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Conformal Invariance of Symplectic Packing Numbers and Relationship with Symplectic Capacities

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Abstract

In this paper, we proved the conformal invariance of symplectic packing numbers and its consequences. We also found the relationship with symplectic capacities and studied the existence of a symplectic embedding from a symplectic ellipsoid into a polydisc.

Keywords: Symplectic packing number; Symplectic capacity; Symplectic ellipsoid; Symplectic embedding

1 Introduction

Symplectic geometry has its own characteristics because of its symplectic structure, a nondegenerate closed 2-form, and is an important rule among geometries such as algebraic and complex geometry, etc., as well as nonlinear functional analysis and dynamics [1, 2].

Symplectic embedding, a map between symplectic manifolds which preserves symplectic structure, satisfies the most fundamental nonsqueezing theorem first established by Gromov (1985).

This theorem shows that $B^{2n}(a)$ embeds symplectically into $Z^{2n}(b)$ only if $a \leq b$ where $B^{2n}(a)$ and $Z^{2n}(b)$ are respectively the symplectic ball and the symplectic cylinder defined as follows:

$$B^{2n}(a) = \Big\{x = (x^1, \cdots, x^n, y^1, \cdots, y^n) \in \mathbb{R}^{2n} \Big| \sum_{i=1}^{n} \left[(x^i)^2 + (y^i)^2\right] < \frac{a}{\pi}\Big\}, \tag{1}$$

$$Z^{2n}(b) = \Big\{x = (x^1, \cdots, x^n, y^1, \cdots, y^n) \in \mathbb{R}^{2n} \Big| (x^1)^2 + (y^1)^2 < \frac{b}{\pi}\Big\}. \tag{2}$$

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Since a and b are the areas of the cross-section by the symplectic plane span$\{e_1, f_1\}$, where $\{e_1, \ldots, e_n, f_1, \ldots, f_n\}$ is the symplectic basis of $(\mathbb{R}^{2n}, \omega_0)$, this theorem reminds us that 2-dimensional measurement of a symplectic manifold contains information for preserving the symplectic structure. Therefore, the concept of symplectic capacity emerged.

Let γ be a family of all $2n$-dimensional symplectic manifolds (M, ω). An symplectic capacity is a map $c : \gamma \to [0, +\infty)$ associating with each manifold a number in such a way that the following axioms are satisfied [2].

(1) **Monotonicity:** $c(M, \omega) \leq c(N, \tau)$, if (M, ω) symplectically embeds into (N, τ).

(2) **Conformality:** $c(M, \alpha\omega) = |\alpha| c(M, \omega)$ for all $\alpha \in \mathbb{R} \setminus \{0\}$.

(3) **Nontriviality:** $c(B^{2n}(1)) = c(Z^{2n}(1)) = 1$.

More generally, nontriviality is replaced by

(3)' **Generalized nontriviality:** For $B^{2n}(1) := B, Z^{2n}(1) := Z, 0 < c(B), c(Z) < \infty$.

In particular, a map that satisfies the first two axioms, monotonicity and conformality, is called a generalised symplectic capacity. Symplectic ellipsoid $E(r_1, \cdots, r_n)$ and polydisc $C^{2n}(a)$ mentioned mainly in this paper is as follows:

$$E(r_1, \cdots, r_n) := \Big\{(x'_1, \cdots, x'_n, y'_1, \cdots, y'_n) \in \mathbb{R}^{2n} \Big| \sum_{i=1}^{n} \frac{\pi(x_i'^2 + y_i'^2)}{r_i} < 1\Big\}, \tag{3}$$

$$C^{2n}(a) := B^2(a) \times \cdots \times B^2(a). \tag{4}$$

Radii $r_i, i = \overline{1,n}$ of ellipsoid are nonzero real numbers such that $r_1 \leq r_2 \leq \cdots \leq r_n$. By definition, $E(a, \cdots, a) = B^{2n}(a)$.

For 4-dimensional symplectic ellipsoid $E(r_1, r_2)$ the Ekeland-Hofer capacities forms a nondecreasing sequence of $kr_1, k \geq 1$ and $lr_2, l \geq 1$ with repetition [3]. Similarly, but

not the same, ECH capacities of symplectic ellipsoids are a nondecreasing sequence of $ma+nb, m,n \geq 0$.

Surprisingly, in [4] they found the necessary and sufficient condition for the existence of a symplectic embedding between symplectic ellipsoids by using ECH capacities. Detailed material comes from [5, 6]. In [7–9], they studied the existence of symplectic embeddings between specific manifolds, and this problem remains as important as ever.

Definition 1 [10] Assume that (M,ω) is connected $2n$-dimensional symplectic manifold of finite volume $Vol(M,\omega)$
$= \frac{1}{n!}\int_M \omega^n$. For each $k \in \mathbb{N}$, the k-th symplectic packing number p_k is defined as follows:

$$p_k(M,\omega) = \sup_a \frac{kVol(B^{2n}(a),\omega_0)}{Vol(M,\omega)}, \tag{5}$$

where the supremum is taken over all those a for which the disjoint union $\bigsqcup_{i=1}^{k} B^{2n}(a)$ of k equal balls symplectically embeds into (M,ω).

This packing number is a symplectic invariant, and it has been studied [11] in 4 dimensions. In [12, 13], they considered the stability of the symplectic packing number.

2 Conformal invariance of symplectic packing numbers

Both the symplectic packing number and symplectic capacities are symplectic invariants, but they are distinct. Therefore, it is natural to view the symplectic packing number in the light of monotonicity and conformality of symplectic capacity.

If you have the definitions of both the symplectic packing number and the symplectic capacity, it seems that comparing them is pointless. Still, in this paper, we ask how different they are to understand the properties of the symplectic packing number.

First, consider the packing number for a symplectic ellipsoid.

Example 1: In $(\mathbb{R}^{2n},\omega_0)$, where $\omega_0 = \sum_{i=1}^{n} dx_i \wedge dy_i$, the first symplectic packing number for $E(a_1,\cdots,a_n)$ is $\frac{a_1^{n-1}}{a_2\cdots a_n}$.

Indeed, from the definition of the symplectic ellipsoid

$$B^{2n}(a_1) \subset E(a_1,\cdots,a_n) \subset Z^{2n}(a_1), \tag{6}$$

and the biggest symplectic ball embedded into the symplectic ellipsoid $E(a_1,\cdots,a_n)$ is $B^{2n}(a_1)$. So,

$$\begin{aligned} p_1\big(E(a_1,\ldots,a_n)\big) &= \frac{\mathrm{Vol}\left(B^{2n}(a_1),\omega_0\right)}{\mathrm{Vol}\left(E(a_1,\ldots,a_n)\right)} \\ &= \frac{a_1^{n-1}}{a_2a_3\cdots a_n}. \end{aligned} \tag{7}$$

From this example, we have got something more about the symplectic packing number: Firstly, $p_1(B^{2n}(a)) = 1$ for each real number a.

Second, the symplectic packing number is not monotonic. Indeed, it is obvious that $B^{2n}(a_1)$ embeds symplectically into $E(a_1,\cdots,a_n)$. But, $p_1\big(E(a_1,\cdots,a_n)\big)$ is less than one except the special $a_1 = a_2 = \cdots = a_n$, that is,

$$p_1\big(B^{2n}(a_1)\big) \geq p_1\big(E(a_1,\cdots,a_n)\big),$$

as desired. And then, how about conformality?

Theorem 1 *For each nonzero real number α and natural number k,*

$$p_k(M,\alpha\omega) = p_k(M,\omega). \tag{8}$$

Proof Suppose that $\bigsqcup_{i=1}^{k}\big(B^{2n}(a),\omega_0\big)$ embeds symplectically into $(M,\alpha\omega)$, this means that there exist symplectic embeddings $\phi_i, i = \overline{1,n}$ such that

$$\phi_i\big(B^{2n}(a)\big) \subset (M,\alpha\omega), \tag{9}$$

and

$$\phi_i\big(B^{2n}(a)\big) \cap \phi_j\big(B^{2n}(a)\big) = \varnothing, \tag{10}$$

for all $i \neq j$. First, we check that $\big(B^{2n}(a),\omega_0\big) \xrightarrow{s} (M,\alpha\omega) \Longleftrightarrow \big(B^{2n}(\frac{a}{|\alpha|}),\omega_0\big) \xrightarrow{s} (M,\omega)$.

Let $\phi : \big(B^{2n}(a),\omega_0\big) \longrightarrow (M,\alpha\omega)$ be symplectic embedding. Then, $\phi^*(\alpha\omega) = \omega_0$ so that $\phi^*(\omega) = \frac{1}{\alpha}\omega_0$. Simplifying $a' = \frac{a}{|\alpha|}$, consider $\psi : B^{2n}(a') \to B^{2n}(a)$ defined by $\psi(x) = \sqrt{|\alpha|}x$. Then, we can see

$$\psi^*(\frac{1}{\alpha}\omega_0) = \frac{|\alpha|}{\alpha}\omega_0. \tag{11}$$

Thus, if $\alpha > 0$, we denote $\hat{\phi}$ as $\hat{\phi} = \phi\circ\psi : \big(B^{2n}(a'),\omega_0\big) \longrightarrow (M,\omega)$, if not, as $\hat{\phi} = \phi\circ\psi\circ\psi_0 : \big(B^{2n}(a'),\omega_0\big) \longrightarrow (M,\omega)$, where ψ_0 is defined as

$$\psi_0 : \big(B^{2n}(a'),\omega_0\big) \longrightarrow \big(B^{2n}(a'),-\omega_0\big), \tag{12}$$

so that $\psi_0(u,v) = (-u,v)$, for all $(u,v) \in \mathbb{R}^{2n}$.

Then, $\hat{\phi}$ is symplectic embedding which maps $\big(B^{2n}(\frac{a}{|\alpha|}),\omega_0\big)$ into (M,ω) for each real number α and we conclude

$$\begin{aligned} p_k(M,\alpha\omega) &= \sup_a\left\{\frac{k\,\mathrm{Vol}\left(B^{2n}(a),\omega_0\right)}{\mathrm{Vol}(M,\alpha\omega)}\right\} \\ &= \sup_a\left\{\frac{k\frac{1}{n!}\int_{B^{2n}(a)}\omega_0^n}{\frac{1}{n!}\int_M(\alpha\omega)^n}\right\} \\ &= \sup_a\left\{\frac{ka^n}{|\alpha|^n\mathrm{Vol}(M,\omega)}\right\} \\ &= \sup_{a'}\left\{\frac{k|\alpha|^n(a')^n}{|\alpha|^n\mathrm{Vol}(M,\omega)}\right\} = p_k(M,\omega). \end{aligned} \tag{13}$$

for all natural number k as claimed.

This theorem shows that the symplectic packing number of a symplectic manifold (M,ω) remains unchanged even though a scalar multiple dilates the symplectic structure. This differs from the conformality of symplectic capacity, which says the symplectic capacity of $(M,\alpha\omega)$ is $|\alpha|$-multiple of the capacity of (M,ω), and that's why we call this property conformal invariance of the symplectic packing number.

Finally, nontriviality is meaningful only for the symplectic ball and meaningless for the symplectic cylinder, since the symplectic packing number is defined for symplectic manifolds of finite volume. Symplectic packing number for symplectic ball is studied in [3, 14].

3 Relationship between symplectic packing numbers and symplectic capacities

Consider maps defined as

$$c_k(M,\omega) := \sup\left\{a\Big|\bigsqcup_{i=1}^{k} B^{2n}(a) \xrightarrow{s} (M,\omega)\right\}, \tag{14}$$

$$c^c(M,\omega) := \inf\left\{a\Big|(M,\omega) \xrightarrow{s} C^{2n}(a)\right\}, \tag{15}$$

for each $2n$-dimensional symplectic manifold (M,ω). Obviously, if $k=1$, c_k is Gromov width and

$$p_k(M,\omega)Vol(M,\omega) = \frac{k}{n!}\big(c_k(M,\omega)\big)^n. \tag{16}$$

The maps c_k and c^c satisfy monotonicity and conformality, so they are generalised symplectic capacities. For 4-dimensional symplectic ellipsoid $E(1,a)$, these have the following properties.

Theorem 2 *For all natural number k and $C := C^4(1)$,*

$$\Big[c^c(E(1,k))\Big]^{-1} = c_k(C), \tag{17}$$

$$\Big[c^c(E(1,k))\Big]^{2} = \frac{k}{2p_k(C)}, \tag{18}$$

Proof The first statement follows from:

By the results

$$\bigsqcup_{i=1}^{k} B^4(1) \to C^4(A) \Leftrightarrow E(1,k) \to C^4(A), \tag{19}$$

of [4, 7], we know

$$\begin{aligned} c^c(E(1,k)) &= \inf\left\{A\Big|E(1,k) \xrightarrow{s} C^{2n}(A)\right\} \\ &= \inf\left\{A\Big|\bigsqcup_{i=1}^{k} B^4(1) \to C^4(A)\right\}, \end{aligned} \tag{20}$$

and remembering $c_U(V) = \big(c^V(U)\big)^{-1}$ for

$$c_U(V) = \sup_{\lambda}\left\{\lambda\,\middle|\,\lambda U \xrightarrow{s} V\right\}, \tag{21}$$

$$c^V(U) = \inf_{\lambda}\left\{\lambda\,\middle|\,U \xrightarrow{s} \lambda V\right\}, \quad U,V \in \mathbb{R}^{2n}, \tag{22}$$

we conclude that

$$\begin{aligned} \Big[c^c(E(1,k))\Big]^{-1} &= c_{E(1,k)}(C^4(1)) \\ &= \sup\left\{a\Big|\sqrt{a}E(1,k) \to C^4(1)\right\} \\ &= \sup\left\{a\Big|E(a,ka) \to C^4(1)\right\} \\ &= \sup\left\{a\Big|\bigsqcup_{i=1}^{k} B^{2n}(a) \to C^4(1)\right\} \\ &= c_k\big(C^4(1)\big). \end{aligned} \tag{23}$$

Let's move on to the second statement of Theorem 2. By using (16) and $Vol\big(C^4(1)\big) = 1$, we get

$$\Big[c^c(E(1,k))\Big]^{2} = \frac{k}{2p_k(C)}, \tag{24}$$

from the first.

Remark 1 The map c_k in Theorem 2 is different from c_k in [15] (page 145). This theorem relates the k-th symplectic packing number p_k to the generalised symplectic capacities c_k and c^c.

Corollary 1 *The value of $c^c(E(1,k))$ is given by Table 1.*

Proof In [15], the k-th symplectic packing numbers p_k for the 4-dimensional box $(0,1)^4$ are given. Since $C^4(1)$ is equivalent symplectically to $(0,1)^4$, we get $p_k\big(C^4(1)\big)$ (Table 2).

Now, by Theorem 2, this corollary is proved.

If the second radius of the symplectic ellipsoid $E(1,a)$ is given in general, that is, $a \geq 1$ is real number, abbreviating $c^c\big(E(1,a)\big)$ by $c^c(a)$, we know

$$c^c(k) \leq c^c(a) \leq c^c(k+1), \tag{25}$$

where k is the greatest integer less than or equal to a. Indeed, it is easy from the relation

$$E(1,k) \subset E(1,a) \subset E(1,k+1), \tag{26}$$

and monotonicity of c^c. Besides, there are many generalized symplectic capacities such as ECH capacities. We suggest that readers review the relationship between the packing number and ECH capacities in detail [15].

Table 1 The values of $c^c(E(1,k))$

k	1	2	3	4	5	6	7	more that 8
c^c	1	1	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{7}{4}$	$\frac{15}{8}$	$\sqrt{\frac{k}{2}}$

Table 2 The values of $p_k\big(C^4(1)\big)$

k	1	2	3	4	5	6	7	more that 8
p_k	$\frac{1}{2}$	1	$\frac{2}{3}$	$\frac{8}{9}$	$\frac{9}{10}$	$\frac{48}{49}$	$\frac{224}{225}$	1

Corollary 2 *Generalized symplectic capacity c^c for $E(1,a)$ is given in some subintervals as follows:*

$$c^c(a) = 1 \quad if \quad a \in [1,2], \tag{27}$$

$$c^c(a) = \frac{a}{2} \quad if \quad a \in [2,3], \tag{28}$$

$$c^c(a) = \frac{3}{2} \quad if \quad a \in [3,4]. \tag{29}$$

Proof The first and third cases follow from Corollary 1 and equation (25). The second case implies that the two ratios of the first and the second radius for both $E(1,a)$ and $E(A,2A)$, respectively, are always in $[2,3)$. That is, $E(1,a)$ and $E(A,2A)$ have the same shape type [16]. Therefore,

$$E(a) \xrightarrow{s} E(a') \Leftrightarrow a_i \leq a'_i, \tag{30}$$

for each i (see, Theorem 3, page 266 of [16]). We can get

$$E(1,a) \xrightarrow{s} E(A,2A) \Leftrightarrow \begin{cases} 1 \leq A \\ a \leq 2A \end{cases} \tag{31}$$

and from this,

$$\frac{a}{2} \leq A \leq 1. \tag{32}$$

Therefore $c^c(a) = \frac{a}{2}$ as desired.

The generalised symplectic capacity c^c in this paper provides an upper bound on symplectic capacities and can be used to estimate the existence of symplectic embeddings. Since symplectic ellipsoid $E(1,a)$ embeds into $C^4(b)$ if and only if $c^c(a) < b$, it is impossible to estimate the existence of symplectic embeddings using symplectic invariants other than c^c.

Example 2. Let's estimate whether $E(1,4)$ embeds into $C^4(\frac{10}{7})$ or not. First of all, the volume constraint holds:

$$\begin{aligned} \text{Vol}\big(E(1,4)\big) &= \frac{1\cdot 4}{2!} = 2, \\ \text{Vol}\big(C^4(\tfrac{10}{7})\big) &= \left(\frac{10}{7}\right)^2 = \frac{100}{49}. \end{aligned} \tag{33}$$

and the Gromov width is preserved

$$\omega_G\big(E(1,4)\big) = 1, \qquad \omega_G\big(C^4(\frac{10}{7})\big) = \frac{10}{7}. \tag{34}$$

Next, the first three terms of the Ekeland-Hofer capacity are also satisfied. Indeed, the first three terms of the Ekeland-Hofer capacity of $E(1,4)$ are $1,2,3$, respectively. But corresponding of $C^4(1)$ are given as follows [17]

$$\begin{aligned} Vol\big(E(1,4)\big) &= \frac{1\cdot 4}{2!} = 2, \\ Vol\big(C^4(\frac{10}{7})\big) &= \left(\frac{10}{7}\right)^2 = \frac{100}{49}. \end{aligned} \tag{35}$$

So, the first three terms of Ekeland-Hofer capacity of $C^4(\frac{10}{7})$ are

$$\begin{aligned} c_1\big(C^4(\tfrac{10}{7})\big) &= \frac{40}{7}, \\ c_2\big(C^4(\tfrac{10}{7})\big) &= \frac{30\sqrt{3}}{7}, \\ c_3\big(C^4(\tfrac{10}{7})\big) &= \frac{80}{7}. \end{aligned} \tag{36}$$

So, the Ekeland-Hofer capacity does not distinguish whether $E(1,4)$ embeds symplectically into $C^4(\frac{10}{7})$. But, $E(1,4)$ could not embed symplectically into $C^4(\frac{10}{7})$. From Corollary 2, we know that

$$c^c\big(E(1,4)\big) = \frac{3}{2} \geq c^c\big(C^4(\frac{10}{7})\big) = \frac{10}{7}, \tag{37}$$

this implies that $E(1,4)$ could not embed symplectically into $C^4(\frac{10}{7})$.

4 Conclusions

In this paper, we studied the conformal invariance of the symplectic packing number and its relationship to symplectic capacities, and we found a counterexample to symplectic embedding from a symplectic ellipsoid into a polydisc.

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