int}_{(z,\sigma)} = &\sum_{n,m=0}^{\infty} \frac{|n\rangle\langle m|}{\sqrt{\rho(n)\rho(m)}}\\ &\times \sum_{s=0}^{n} \binom{\rho(n)}{\rho(s)} \sum_{k=0}^{m} \binom{\rho(m)}{\rho(k)}\\ &\times \frac{1}{\sqrt{2^n}} \rho(n-k) \frac{1}{\sqrt{2^m}} \rho(k)\\ = &\sum_{n=0}^{\infty} \frac{|n\rangle\langle n|}{2^n} \sum_{k=0}^{n} \binom{\rho(n)}{\rho(k)}.\end{aligned} \tag{A.14}$$

because it is evident that, in order to accomplish the completeness relation, it must be $m = n$.

Taking into account that

$$(1+1)^n = \sum_{k=0}^{n} \binom{\rho(n)}{\rho(k)} 1^{n-k} 1^k = 2^n, \tag{A.15}$$

finally, we find that the completeness relation is satisfied:

$$\sum_{n=0}^{\infty} |n\rangle\langle n| = \hat{I}. \tag{A.16}$$

and, implicitly, the resolution of the identity operator.

Appendix 2

In many situations, products of two complex-conjugate hypergeometric functions arise—and thus, ultimately, products of two complex-conjugate numbers $\{Z\}$ raised to different powers.

$$\begin{aligned}&{}_pF_q(\{Z\}\hat{A}_+)\, {}_pF_q(\{Z^*\}\hat{A}_-)\\ &= \sum_{n=0}^{\infty} \frac{(\hat{A}_+)^n}{\rho(n)} \sum_{m=0}^{\infty} \frac{(\hat{A}_-)^m}{\rho(m)} \{Z\}^n \{Z^*\}^m.\end{aligned} \tag{A.17}$$

In *Perspective 1* we have:

$$\begin{aligned}\{Z\}^n\{Z^*\}^m = &\{\varepsilon z + \lambda\sigma\}^n \{\varepsilon z^* + \lambda\sigma^*\}^m\\ = &\sum_{s=0}^{n} \binom{\rho(n)}{\rho(s)} \sum_{k=0}^{m} \binom{\rho(m)}{\rho(k)}\\ &\times (\varepsilon z)^{n-s} (\lambda\sigma)^s (\varepsilon z^*)^{m-k} (\lambda\sigma^*)^k.\end{aligned} \tag{A.18}$$

In an integral over the entire space, of the form

$$
\begin{aligned}
\mathrm{Int}_{(z,\sigma)}^{(m,n)} \equiv & \int d\mu(z,\sigma) \\
& \times \{\varepsilon z + \lambda\sigma\}^n \{\varepsilon z^* + \lambda\sigma^*\}^m \\
& = \sum_{s=0}^{n} \binom{\rho(n)}{\rho(s)} \sum_{k=0}^{m} \binom{\rho(m)}{\rho(k)} \\
& \times \int d\mu(z,\sigma) \\
& \times (\varepsilon z)^{n-s} (\varepsilon z^*)^{m-k} (\lambda\sigma)^s (\lambda\sigma^*)^k .
\end{aligned}
\tag{A.19}
$$

using the variables $z = |z|e^{i\phi_z}$ and $\sigma = |\sigma|e^{i\phi_\sigma}$, as well as the expression of two complex variable integration measure

$$
\begin{aligned}
d\mu(Z) \equiv & \, d\mu(z,\sigma) \\
& = d\mu(\varepsilon z)\, d\mu(\sigma)\, {}_pF_q(|\{\varepsilon z + \lambda\sigma\}|^2) \\
& = d(|\varepsilon z|^2)\, h(|\varepsilon z|^2) \frac{d\phi_z}{2\pi} \\
& \times d(|\lambda\sigma|^2)\, h(|\lambda\sigma|^2) \frac{d\phi_\sigma}{2\pi} \\
& \times {}_pF_q(|\{\varepsilon z + \lambda\sigma\}|^2).
\end{aligned}
\tag{A.20}
$$

we will have

$$
\begin{aligned}
\left(|Z|^2\right)^l & = \left(|\varepsilon z + \lambda\sigma|^2\right)^l \\
& = (\varepsilon z + \lambda\sigma)^l (\varepsilon z^* + \lambda\sigma^*)^l \\
& = \sum_{n=0}^{l} \binom{l}{n} \sum_{m=0}^{l} \binom{l}{m} \\
& \times (\varepsilon z)^{l-n} (\lambda\sigma)^n (\varepsilon z^*)^{l-m} (\lambda\sigma^*)^m \\
& = \sum_{n=0}^{l} \binom{l}{n} \sum_{m=0}^{l} \binom{l}{m} \\
& \times (\varepsilon |z|)^{2l-n-m} (\lambda|\sigma|)^{n+m} \\
& \times e^{i(m-n)\phi_z} e^{-i(m-n)\phi_\sigma} .
\end{aligned}
\tag{A.21}
$$

Situations involving angular integrations are of particular interest. For example:

From the point of view of *Perspective 1*, we have:

$$
\begin{aligned}
& \int_0^{2\pi} \frac{d\phi_Z}{2\pi} {}_pF_q([Z]\hat{A}_+)\, {}_pF_q([Z^*]\hat{A}_-) \\
& = \sum_{n,m=0}^{\infty} \frac{(\hat{A}_+)^n}{\rho(n)} \frac{(\hat{A}_-)^m}{\rho(m)} \int_0^{2\pi} \frac{d\phi_Z}{2\pi} ([Z])^n ([Z^*])^m \\
& = \sum_{n,m=0}^{\infty} \frac{(\hat{A}_+)^n}{\rho(n)} \frac{(\hat{A}_-)^m}{\rho(m)} \# \cdot \mathrm{Int}_{\phi_z} \cdot \mathrm{Int}_{\phi_\sigma} .
\end{aligned}
\tag{A.22}
$$

The last line follows from the fact that the DOOT rules require $m = n$.

$$
\begin{aligned}
\mathrm{Int}_{\phi_z} & = \sum_{s=0}^{n} \binom{\rho(n)}{\rho(s)} (|\varepsilon z|)^{n-s} (|\varepsilon z|)^{m-k}, \\
\int_0^{2\pi} & \frac{d\phi_z}{2\pi} e^{i(n-s-m+k)\phi_z} = \\
& \sum_{s=0}^{n} \binom{\rho(n)}{\rho(s)} (|\varepsilon z|^2)^{n-s} \delta_{n-s,\, m-k} .
\end{aligned}
\tag{A.23}
$$

$$
\begin{aligned}
\mathrm{Int}_{\phi_\sigma} & = \sum_{k=0}^{m} \binom{\rho(m)}{\rho(k)} (|\lambda\sigma|)^s (|\lambda\sigma|)^k, \\
\int_0^{2\pi} & \frac{d\phi_\sigma}{2\pi} e^{i(s-k)\phi_\sigma} = \sum_{k=0}^{m} \binom{\rho(m)}{\rho(k)} (|\lambda\sigma|^2)^k \delta_{sk} .
\end{aligned}
\tag{A.24}
$$

from which it follows that $s = k$ and, consequently, $m = n$. This is in agreement with the DOOT rules, in which, into an ordered product of creation and annihilation operators, inside of sign $\# \cdots \#$, the powers of these operators must be equal. Then, the integral becomes

$$
\begin{aligned}
& \int_0^{2\pi} \frac{d\phi_Z}{2\pi} {}_pF_q([Z]\hat{A}_+)\, {}_pF_q([Z^*]\hat{A}_-) \\
& = \sum_{n,m=0}^{\infty} \frac{(\hat{A}_+)^n}{\rho(n)} \frac{(\hat{A}_-)^m}{\rho(m)} \# \\
& \quad \times \sum_{k=0}^{n} \binom{\rho(n)}{\rho(k)} (|\varepsilon z|^2)^{n-s} (|\lambda\sigma|^2)^k \\
& \quad \times \sum_{k=0}^{m} \binom{\rho(m)}{\rho(k)} (|\varepsilon z|^2)^{m-k} (|\lambda\sigma|^2)^k \\
& = \sum_{n=0}^{\infty} \frac{(\hat{A}_+\hat{A}_-)^n}{[\rho(n)]^2} \left((|\varepsilon z|^2 + |\lambda\sigma|^2)^2\right)^n .
\end{aligned}
\tag{A.25}
$$

The last line results from the following calculation:

$$
\begin{aligned}
\left[|Z|^2\right]^n & = \left[|\varepsilon z|^2 + |\lambda\sigma|^2 \right. \\
& \qquad \left. + 2|\varepsilon z||\lambda\sigma| \cos(\phi_z - \phi_\sigma)\right]^n \\
& = \sum_{j=0}^{n} \binom{n}{j} \left(|\varepsilon z|^2 + |\lambda\sigma|^2\right)^{n-j} \\
& \quad \times (2|\varepsilon z||\lambda\sigma|)^j \cos^j(\phi_z - \phi_\sigma).
\end{aligned}
\tag{A.26}
$$

$$
\begin{aligned}
\cos^j(\phi_z - \phi_\sigma) & = \frac{1}{2^j} \left[e^{i(\phi_z - \phi_\sigma)} + e^{-i(\phi_z - \phi_\sigma)}\right]^j \\
& = \frac{1}{2^j} \sum_{k=0}^{j} \binom{j}{k} e^{i(\phi_z - \phi_\sigma)(j-k)} e^{-i(\phi_z - \phi_\sigma)k} \\
& = \frac{1}{2^j} \sum_{k=0}^{j} \binom{j}{k} e^{i\phi_z(j-2k)} e^{-i\phi_\sigma(j-2k)} .
\end{aligned}
\tag{A.27}
$$

$$\int_0^{2\pi} \frac{d\phi_Z}{2\pi} {}_pF_q([Z]\hat{A}_+)\, {}_pF_q([Z^*]\hat{A}_-)$$
$$= \sum_{n=0}^{\infty} \frac{(\hat{A}_+\hat{A}_-)^n}{[\rho(n)]^2} \int_0^{2\pi} \frac{d\phi_Z}{2\pi} \left[|Z|^2\right]^n \tag{A.28}$$
$$= \sum_{n=0}^{\infty} \frac{(\hat{A}_+\hat{A}_-)^n}{[\rho(n)]^2} \mathrm{Int}_{|Z|^2},$$

where

$$\mathrm{Int}_{|Z|^2} = \int_0^{2\pi} \frac{d\phi_z}{2\pi} \int_0^{2\pi} \frac{d\phi_\sigma}{2\pi} \left[|Z|^2\right]^n$$
$$= \sum_{j=0}^{n} \binom{n}{j} \left(|\varepsilon z|^2 + |\lambda\sigma|^2\right)^{n-j} \left(|\varepsilon z||\lambda\sigma|\right)^j$$
$$\times \sum_{k=0}^{j} \binom{j}{k} \int_0^{2\pi} \frac{d\phi_z}{2\pi} e^{i\phi_z(j-2k)} \tag{A.29}$$
$$\times \int_0^{2\pi} \frac{d\phi_\sigma}{2\pi} e^{-i\phi_\sigma(j-2k)}$$
$$= \sum_{k=0}^{j} \binom{j}{k} \delta_{j-2k,0}\delta_{j-2k,0} \sum_{k=0}^{2k} \binom{2k}{k} = 2^{2k},$$

because $j = 2k$, i.e. j must be an even number.

$$\int_0^{2\pi} \frac{d\phi_z}{2\pi} e^{i\phi_z(j-2k)} = \delta_{j-2k,\,0}, \qquad \Rightarrow \qquad j = 2k. \tag{A.30}$$

Finally, we obtain

$$\int_0^{2\pi} \frac{d\phi_Z}{2\pi} {}_pF_q(\{Z\}\hat{A}_+)\, {}_pF_q(\{Z^*\}\hat{A}_-)$$
$$= \sum_{n=0}^{\infty} \frac{(\hat{A}_+\hat{A}_-)^n}{[\rho(n)]^2} \sum_{k=0}^{n} \binom{n}{2k}$$
$$\times \left(|\varepsilon z|^2 + |\lambda\sigma|^2\right)^{n-2k} \left(2|\varepsilon z||\lambda\sigma|\right)^{2k}$$
$$= \sum_{n=0}^{\infty} \frac{(\hat{A}_+\hat{A}_-)^n}{[\rho(n)]^2} \left(|\varepsilon z|^2 + |\lambda\sigma|^2 + 2|\varepsilon z||\lambda\sigma|\right)^n \tag{A.31}$$
$$= \sum_{n=0}^{\infty} \frac{(\hat{A}_+\hat{A}_-)^n}{[\rho(n)]^2} \left((|\varepsilon z| + |\lambda\sigma|)^2\right)^n$$
$$= \sum_{n=0}^{\infty} \frac{(\hat{A}_+\hat{A}_-)^n}{[\rho(n)]^2} \left(|Z|^2\right)^n.$$

$$\int_0^{2\pi} \frac{d\phi_Z}{2\pi} {}_pF_q(\{Z\}\hat{A}_+)\, {}_pF_q(\{Z^*\}\hat{A}_-)$$
$$= \sum_{n,m=0}^{\infty} \frac{(\hat{A}_+)^n}{\rho(n)} \frac{(\hat{A}_-)^m}{\rho(m)} \#$$
$$\times \sum_{k=0}^{n} \binom{\rho(n)}{\rho(k)} (|\varepsilon z|^2)^{n-s} (|\lambda\sigma|^2)^k \tag{A.32}$$
$$\times \sum_{k=0}^{m} \binom{\rho(m)}{\rho(k)} (|\varepsilon z|^2)^{m-k} (|\lambda\sigma|^2)^k$$
$$= \sum_{n=0}^{\infty} \frac{(\hat{A}_+\hat{A}_-)^n}{[\rho(n)]^2} \left((|\varepsilon z| + |\lambda\sigma|)^2\right)^n.$$

In *Perspective 2* we have:

$$\mathrm{Int}_{(Z)} \equiv \int_0^{2\pi} \frac{d\phi_Z}{2\pi} {}_pF_q(\{Z\}\hat{A}_+)\, {}_pF_q(\{Z^*\}\hat{A}_-)$$
$$= \sum_{n,m=0}^{\infty} \frac{(\hat{A}_+)^n}{\rho(n)} \frac{(\hat{A}_-)^m}{\rho(m)}$$
$$\times \int_0^{2\pi} \frac{d\phi_Z}{2\pi} ([Z])^n ([Z^*])^m$$
$$= \sum_{n,m=0}^{\infty} \frac{(\hat{A}_+)^n}{\rho(n)} \frac{(\hat{A}_-)^m}{\rho(m)} \tag{A.33}$$
$$\times |Z|^n |Z|^m \int_0^{2\pi} \frac{d\phi_Z}{2\pi} e^{i(n-m)\phi_Z}$$
$$= \sum_{n=0}^{\infty} \frac{(\hat{A}_+\hat{A}_-)^n}{[\rho(n)]^2} \left(|Z|^2\right)^n.$$

In conclusion, the same result is obtained from both *Perspective 1* and *Perspective 2*. However, given that the calculations for *Perspective 2* are simpler and more direct, it is preferable to use that approach.

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