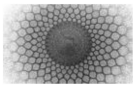
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Some properties of coherent states with singular complex matrix argument

Dušan Popov[1,2,a]

[1] University Politehnica Timişoara, Department of Physical Foundations of Engineering, Bd. V. Pârvan No. 2, 300223 Timişoara, Romania
[2] Serbian Academy of Nonlinear Sciences (SANS), Kneza Mihaila 36, Beograd-Stari Grad, Belgrade, Serbia

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Abstract In this paper, we studied the properties of a new version of coherent states whose argument is a linear combination of two special singular 2×2 matrices, each with a single nonzero element equal to 1, and two complex labelling variables as expansion coefficients. We show that this new version of coherent states satisfies all conditions imposed on coherent states, for both pure and mixed (thermal) states characterised by the density operator. As applications, we examined the connection between these coherent states and qubits and von Neumann entropy.

keywords: coherent states, singular matrix, Hamel basis, density operator, entropy.

1 Introduction

Coherent states were discovered a century ago by Schrödinger in 1926, although he did not use this name [1]. They have proven useful in various branches of theoretical physics, especially quantum optics and quantum information. Starting with the linear quantum harmonic oscillator, coherent states were later formulated for more complex quantum systems, including nonlinear oscillators. Methods for describing the properties of these states have been diverse, including phase-space considerations, operator algebra, semiclassical approximations, geometric quantisation, and more [2–6], and the references therein. Among the newest methods, one can consider bicomplex algebra, which highlights the properties of bicomplex numbers. In particular, a new generation of coherent states has been constructed using generalized hypergeometric functions with bicomplex arguments [7, 8].

[a]e-mail: dusan_popov@yahoo.co.uk

Another relatively new direction in examining the properties of coherent states involves states with matrix-valued arguments. In this approach, the parameter labelling the coherent states is replaced by a matrix-valued function [9, 10].

Coherent states with a shifted argument, as well as displaced number states, have also begun to attract attention [11–13].

On the other hand, consider the vector space of square diagonal matrices. A basis for this space can be described as a Hamel basis [14]. Every matrix in this space can be uniquely written as a finite linear combination of the standard diagonal basis matrices. These are singular elementary matrices with one diagonal element equal to 1 and all remaining elements equal to zero:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + d \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \quad (1)$$

The matrix elements are real or complex numbers.

By definition, a matrix is singular if its determinant is zero. Consequently, such a matrix is non-invertible. For diagonal square matrices, a matrix is singular if at least one diagonal element is zero.

In this paper, we examine a particular case of a singular diagonal matrix: one with a single nonzero element on the main diagonal, equal to 1, and zeros elsewhere.

We examine coherent states labelled by a linear combination of two singular diagonal 2×2 matrices, with two complex variables as expansion coefficients.

We therefore consider the following two basic singular matrices:

$$u_0 \equiv \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad u_1 \equiv \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \quad (2)$$

A diagonal square matrix, viewed as an element of this vector space, can be written as a linear combination of these basis matrices with complex coefficients:

$$Z \equiv \begin{pmatrix} z & 0 \\ 0 & \sigma \end{pmatrix} = zu_0 + \sigma u_1 = z \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \sigma \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{3}$$

For the two-dimensional case examined here, we use the argument specified in Eq. (3). This leads to functions satisfying Cauchy's functional equation [15], but involving complex variables:

$$\begin{aligned} \mathcal{F}(zu_0 + \sigma u_1) &= \mathcal{F}(zu_0) + \mathcal{F}(\sigma u_1) \\ &= \mathcal{F}(z)u_0 + \mathcal{F}(\sigma)u_1. \end{aligned} \tag{4}$$

In this paper, we study the properties of coherent states from two perspectives: as coherent states with a shifted complex argument, and as coherent states whose argument is a linear combination of special singular 2×2 matrices forming a Hamel basis. This is the main purpose of the present paper. Owing to the properties of these matrices, functions labelled by two complex variables decompose into two functions, each depending on only one variable. Our 2×2 matrices are somewhat similar to a matrix representation of quaternions, which extend complex numbers through one real component and three imaginary components. However, the matrices used here have specific properties.

Complex representations of quaternions have also been used to construct vector coherent states [9–11, 16, 17].

2 Some properties of special square orthogonal matrices

As is known, any quantum state of a system is represented by a linear combination of state vectors, the set of these vectors forming an orthonormal basis in Hilbert space (the vectors being mutually perpendicular and of norm equal to one). One of the most useful bases is the Fock vector basis, which uses occupation-number states. Each individual state is a single-particle state, an eigenstate of the particle number operator, orthogonal, and part of a complete set for representing any arbitrary quantum state in Fock space. In matrix form, each Fock vector is represented as a column vector with all terms zero except one, which is 1 and represents the state containing a single particle.

Thus, the column vector $|n\rangle$, of dimensions $n_{\max}\times1$, and its transpose (row vector) $\langle n| = |n\rangle^T$, with dimensions $1 \times n_{\max}$, have all elements equal to zero, except for the element in position $n+1$, which is equal to 1:

$$\begin{aligned} &\mathcal{M}(|n\rangle)_{n_{\max}\times1} \equiv (\delta_{in})_{n_{\max}\times1}, \quad \delta_{in} = \begin{cases} 0, & i \neq n, \\ 1, & i = n, \end{cases} \\ &\mathcal{M}(\langle n|)_{1\times n_{\max}} \equiv (\delta_{jn})_{1\times n_{\max}}, \quad \delta_{jn} = \begin{cases} 0, & j \neq n, \\ 1, & j = n. \end{cases} \end{aligned} \tag{5}$$

Their product satisfies

$$\mathcal{M}(\langle n|)_{1\times n_{\max}} \mathcal{M}(|n\rangle)_{n_{\max}\times1} = 1. \tag{6}$$

In expanded form, the Fock vectors are expressed as follows:

$$|0\rangle \equiv \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad |1\rangle \equiv \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \quad \ldots, \quad |n_{\max}\rangle \equiv \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}. \tag{7}$$

Consequently, the Fock-vector projector $|n\rangle\langle n|$ is a square singular diagonal matrix of dimensions $n_{\max} \times n_{\max}$, which has only one element equal to 1 on the main diagonal, the remaining off-diagonal elements being equal to zero. We denote this projector by u_n:

$$\begin{aligned} u_n \equiv |n\rangle\langle n| &= \mathcal{M}(|n\rangle)_{n_{\max}\times1} \mathcal{M}(\langle n|)_{1\times n_{\max}} \\ &\equiv (\delta^n_{ij})_{n_{\max}\times n_{\max}}, \quad \delta^n_{ij} = \begin{cases} 0, & i \neq n \text{ or } j \neq n, \\ 1, & i = j = n. \end{cases} \end{aligned} \tag{8}$$

It is observed that the matrices u_j, $j = 0, 1, \ldots, n_{\max}$, have a special structure: they have a single element on the main diagonal equal to 1, with all other elements equal to zero. These matrices are positive semidefinite, idempotent, and mutually orthogonal. Their determinant is zero; therefore, they are singular and noninvertible. Because they are diagonal, they are symmetric. The product of any two different matrices is the zero matrix. Their trace is equal to 1.

$$\begin{aligned} u_i(u_j)^m = (u_i)^m u_j = \begin{cases} u_j, & i = j, \\ 0, & i \neq j, \end{cases} \\ i, j = 0, 1, \ldots, n_{\max}, \qquad m = 1, 2, \ldots \end{aligned} \tag{9}$$

$$\sum_{n=0}^{n_{\max}-1} u_n = \sum_{n=0}^{n_{\max}-1} |n\rangle\langle n| = I_{n_{\max}}. \tag{10}$$

The 2×2 matrices of the form shown above are 2×2 projectors u_i acting on a two-dimensional Hilbert space $\mathcal{H}_2$. In most cases, this space corresponds to two-level energy systems or, in computational physics, qubits. On the other hand, the Fock vectors $|n\rangle$ belong to the infinite-dimensional Hilbert space known in quantum mechanics as Fock space $\mathcal{H}_F$. The total space is formed

by the elements obtained as the tensor product of the two spaces: $\mathcal{H}_{\text{total}} = \mathcal{H}_2 \otimes \mathcal{H}_F$. An example is $u_i \otimes |n\rangle$.

As an example often used in quantum mechanics, consider a system with $n_{\max}$ states. If the system is in thermodynamic equilibrium with the environment at temperature $T = 1/(\beta k_{\text{B}})$, its states are mixed and described by the normalised canonical density operator. This operator can be represented as a linear combination of diagonal square matrices, each having only one nonzero element on the main diagonal, equal to 1:

$$\begin{aligned}\rho &= \frac{1}{Z(\beta)} \sum_{n=0}^{n_{\max}-1} e^{-\beta E_n} |n\rangle\langle n| \\ &= \frac{1}{Z(\beta)} \sum_{n=0}^{n_{\max}-1} e^{-\beta E_n} u_n, \\ Z(\beta) &= \sum_{n=0}^{n_{\max}-1} e^{-\beta E_n}.\end{aligned} \tag{11}$$

Hence the interest in this type of matrix and, at the same time, the motivation to examine their properties in connection with applications in quantum mechanics, especially in the coherent-state approach. We limit ourselves, for reasons of clarity and brevity, to 2×2 matrices, although the results can be generalised relatively easily to matrices of dimensions $n_{\max} \times n_{\max}$.

In particular, consider a two-level energy system, where E_0 is the energy eigenvalue in the ground state $n = 0$, and E_1 is the energy eigenvalue in the excited state $n = 1$. The corresponding density operator in $\mathcal{H}_2$ has the spectral decomposition

$$\begin{aligned}\rho &= \frac{1}{Z(\beta)} \sum_{n=0}^{1} e^{-\beta E_n} |n\rangle\langle n| \\ &= \frac{e^{-\beta E_0}}{Z(\beta)} |0\rangle\langle 0| + \frac{e^{-\beta E_1}}{Z(\beta)} |1\rangle\langle 1|.\end{aligned} \tag{12}$$

The partition function, given by the sum of the diagonal elements of $e^{-\beta H}$, is

$$Z(\beta) = \operatorname{Tr}\left(e^{-\beta H}\right) = e^{-\beta E_0} + e^{-\beta E_1}. \tag{13}$$

In matrix form, the two energy states are

$$\begin{aligned}|0\rangle &\equiv \begin{pmatrix}1\\0\end{pmatrix}, \qquad \langle 0| \equiv \begin{pmatrix}1 & 0\end{pmatrix}, \\ |0\rangle\langle 0| &= \begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}1 & 0\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix} \equiv u_0.\end{aligned} \tag{14}$$

$$\begin{aligned}|1\rangle &\equiv \begin{pmatrix}0\\1\end{pmatrix}, \qquad \langle 1| \equiv \begin{pmatrix}0 & 1\end{pmatrix}, \\ |1\rangle\langle 1| &= \begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}0 & 1\end{pmatrix} = \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix} \equiv u_1.\end{aligned} \tag{15}$$

In quantum mechanics, when an observable has a spectrum of discrete eigenvalues and the set $\{|n\rangle\}$ is complete, any ket $|\Psi\rangle$ can be uniquely expanded as a linear combination, with the sum extending over the entire range of values of n, namely $|\Psi\rangle = \sum_n c_n |n\rangle$. The same situation occurs for singular diagonal square matrices, which can be written as a linear combination of matrices with all elements zero except one:

$$\begin{aligned}\mathcal{M}(z_n)_{n_{\max}\times n_{\max}} &\equiv (z_n \delta_{ij})_{n_{\max}\times n_{\max}}, \\ \delta_{ij} &= \begin{cases}0, & i,j \neq n, \\ z_n, & i = j = n,\end{cases} \\ & \quad n = 0, 1, \ldots, n_{\max}.\end{aligned} \tag{16}$$

$$\mathcal{M}(z_n)_{n_{\max}\times n_{\max}} = \sum_{n=0}^{n_{\max}-1} z_n u_n. \tag{17}$$

This is the linear expansion in the Hamel basis [14], that is, a maximal linearly independent set spanning a vector space through finite combinations. Here, the basis consists of linearly independent diagonal square matrices spanning the entire space of matrices of this type. The expansion is unique, and the coefficients are complex numbers, namely the elements of the singular diagonal matrix.

Let us recall the properties of these matrices:

$$\begin{aligned}&(u_0)^2 = u_0, \qquad (u_1)^2 = u_1, \\ &u_i u_j = \begin{cases}1, & i = j, \\ 0, & i \neq j,\end{cases} \qquad i,j = 0,1, \\ &u_0 + u_1 = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} \equiv I.\end{aligned} \tag{18}$$

As a geometrical interpretation, the matrices u_j are orthogonal projection matrices that project vectors onto the real coordinate axes, with $u_0 = P_x$ and $u_1 = P_y$:

$$P_x \begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x\\0\end{pmatrix}, \qquad P_y \begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}0\\y\end{pmatrix}. \tag{19}$$

These projection matrices represent the probability density that a system with only two states, or two energy levels, will end up in one of these states.

Finally, any 2×2 diagonal matrix can be written as a linear combination of these two projection matrices:

$$\begin{aligned}\begin{pmatrix}z & 0\\0 & \sigma\end{pmatrix} &= z\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix} + \sigma\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix} \\ &= z u_0 + \sigma u_1.\end{aligned} \tag{20}$$

This can be related to Cauchy's functional equation:

$$\mathcal{F}(z u_0 + \sigma u_1) = \mathcal{F}(z u_0) + \mathcal{F}(\sigma u_1). \tag{21}$$

In particular, an important property of these 2×2 diagonal matrices can be formulated for the following two situations.

If $\mathcal{F}(0) = 0$, we have

$$\mathcal{F}(xu_j) = \mathcal{F}(x)u_j, \qquad j = 0, 1. \tag{22}$$

Alternatively, if $\mathcal{F}(0) \neq 0$, the power-series expansion of the function about $x = 0$ gives

$$\begin{aligned}\mathcal{F}(xu_j) &= \sum_{n=0}^{\infty} c_n (xu_j)^n \\ &= c_0 I + \sum_{n=1}^{\infty} c_n x^n u_j \\ &= \mathcal{F}(0)I + u_j \sum_{n=1}^{\infty} c_n x^n \\ &= \mathcal{F}(0)I + \big[\mathcal{F}(x) - \mathcal{F}(0)\big]u_j \\ &= \mathcal{F}(x)u_j + \mathcal{F}(0)(I - u_j).\end{aligned} \tag{23}$$

Since the projectors are complementary, $u_0 + u_1 = I$ and $u_0 u_1 = u_1 u_0 = 0$, we obtain, in particular, for $\mathcal{F}(0) \neq 0$,

$$\begin{aligned}\mathcal{F}(xu_0) &= \mathcal{F}(x)u_0 + \mathcal{F}(0)u_1 = \begin{pmatrix} \mathcal{F}(x) & 0 \\ 0 & \mathcal{F}(0) \end{pmatrix}, \\ \mathcal{F}(xu_1) &= \mathcal{F}(0)u_0 + \mathcal{F}(x)u_1 = \begin{pmatrix} \mathcal{F}(0) & 0 \\ 0 & \mathcal{F}(x) \end{pmatrix}.\end{aligned} \tag{24}$$

If the argument of an analytic function is a linear combination of two complementary projectors u_j, the projectors can be extracted from the function:

$$\mathcal{F}(xu_0 + yu_1) = \mathcal{F}(x)u_0 + \mathcal{F}(y)u_1. \tag{25}$$

Returning to the density operator of a two-level system, we can write, using the new notation,

$$\begin{aligned}\rho &= \frac{1}{Z(\beta)} \sum_{n=0}^{1} e^{-\beta E_n} |n\rangle\langle n| \\ &= \frac{1}{Z(\beta)} \left(e^{-\beta E_0} u_0 + e^{-\beta E_1} u_1\right) \\ &= \frac{1}{Z(\beta)} \begin{pmatrix} e^{-\beta E_0} & 0 \\ 0 & e^{-\beta E_1} \end{pmatrix}.\end{aligned} \tag{26}$$

The partition function $Z(\beta)$ is obtained from the normalization condition $\operatorname{Tr} \rho = 1$:

$$\begin{aligned}Z(\beta) &= \sum_{m=0}^{1} \langle m|e^{-\beta H}|m\rangle \\ &= \sum_{n=0}^{1} e^{-\beta E_n} = e^{-\beta E_0} + e^{-\beta E_1}.\end{aligned} \tag{27}$$

For this case, the density operator is a 2×2 diagonal matrix.

$$\begin{aligned}\rho &= \frac{1}{1 + e^{-\beta(E_1 - E_0)}} u_0 + \frac{1}{1 + e^{\beta(E_1 - E_0)}} u_1 \\ &= \begin{pmatrix} \dfrac{1}{1 + e^{-\beta(E_1 - E_0)}} & 0 \\ 0 & \dfrac{1}{1 + e^{\beta(E_1 - E_0)}} \end{pmatrix}.\end{aligned} \tag{28}$$

From this expression, the trace of the operator is seen to be equal to unity:

$$\operatorname{Tr} \rho = \frac{1}{1 + e^{-\beta(E_1 - E_0)}} + \frac{1}{1 + e^{\beta(E_1 - E_0)}} = 1. \tag{29}$$

3 Coherent states with shifted arguments

In a previous paper [13], we derived the expression for generalized coherent states with a displaced or shifted argument. This type of coherent state has as its argument a linear combination of two complex variables z and σ, with two real numbers ε and λ as coefficients. Below, we present the main results on coherent states with a shifted argument obtained in that paper. These results will help us examine the case considered here.

We use the following pair of nonlinear adjoint annihilation and creation operators:

$$\widehat{A} = \widehat{a}\, {}_p f_q(\widehat{N}), \qquad \widehat{A}^\dagger = {}_p f_q(\widehat{N})\, \widehat{a}^\dagger. \tag{30}$$

Here, ${}_p f_q(\widehat{N})$ is the deformation function depending on the number operator $\widehat{N} = \widehat{a}^\dagger \widehat{a}$, and p and q are positive integers. The deformation function changes the usual canonical commutation relation $[\widehat{a}, \widehat{a}^\dagger] = 1$ into a deformed Heisenberg algebra. It satisfies

$${}_p f_q(\widehat{N} + m)|n + s\rangle = {}_p f_q(n + s + m)|n + s\rangle. \tag{31}$$

$$\begin{aligned}{}_p f_q(\widehat{N}) &= \sqrt{\frac{\prod_{j=1}^{q} (b_j - 1 + \widehat{N})}{\prod_{i=1}^{p} (a_i - 1 + \widehat{N})}}, \\ {}_p f_q(\widehat{N})|n\rangle &= \sqrt{\frac{\prod_{j=1}^{q} (b_j - 1 + n)}{\prod_{i=1}^{p} (a_i - 1 + n)}}\, |n\rangle.\end{aligned} \tag{32}$$

The actions of these operators and their adjoints on the Fock vectors are

$$\begin{aligned}
&\widehat{A}|n\rangle = \sqrt{{}_pe_q(n)}\,|n-1\rangle,\\
&\langle n|\widehat{A}^\dagger = \sqrt{{}_pe_q(n)}\,\langle n-1|,\\
&\widehat{A}^\dagger|n\rangle = \sqrt{{}_pe_q(n+1)}\,|n+1\rangle,\\
&\langle n|\widehat{A} = \sqrt{{}_pe_q(n+1)}\,\langle n+1|,\\
&\langle n|\widehat{A}^\dagger\widehat{A}|n\rangle = {}_pe_q(n).
\end{aligned} \tag{33}$$

We use the notation

$${}_pe_q(n) \equiv n\big[{}_pf_q(n)\big]^2 = n\,\frac{\prod_{j=1}^{q}(b_j-1+n)}{\prod_{i=1}^{p}(a_i-1+n)}. \tag{34}$$

The repeated action of the creation operator on the vacuum state leads to

$$\begin{aligned}
(\widehat{A}^\dagger)^n|0\rangle &= \sqrt{\prod_{s=1}^{n}{}_pe_q(s)}\,|n\rangle = \sqrt{{}_p\rho_q(n)}\,|n\rangle,\\
\langle 0|(\widehat{A})^n &= \langle n|\sqrt{\prod_{s=1}^{n}{}_pe_q(s)} = \langle n|\sqrt{{}_p\rho_q(n)}.
\end{aligned} \tag{35}$$

The newly introduced quantity is called the structure function because it determines the internal structure of coherent states:

$$\begin{aligned}
{}_p\rho_q(n) &\equiv \prod_{s=1}^{n}{}_pe_q(s) = n!\,\frac{\prod_{j=1}^{q}(b_j)_n}{\prod_{i=1}^{p}(a_i)_n}\\
&= \Gamma(a/b)\,\Gamma(n+1)\,\frac{\prod_{j=1}^{q}\Gamma(b_j+n)}{\prod_{i=1}^{p}\Gamma(a_i+n)},
\end{aligned} \tag{36}$$

$$\Gamma(a/b) \equiv \frac{\prod_{i=1}^{p}\Gamma(a_i)}{\prod_{j=1}^{q}\Gamma(b_j)}.$$

Instead of the usual binomial coefficients, generalized binomial coefficients appear in the calculations. They are defined as follows:

$$\binom{{}_p\rho_q(l)}{{}_p\rho_q(n)} \equiv \frac{{}_p\rho_q(l)}{{}_p\rho_q(n)\,{}_p\rho_q(l-n)}. \tag{37}$$

Generalized binomial coefficients enter the calculation of coherent states with a shifted argument when a product of two shift operators, each depending on a single complex variable, is applied to the vacuum state [13]. A generalized Newton binomial expression is also used. This quantity is denoted by square brackets, $[x+y]^l$, and defined as

$$[x+y]^l \equiv \sum_{m=0}^{l}\binom{{}_p\rho_q(l)}{{}_p\rho_q(m)}x^{l-m}y^m. \tag{38}$$

The coherent states with a shifted argument are labelled by a complex number Z, which is a linear combination of two ordinary complex variables z and σ, with real coefficients ε and λ:

$$\begin{aligned}
Z &= \varepsilon z + \lambda\sigma\\
&= \operatorname{Re} Z + i\operatorname{Im} Z = |Z|\exp(i\varphi_Z).
\end{aligned} \tag{39}$$

In deriving the expression for generalized coherent states with shifted arguments [13], we used the generalized displacement operator

$$\widehat{\mathcal{D}}(z) \equiv \frac{{}_pF_q(z\widehat{A}^\dagger)}{\sqrt{{}_pF_q(|z|^2)}}, \tag{40}$$

which agrees with the construction discussed in Ref. [11].

To simplify the notation, whenever there is no risk of confusion, we omit the parameter sets $a = \{a_i\}_{i=1}^{p}$ and $b = \{b_j\}_{j=1}^{q}$ from the generalized hypergeometric functions. Thus, we write ${}_pF_q(x)$ instead of ${}_pF_q(a;b;x)$, where

$$\begin{aligned}
{}_pF_q(a;b;x) &\equiv {}_pF_q(x)\\
&= \sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}(a_i)_n}{\prod_{j=1}^{q}(b_j)_n}\frac{x^n}{n!} = \sum_{n=0}^{\infty}\frac{x^n}{{}_p\rho_q(n)}.
\end{aligned} \tag{41}$$

Consequently, coherent states whose argument is a linear combination of complex numbers are defined using the generalized displacement operator $\widehat{\mathcal{D}}(\varepsilon z+\lambda\sigma) = \widehat{\mathcal{D}}(\lambda\sigma)\widehat{\mathcal{D}}(\varepsilon z)$, which acts on the vacuum state:

$$|Z\rangle \equiv |\varepsilon z+\lambda\sigma\rangle \longrightarrow \widehat{\mathcal{D}}(\lambda\sigma)\widehat{\mathcal{D}}(\varepsilon z)|0\rangle. \tag{42}$$

After simple calculations, the final expression for the generalized displaced, or shifted, argument is obtained as

$$\begin{aligned}
|\varepsilon z+\lambda\sigma\rangle = &\sqrt{\frac{{}_pF_q(|\lambda\sigma|^2)\,{}_pF_q(|\varepsilon z|^2)}{{}_pF_q(|\varepsilon z+\lambda\sigma|^2)}}\\
&\times\widehat{\mathcal{D}}(\lambda\sigma)\widehat{\mathcal{D}}(\varepsilon z)|0\rangle.
\end{aligned} \tag{43}$$

Equivalently, its expansion with respect to the Fock vectors is

$$|\varepsilon z+\lambda\sigma\rangle = \frac{1}{\sqrt{{}_pF_q(|\varepsilon z+\lambda\sigma|^2)}} \times \sum_{n=0}^{\infty} \frac{[\varepsilon z+\lambda\sigma]^n}{\sqrt{{}_p\rho_q(n)}}\,|n\rangle. \tag{44}$$

This can also be written in the following form:

$$|\varepsilon z+\lambda\sigma\rangle = \frac{1}{\sqrt{{}_pF_q(|\varepsilon z+\lambda\sigma|^2)}} \times {}_pF_q\big([\varepsilon z+\lambda\sigma]\widehat{A}^\dagger\big)|0\rangle, \tag{45}$$

and similarly for its complex-conjugate counterpart.

Using the variable $Z=\varepsilon z+\lambda\sigma$, the coherent states can be written as

$$\begin{aligned}|Z\rangle &= \frac{1}{\sqrt{{}_pF_q(|Z|^2)}}\sum_{n=0}^{\infty}\frac{[Z]^n}{\sqrt{{}_p\rho_q(n)}}\,|n\rangle \\ &= \frac{1}{\sqrt{{}_pF_q(|Z|^2)}}{}_pF_q\big([Z]\widehat{A}^\dagger\big)|0\rangle,\end{aligned} \tag{46}$$

$$\begin{aligned}\langle Z| &= \frac{1}{\sqrt{{}_pF_q(|Z|^2)}}\sum_{n=0}^{\infty}\frac{[Z^*]^n}{\sqrt{{}_p\rho_q(n)}}\,\langle n| \\ &= \frac{1}{\sqrt{{}_pF_q(|Z|^2)}}\langle 0|\,{}_pF_q\big([Z^*]\widehat{A}\big).\end{aligned} \tag{47}$$

4 Coherent states with matrix arguments

Let us consider a complex variable Z with a diagonal matrix structure, that is, a sum of two complex numbers z and σ, each multiplied by a 2×2 matrix:

$$\begin{aligned}Z &= \begin{pmatrix} z & 0\\ 0 & \sigma\end{pmatrix} \\ &= z\begin{pmatrix}1&0\\0&0\end{pmatrix}+\sigma\begin{pmatrix}0&0\\0&1\end{pmatrix} \\ &= zu_0+\sigma u_1.\end{aligned} \tag{48}$$

Thus, Z is a 2×2 diagonal matrix whose main diagonal entries are the ordinary complex numbers z and σ. These are also the expansion coefficients of a linear combination of the two singular diagonal square matrices u_j, $j=0,1$.

Using the results obtained for coherent states with a shifted argument, let us examine the corresponding properties of coherent states whose argument is a linear combination of two complex numbers multiplying singular diagonal square 2×2 matrices.

First, let us make the following identifications:

$$\begin{aligned}&\varepsilon z+\lambda\sigma \equiv zu_0+\sigma u_1,\\ &\varepsilon \equiv u_0 = \begin{pmatrix}1&0\\0&0\end{pmatrix},\\ &\lambda \equiv u_1 = \begin{pmatrix}0&0\\0&1\end{pmatrix}.\end{aligned} \tag{49}$$

For this form of the complex number Z, the generalized Newton binomial expression simplifies considerably, taking into account the properties of the matrices u_j:

$$\begin{aligned}[Z]^l &\equiv [zu_0+\sigma u_1]^l \\ &= \sum_{m=0}^{l}\binom{{}_p\rho_q(l)}{{}_p\rho_q(m)}(zu_0)^{l-m}(\sigma u_1)^m \\ &= (zu_0)^l+(\sigma u_1)^l \\ &= z^l u_0+\sigma^l u_1.\end{aligned} \tag{50}$$

Only the terms involving a single projector survive, because the mixed products of u_0 and u_1 vanish.

In particular, we have

$$\begin{aligned}|Z|^2 &\equiv ZZ^* \\ &= (zu_0+\sigma u_1)(z^*u_0+\sigma^*u_1) \\ &= |z|^2u_0+|\sigma|^2u_1.\end{aligned} \tag{51}$$

The powers of Z can therefore be written as 2×2 diagonal matrices:

$$Z^m = z^m u_0+\sigma^m u_1 = \begin{pmatrix}z^m&0\\0&\sigma^m\end{pmatrix}. \tag{52}$$

We can apply the same procedure to the generalized hypergeometric function.

According to the considerations in Ref. [13], the expression for coherent states with a matrix argument is

$$|zu_0+\sigma u_1\rangle = \frac{1}{\sqrt{{}_pF_q(|zu_0+\sigma u_1|^2)}} \times \sum_{n=0}^{\infty}\frac{(zu_0+\sigma u_1)^n}{\sqrt{{}_p\rho_q(n)}}\,|n\rangle. \tag{53}$$

$$\begin{aligned}{}_pF_q(|Z|^2) &= \sum_{n=0}^{\infty}c_n(|Z|^2)^n \\ &= \sum_{n=0}^{\infty}c_n\big(|z|^2u_0+|\sigma|^2u_1\big)^n \\ &= u_0\sum_{n=0}^{\infty}c_n(|z|^2)^n+u_1\sum_{n=0}^{\infty}c_n(|\sigma|^2)^n \\ &= u_0\,{}_pF_q(|z|^2)+u_1\,{}_pF_q(|\sigma|^2).\end{aligned} \tag{54}$$

Using the properties of u_0 and u_1, namely $(u_j)^m = u_j$ for $m \geq 1$ and $u_i u_j = u_j u_i = 0$ for $i \neq j$, the normalization function of the coherent states with shifted arguments separates into two parts, each depending on one variable:

$$
\begin{aligned}
{}_pF_q(|zu_0 + \sigma u_1|^2) = u_0\, {}_pF_q(|z|^2) \\
+ u_1\, {}_pF_q(|\sigma|^2).
\end{aligned}
\tag{55}
$$

From a mathematical point of view, this relation represents a matrix version of Cauchy's functional equation, whose general expression is

$$
\mathcal{F}(x+y) = \mathcal{F}(x) + \mathcal{F}(y). \tag{56}
$$

In the case examined here, where the argument of the coherent states is matrix-valued, these relations can be written as follows.

$$
\begin{aligned}
\mathcal{F}(xu_0 + yu_1) &= \mathcal{F}(xu_0) + \mathcal{F}(yu_1) \\
&= \mathcal{F}(x)u_0 + \mathcal{F}(y)u_1.
\end{aligned}
\tag{57}
$$

The product of two such functions is

$$
\begin{aligned}
&\mathcal{F}(xu_0 + yu_1)\mathcal{G}(su_0 + \xi u_1) \\
&\quad = \big[\mathcal{F}(x)u_0 + \mathcal{F}(y)u_1\big]\big[\mathcal{G}(s)u_0 + \mathcal{G}(\xi)u_1\big] \\
&\quad = \mathcal{F}(x)\mathcal{G}(s)u_0 + \mathcal{F}(y)\mathcal{G}(\xi)u_1.
\end{aligned}
\tag{58}
$$

Since the matrices u_0 and u_1 are idempotent, they can be removed from the arguments of the respective functions, so that expressions involving two independent complex variables decouple:

$$
f\big(|z|^2u_0 + |\sigma|^2u_1\big) = u_0 f(|z|^2) + u_1 f(|\sigma|^2). \tag{59}
$$

Writing the complete coherent state $|zu_0 + \sigma u_1\rangle$ as an expansion over the basis vectors $|n\rangle$, we obtain

$$
\begin{aligned}
|zu_0 + \sigma u_1\rangle &= \big[{}_pF_q(|z|^2)u_0 + {}_pF_q(|\sigma|^2)u_1\big]^{-1/2} \\
&\quad \times \left(u_0 \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{{}_p\rho_q(n)}}|n\rangle \right. \\
&\quad \left. + u_1 \sum_{n=0}^{\infty} \frac{\sigma^n}{\sqrt{{}_p\rho_q(n)}}|n\rangle \right) \\
&= u_0 \frac{\sqrt{{}_pF_q(|z|^2)}\, u_0}{\sqrt{{}_pF_q(|z|^2)u_0 + {}_pF_q(|\sigma|^2)u_1}}\,|z\rangle \\
&\quad + u_1 \frac{\sqrt{{}_pF_q(|\sigma|^2)}\, u_1}{\sqrt{{}_pF_q(|z|^2)u_0 + {}_pF_q(|\sigma|^2)u_1}}\,|\sigma\rangle,
\end{aligned}
\tag{60}
$$

where the coherent states associated with the individual variables are

$$
\begin{aligned}
|zu_0\rangle &\longrightarrow u_0 \frac{1}{\sqrt{{}_pF_q(|z|^2)}} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{{}_p\rho_q(n)}}|n\rangle, \\
|\sigma u_1\rangle &\longrightarrow u_1 \frac{1}{\sqrt{{}_pF_q(|\sigma|^2)}} \sum_{n=0}^{\infty} \frac{\sigma^n}{\sqrt{{}_p\rho_q(n)}}|n\rangle.
\end{aligned}
\tag{61}
$$

The normalization of each component is projector-valued:

$$
\begin{aligned}
&\langle zu_0|zu_0\rangle = \langle z|u_0^2|z\rangle = u_0, \\
&\langle \sigma u_1|\sigma u_1\rangle = \langle \sigma|u_1^2|\sigma\rangle = u_1, \\
&\langle zu_0|\sigma u_1\rangle = \langle z|u_0u_1|\sigma\rangle = 0.
\end{aligned}
\tag{62}
$$

Thus,

$$
\langle zu_0|zu_0\rangle + \langle \sigma u_1|\sigma u_1\rangle = u_0 + u_1 = I. \tag{63}
$$

In the above relations, the coherent states $|z\rangle$ are not ordinary vectors but matrix-valued vectors. The Fock vectors $|n\rangle$ are column vectors, whereas the coherent states are linear combinations with matrix coefficients. In expanded form, we can write

$$
\begin{aligned}
|z\rangle \longrightarrow \frac{1}{\sqrt{{}_pF_q(|z|^2)}} \left[\frac{1}{\sqrt{{}_p\rho_q(0)}}|0\rangle \right. \\
+ \frac{z}{\sqrt{{}_p\rho_q(1)}}|1\rangle + \cdots \\
\left. + \frac{z^n}{\sqrt{{}_p\rho_q(n)}}|n\rangle + \cdots \right].
\end{aligned}
\tag{64}
$$

The partial conclusion is that coherent states whose argument is a sum of two terms can be written as a linear combination of coherent states associated with the separate terms, with matrix-valued weights.

$$
\begin{aligned}
\mathcal{P}_{zu_0} &\equiv \frac{u_0\sqrt{{}_pF_q(|z|^2)}}{\sqrt{{}_pF_q(|z|^2)u_0 + {}_pF_q(|\sigma|^2)u_1}}, \\
\mathcal{P}_{\sigma u_1} &\equiv \frac{u_1\sqrt{{}_pF_q(|\sigma|^2)}}{\sqrt{{}_pF_q(|z|^2)u_0 + {}_pF_q(|\sigma|^2)u_1}}.
\end{aligned}
\tag{65}
$$

In short, we write

$$
|zu_0 + \sigma u_1\rangle = \mathcal{P}_{zu_0}|z\rangle u_0 + \mathcal{P}_{\sigma u_1}|\sigma\rangle u_1. \tag{66}
$$

It is not difficult to verify that these coherent states are normalized:

$$
\begin{aligned}
&\langle zu_0 + \sigma u_1|zu_0 + \sigma u_1\rangle \\
&= \big(\mathcal{P}_{zu_0}u_0\langle z| + \mathcal{P}_{\sigma u_1}u_1\langle\sigma|\big) \\
&\quad \times \big(\mathcal{P}_{zu_0}|z\rangle u_0 + \mathcal{P}_{\sigma u_1}|\sigma\rangle u_1\big) \\
&= (\mathcal{P}_{zu_0})^2 + (\mathcal{P}_{\sigma u_1})^2 = I.
\end{aligned}
\tag{67}
$$

Their associated projector is

$$
\begin{aligned}
&|zu_0 + \sigma u_1\rangle\langle zu_0 + \sigma u_1| \\
&= \big(\mathcal{P}_{zu_0}|z\rangle u_0 + \mathcal{P}_{\sigma u_1}|\sigma\rangle u_1\big) \\
&\quad \times \big(\mathcal{P}_{zu_0}\langle z|u_0 + \mathcal{P}_{\sigma u_1}\langle\sigma|u_1\big) \\
&= (\mathcal{P}_{zu_0})^2|z\rangle\langle z|u_0 + (\mathcal{P}_{\sigma u_1})^2|\sigma\rangle\langle\sigma|u_1.
\end{aligned}
\tag{68}
$$

The following limits reduce the coherent states with a matrix argument to coherent states labelled by a single variable:

$$
\begin{aligned}
&\lim_{z\to 0} |zu_0 + \sigma u_1\rangle = u_1|\sigma\rangle, \\
&\lim_{\sigma\to 0} |zu_0 + \sigma u_1\rangle = u_0|z\rangle.
\end{aligned}
\tag{69}
$$

Particular case 1: One-dimensional quantum harmonic oscillator. For $p = q = 0$, with empty parameter sets $\{a_i\}$ and $\{b_j\}$, the generalized hypergeometric function becomes ${}_0F_0(x) = \exp(x)$. The coherent states with a matrix argument are then written as

$$\begin{aligned}|zu_0+\sigma u_1\rangle &= \frac{1}{\sqrt{e^{|z|^2}u_0+e^{|\sigma|^2}u_1}} \\ &\quad\times\left(u_0\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{n!}}|n\rangle\right. \\ &\quad\left.+u_1\sum_{n=0}^{\infty}\frac{\sigma^n}{\sqrt{n!}}|n\rangle\right) \\ &= \frac{1}{\sqrt{e^{|z|^2}u_0+e^{|\sigma|^2}u_1}} \\ &\quad\times\left(u_0e^{|z|^2/2}|z\rangle+u_1e^{|\sigma|^2/2}|\sigma\rangle\right) \\ &= \frac{e^{|z|^2/2}}{\sqrt{e^{|z|^2}u_0+e^{|\sigma|^2}u_1}}\,|z\rangle u_0 \\ &\quad+\frac{e^{|\sigma|^2/2}}{\sqrt{e^{|z|^2}u_0+e^{|\sigma|^2}u_1}}\,|\sigma\rangle u_1.\end{aligned} \tag{70}$$

The normalization follows by calculating the scalar product:

$$\begin{aligned}&\langle zu_0+\sigma u_1|zu_0+\sigma u_1\rangle \\ &= \frac{1}{e^{|z|^2}u_0+e^{|\sigma|^2}u_1} \\ &\quad\times\left(u_0e^{|z|^2/2}\langle z|+u_1e^{|\sigma|^2/2}\langle\sigma|\right) \\ &\quad\times\left(u_0e^{|z|^2/2}|z\rangle+u_1e^{|\sigma|^2/2}|\sigma\rangle\right) \\ &= \frac{e^{|z|^2}u_0+e^{|\sigma|^2}u_1}{e^{|z|^2}u_0+e^{|\sigma|^2}u_1} = I.\end{aligned} \tag{71}$$

This case corresponds to the one-dimensional quantum harmonic oscillator and hence to canonical coherent states.

As a side note, the denominator above is a full diagonal matrix. In accordance with the general power-series expansion,

$$\mathcal{F}(xu_j) = \mathcal{F}(x)u_j + \mathcal{F}(0)(I-u_j), \tag{72}$$

we recall the following useful relations:

$$e^xu_0+e^yu_1 = \begin{pmatrix}e^x&0\\0&0\end{pmatrix}+\begin{pmatrix}0&0\\0&e^y\end{pmatrix} = \begin{pmatrix}e^x&0\\0&e^y\end{pmatrix}. \tag{73}$$

$$\begin{aligned}e^{xu_j} &= I+(e^x-1)u_j, \qquad j=0,1, \\ e^{xu_0} &= e^xu_0+u_1 = \begin{pmatrix}e^x&0\\0&1\end{pmatrix}, \\ e^{xu_1} &= u_0+e^xu_1 = \begin{pmatrix}1&0\\0&e^x\end{pmatrix}.\end{aligned} \tag{74}$$

$$\begin{aligned}e^{xu_0}+e^{yu_1} &= \begin{pmatrix}e^x&0\\0&1\end{pmatrix}+\begin{pmatrix}1&0\\0&e^y\end{pmatrix} \\ &= \begin{pmatrix}e^x+1&0\\0&e^y+1\end{pmatrix}.\end{aligned} \tag{75}$$

$$\begin{aligned}e^{xu_0}+e^{xu_1} &= \begin{pmatrix}e^x+1&0\\0&e^x+1\end{pmatrix} \\ &= (e^x+1)I.\end{aligned} \tag{76}$$

For real x and y, using the principal matrix logarithm,

$$\ln(e^{xu_j}) = \ln(e^x)\,u_j = xu_j. \tag{77}$$

$$\ln(e^xu_0+e^yu_1) = xu_0+yu_1. \tag{78}$$

Particular case 2—A two-level system.

For a two-level system, we have

$$\begin{aligned}|zu_0+\sigma u_1\rangle &= \frac{1}{\sqrt{u_0\,{}_pF_q(|z|^2)+u_1\,{}_pF_q(|\sigma|^2)}} \\ &\quad\times\left(u_0\sum_{n=0}^{1}\frac{z^n}{\sqrt{{}_p\rho_q(n)}}|n\rangle+u_1\sum_{n=0}^{1}\frac{\sigma^n}{\sqrt{{}_p\rho_q(n)}}|n\rangle\right).\end{aligned} \tag{79}$$

$$\begin{aligned}\langle zu_0+\sigma u_1| &= \frac{1}{\sqrt{u_0\,{}_pF_q(|z|^2)+u_1\,{}_pF_q(|\sigma|^2)}} \\ &\quad\times\left(u_0\sum_{n=0}^{1}\frac{(z^*)^n}{\sqrt{{}_p\rho_q(n)}}\langle n|+u_1\sum_{n=0}^{1}\frac{(\sigma^*)^n}{\sqrt{{}_p\rho_q(n)}}\langle n|\right).\end{aligned} \tag{80}$$

The corresponding inner product is

$$\begin{aligned}&\langle zu_0+\sigma u_1\mid zu_0+\sigma u_1\rangle \\ &\quad= \frac{1}{u_0\,{}_pF_q(|z|^2)+u_1\,{}_pF_q(|\sigma|^2)} \\ &\qquad\times\left(u_0\sum_{n=0}^{1}\frac{(|z|^2)^n}{{}_p\rho_q(n)}+u_1\sum_{n=0}^{1}\frac{(|\sigma|^2)^n}{{}_p\rho_q(n)}\right),\end{aligned} \tag{81}$$

where

$${}_p\rho_q(0) = 1, \qquad {}_p\rho_q(1) = \frac{\prod_{j=1}^{q}b_j}{\prod_{i=1}^{p}a_i} \neq 0. \tag{82}$$

Coherent states with a shifted argument satisfy the completeness relation, or resolution of the identity operator, which is a fundamental property in constructing coherent states:

$$\int \mathrm{d}\mu(\varepsilon z+\lambda\sigma)\,|\varepsilon z+\lambda\sigma\rangle\langle\varepsilon z+\lambda\sigma| = I. \tag{83}$$

With $Z = \varepsilon z + \lambda\sigma$, this condition becomes

$$\int \mathrm{d}\mu(Z)\,|Z\rangle\langle Z| = I. \tag{84}$$

The corresponding integration measure on the integration domain is [23]

$$\begin{aligned}\mathrm{d}\mu(Z) &= \Gamma(a/b)\,\frac{\mathrm{d}^2 Z}{\pi}\\ &\times G^{q+1,0}_{p,q+1}\left(|Z|^2\,\middle|\,\begin{matrix}-;\,\{a_i-1\}_{i=1}^{p}\\ 0,\{b_j-1\}_{j=1}^{q};\,-\end{matrix}\right){}_pF_q(|Z|^2).\end{aligned} \tag{85}$$

Equivalently, in terms of the shifted argument,

$$\begin{aligned}\mathrm{d}\mu(\varepsilon z+\lambda\sigma) &= \Gamma(a/b)\,\mathrm{d}\big(|\varepsilon z+\lambda\sigma|^2\big)\,\frac{\mathrm{d}\varphi_Z}{2\pi}\\ &\times G^{q+1,0}_{p,q+1}\left(|\varepsilon z+\lambda\sigma|^2\,\middle|\,\begin{matrix}-;\,\{a_i-1\}_{i=1}^{p}\\ 0,\{b_j-1\}_{j=1}^{q};\,-\end{matrix}\right)\\ &\times {}_pF_q\big(|\varepsilon z+\lambda\sigma|^2\big)\,.\end{aligned} \tag{86}$$

To simplify the formulas, whenever there is no risk of confusion, we use a shorthand notation for Meijer's G-function, suppressing its parameters and displaying only its argument:

$$\begin{aligned}G^{q+1,0}_{p,q+1}\left(x\,\middle|\,\begin{matrix}-;\,\{a_i-1\}_{i=1}^{p}\\ 0,\{b_j-1\}_{j=1}^{q};\,-\end{matrix}\right) &\equiv G^{q+1,0}_{p,q+1}(x\mid\cdots),\\ \Phi(s;a-1,b-1) &= \frac{\Gamma(-s)\prod_{j=1}^{q}\Gamma(b_j-1-s)}{\prod_{i=1}^{p}\Gamma(a_i-1-s)}.\end{aligned} \tag{87}$$

We now consider the completeness condition for coherent states with a matrix argument. In this case, the projectors u_0 and u_1 replace the constants ε and λ, respectively: $\varepsilon = u_0$ and $\lambda = u_1$. Thus,

$$\begin{aligned}&\int \mathrm{d}\mu(zu_0+\sigma u_1)\,|zu_0+\sigma u_1\rangle\langle zu_0+\sigma u_1|\\ &\quad=\int \mathrm{d}\mu(zu_0+\sigma u_1)\Big[\big(\mathcal{P}_{zu_0}\big)^2|z\rangle\langle z|u_0\\ &\qquad\qquad+\big(\mathcal{P}_{\sigma u_1}\big)^2|\sigma\rangle\langle\sigma|u_1\Big] = I.\end{aligned} \tag{88}$$

With these substitutions, the corresponding integration measure is written as

$$\begin{aligned}\mathrm{d}\mu(zu_0+\sigma u_1) &= \Gamma(a/b)\,\frac{\mathrm{d}^2(zu_0+\sigma u_1)}{\pi}\\ &\times G^{q+1,0}_{p,q+1}\left(|zu_0|^2+|\sigma u_1|^2\,\middle|\cdots\right)\\ &\times {}_pF_q\big(u_0|z|^2+u_1|\sigma|^2\big)\,.\end{aligned} \tag{89}$$

The differential associated with $Z = zu_0 + \sigma u_1$ is written formally in terms of the two independent complex variables z and σ as follows:

$$\begin{aligned}\frac{\mathrm{d}^2 Z}{\pi} &\equiv \frac{\mathrm{d}^2(zu_0+\sigma u_1)}{\pi}\\ &= \frac{\partial}{\partial z}(zu_0+\sigma u_1)\frac{\mathrm{d}^2 z}{\pi}\\ &\quad+\frac{\partial}{\partial\sigma}(zu_0+\sigma u_1)\frac{\mathrm{d}^2\sigma}{\pi}\\ &= \frac{\mathrm{d}^2 z}{\pi}u_0+\frac{\mathrm{d}^2\sigma}{\pi}u_1\\ &= \frac{\mathrm{d}\varphi_z}{2\pi}\mathrm{d}(|z|^2)u_0+\frac{\mathrm{d}\varphi_\sigma}{2\pi}\mathrm{d}(|\sigma|^2)u_1.\end{aligned} \tag{90}$$

First, we examine Meijer's G-function appearing in the integration measure. Using its integral representation [18], we obtain

$$\begin{aligned}&G^{q+1,0}_{p,q+1}\left(x\,\middle|\,\begin{matrix}-;\,\{a_i-1\}_{i=1}^{p}\\ 0,\{b_j-1\}_{j=1}^{q};\,-\end{matrix}\right)\\ &\quad=\frac{1}{2\pi\mathrm{i}}\int_{\mathcal{L}}\frac{\Gamma(-s)\prod_{j=1}^{q}\Gamma(b_j-1-s)}{\prod_{i=1}^{p}\Gamma(a_i-1-s)}x^s\,\mathrm{d}s\\ &\quad\equiv\frac{1}{2\pi\mathrm{i}}\int_{\mathcal{L}}\Phi(s;a-1,b-1)\,x^s\,\mathrm{d}s.\end{aligned} \tag{91}$$

In our case, this gives

$$\begin{aligned}&G^{q+1,0}_{p,q+1}(|Z|^2\mid\cdots)\\ &\quad=G^{q+1,0}_{p,q+1}\left(|z|^2u_0+|\sigma|^2u_1\,\middle|\cdots\right)\\ &\quad=\frac{1}{2\pi\mathrm{i}}\int_{\mathcal{L}}\Phi(s;a-1,b-1)\left(|z|^2u_0+|\sigma|^2u_1\right)^s\mathrm{d}s\\ &\quad=\frac{u_0}{2\pi\mathrm{i}}\int_{\mathcal{L}}\Phi(s;a-1,b-1)(|z|^2)^s\,\mathrm{d}s\\ &\qquad+\frac{u_1}{2\pi\mathrm{i}}\int_{\mathcal{L}}\Phi(s;a-1,b-1)(|\sigma|^2)^s\,\mathrm{d}s\\ &\quad=u_0G^{q+1,0}_{p,q+1}(|z|^2\mid\cdots)+u_1G^{q+1,0}_{p,q+1}(|\sigma|^2\mid\cdots).\end{aligned} \tag{92}$$

Thus, the matrix-valued Meijer G-function decomposes into two scalar functions, depending on $|z|^2$ and $|\sigma|^2$, respectively.

Using these results, the integration measure becomes

$$\begin{aligned}&\mathrm{d}\mu(zu_0+\sigma u_1)\\ &\quad=\Gamma(a/b)\left[u_0\frac{\mathrm{d}\varphi_z}{2\pi}\mathrm{d}(|z|^2)+u_1\frac{\mathrm{d}\varphi_\sigma}{2\pi}\mathrm{d}(|\sigma|^2)\right]\\ &\qquad\times\left[u_0\,{}_pF_q(|z|^2)+u_1\,{}_pF_q(|\sigma|^2)\right]\\ &\qquad\times\Big[u_0G^{q+1,0}_{p,q+1}(|z|^2\mid\cdots)\\ &\qquad\qquad+u_1G^{q+1,0}_{p,q+1}(|\sigma|^2\mid\cdots)\Big]\,.\end{aligned} \tag{93}$$

Using the properties of the projectors, this expression can be simplified further.

$$\begin{aligned}\mathrm{d}\mu(zu_0+\sigma u_1) &= u_0\Gamma(a/b)\left[\frac{\mathrm{d}\varphi_z}{2\pi}\mathrm{d}(|z|^2)\right.\\ &\quad\left.\times\, G^{q+1,0}_{p,q+1}(|z|^2\mid\cdots)\,{}_pF_q(|z|^2)\right]\\ &\quad+u_1\Gamma(a/b)\left[\frac{\mathrm{d}\varphi_\sigma}{2\pi}\mathrm{d}(|\sigma|^2)\right.\\ &\quad\left.\times\, G^{q+1,0}_{p,q+1}(|\sigma|^2\mid\cdots)\,{}_pF_q(|\sigma|^2)\right]\\ &\equiv u_0\,\mathrm{d}\mu(z)+u_1\,\mathrm{d}\mu(\sigma).\end{aligned}\tag{94}$$

The resolution of the identity therefore becomes

$$\begin{aligned}&u_0\left[\int \mathrm{d}\mu(z)\left(\mathcal{P}_{zu_0}\right)^2|z\rangle\langle z|\right]\\ &\quad+u_1\left[\int \mathrm{d}\mu(\sigma)\left(\mathcal{P}_{\sigma u_1}\right)^2|\sigma\rangle\langle\sigma|\right]=I.\end{aligned}\tag{95}$$

The projector associated with the coherent state labelled by z is

$$\begin{aligned}|z\rangle\langle z| &= \frac{1}{{}_pF_q(|z|^2)}\\ &\quad\times\left[\sum_{n,m=0}^{\infty}\frac{z^n(z^*)^m}{\sqrt{{}_p\rho_q(n)}\sqrt{{}_p\rho_q(m)}}|n\rangle\langle m|\right],\end{aligned}\tag{96}$$

with an analogous expression for the variable σ.

We now integrate the first component:

$$\begin{aligned}\mathrm{Int}_z &\equiv u_0\left[\int \mathrm{d}\mu(z)\left(\mathcal{P}_{zu_0}\right)^2|z\rangle\langle z|\right]\\ &= u_0\Gamma(a/b)\sum_{n,m=0}^{\infty}\frac{|n\rangle\langle m|}{\sqrt{{}_p\rho_q(n)}\sqrt{{}_p\rho_q(m)}}\\ &\quad\times\int_0^{\infty}\mathrm{d}(|z|^2)\,G^{q+1,0}_{p,q+1}(|z|^2\mid\cdots)\\ &\quad\times\int_0^{2\pi}\frac{\mathrm{d}\varphi_z}{2\pi}\,z^n(z^*)^m\\ &= \Gamma(a/b)\left[\sum_{n=0}^{\infty}\frac{|n\rangle\langle n|}{{}_p\rho_q(n)}\frac{{}_p\rho_q(n)}{\Gamma(a/b)}\right]u_0\\ &= u_0\sum_{n=0}^{\infty}|n\rangle\langle n|=u_0.\end{aligned}\tag{97}$$

The angular integral satisfies

$$\int_0^{2\pi}\frac{\mathrm{d}\varphi_z}{2\pi}\,z^n(z^*)^m=(|z|^2)^n\delta_{nm}.\tag{98}$$

Thus, the off-diagonal terms vanish upon integration over the phase of z. The remaining radial integral is [23]

$$\begin{aligned}&\int_0^{\infty}\mathrm{d}(|z|^2)\,(|z|^2)^n\\ &\quad\times G^{q+1,0}_{p,q+1}\left(|z|^2\,\middle|\,\begin{matrix}-;\,\{a_i-1\}_{i=1}^{p}\\ 0,\{b_j-1\}_{j=1}^{q};-\end{matrix}\right)\\ &\quad=\frac{{}_p\rho_q(n)}{\Gamma(a/b)}.\end{aligned}\tag{99}$$

Consequently,

$$\begin{aligned}\mathrm{Int}_z &= \Gamma(a/b)\left[\sum_{n=0}^{\infty}\frac{|n\rangle\langle n|}{{}_p\rho_q(n)}\frac{{}_p\rho_q(n)}{\Gamma(a/b)}\right]u_0\\ &= u_0\sum_{n=0}^{\infty}|n\rangle\langle n|=u_0.\end{aligned}\tag{100}$$

Similarly, integration over σ yields $\mathrm{Int}_\sigma=u_1$. Hence,

$$\mathrm{Int}_z+\mathrm{Int}_\sigma=u_0+u_1=I.\tag{101}$$

The integral therefore reproduces the identity operator.

The preceding calculation gives the identity matrix as the sum of the two components. Thus, with the componentwise interpretation of the measure, the coherent states satisfy the resolution of the identity. This is the third condition in Klauder's prescription [19].

The first two conditions concern normalization and continuity. The states are normalized and generally nonorthogonal:

$$\begin{aligned}&\langle zu_0+\sigma u_1\mid z'u_0+\sigma'u_1\rangle\\ &\quad=\langle z|z'\rangle u_0+\langle\sigma|z'\rangle u_1u_0\\ &\qquad+\langle z|\sigma'\rangle u_0u_1+\langle\sigma|\sigma'\rangle u_1\\ &\quad=\langle z|z'\rangle u_0+\langle\sigma|\sigma'\rangle u_1.\end{aligned}\tag{102}$$

For $z'=z$ and $\sigma'=\sigma$, this expression equals $I=u_0+u_1$. For distinct labels, the overlap is generally nonzero, so the states are generally nonorthogonal.

The coherent states depend continuously on the complex labels z and σ. Indeed, using the operator norm for the corresponding maps from $\mathbb{C}^2$ into $\mathbb{C}^2\otimes\mathcal{H}_{\mathrm{Fock}}$, we have

$$\begin{aligned}&\big\|\,|zu_0+\sigma u_1\rangle-|z'u_0+\sigma'u_1\rangle\,\big\|\\ &\quad=\max\left\{\big\||z\rangle-|z'\rangle\big\|,\big\||\sigma\rangle-|\sigma'\rangle\big\|\right\}\longrightarrow 0\\ &\qquad z\longrightarrow z',\quad \sigma\longrightarrow\sigma'.\end{aligned}\tag{103}$$

Finally, the states satisfy the resolution of the identity, as established above:

$$\begin{aligned}&\int\mathrm{d}\mu(zu_0+\sigma u_1)\,|zu_0+\sigma u_1\rangle\langle zu_0+\sigma u_1|\\ &\quad=u_0\left[\int\mathrm{d}\mu(z)\left(\mathcal{P}_{zu_0}\right)^2|z\rangle\langle z|\right]\\ &\qquad+u_1\left[\int\mathrm{d}\mu(\sigma)\left(\mathcal{P}_{\sigma u_1}\right)^2|\sigma\rangle\langle\sigma|\right]\\ &\quad=u_0+u_1=I.\end{aligned}\tag{104}$$

For applications, we consider the expectation values of observables $\mathcal{A}$ characterizing the system. For an observable acting on the Fock-space factor, its matrix-valued coherent-state expectation is

$$\begin{aligned}\langle\mathcal{A}\rangle_{z,\sigma} &\equiv \langle zu_0+\sigma u_1|\mathcal{A}|zu_0+\sigma u_1\rangle \\ &= \left[u_0\,{}_pF_q(|z|^2)+u_1\,{}_pF_q(|\sigma|^2)\right]^{-1} \\ &\quad\times\sum_{n,n'=0}^{\infty}\frac{u_0(z^*)^n z^{n'}+u_1(\sigma^*)^n\sigma^{n'}}{\sqrt{{}_p\rho_q(n)\,{}_p\rho_q(n')}} \\ &\quad\times\langle n|\mathcal{A}|n'\rangle.\end{aligned} \tag{105}$$

If the observable is diagonal in the Fock basis, $\mathcal{A}|n\rangle = A(n)|n\rangle$, and its eigenvalues admit a power-series expansion $A(n)=\sum_{\ell=0}^{\infty}c_\ell n^\ell$, the preceding expression can be simplified further.

$$\begin{aligned}\langle\mathcal{A}\rangle_{z,\sigma} &= \left[u_0\,{}_pF_q(|z|^2)+u_1\,{}_pF_q(|\sigma|^2)\right]^{-1} \\ &\quad\times\sum_{n=0}^{\infty}\frac{u_0(|z|^2)^n+u_1(|\sigma|^2)^n}{{}_p\rho_q(n)}\sum_{\ell=0}^{\infty}c_\ell n^\ell \\ &= \left[u_0\,{}_pF_q(|z|^2)+u_1\,{}_pF_q(|\sigma|^2)\right]^{-1} \\ &\quad\times\sum_{\ell=0}^{\infty}c_\ell\left[u_0\sum_{n=0}^{\infty}\frac{n^\ell(|z|^2)^n}{{}_p\rho_q(n)}+u_1\sum_{n=0}^{\infty}\frac{n^\ell(|\sigma|^2)^n}{{}_p\rho_q(n)}\right] \\ &= \left[u_0\,{}_pF_q(|z|^2)+u_1\,{}_pF_q(|\sigma|^2)\right]^{-1} \\ &\quad\times\left[u_0\sum_{\ell=0}^{\infty}c_\ell\Theta^\ell_{|z|^2}\,{}_pF_q(|z|^2)\right. \\ &\qquad\left.+u_1\sum_{\ell=0}^{\infty}c_\ell\Theta^\ell_{|\sigma|^2}\,{}_pF_q(|\sigma|^2)\right] \\ &= \left[u_0\,{}_pF_q(|z|^2)+u_1\,{}_pF_q(|\sigma|^2)\right]^{-1} \\ &\quad\times\left[u_0A(\Theta_{|z|^2})\,{}_pF_q(|z|^2)\right. \\ &\qquad\left.+u_1A(\Theta_{|\sigma|^2})\,{}_pF_q(|\sigma|^2)\right].\end{aligned} \tag{106}$$

The last equality follows by replacing n with the Euler differential operator, $A(n)\mapsto A(\Theta_x)$, where $\Theta_x = x\,\partial/\partial x$ and $x=|z|^2$ or $x=|\sigma|^2$. The interchange of sums and derivatives is understood where the corresponding series converge.

For a coherent state labelled by a single complex variable, for example z, the expectation value is

$$\begin{aligned}\langle\mathcal{A}\rangle_z &\equiv \langle z|\mathcal{A}|z\rangle \\ &= \frac{1}{{}_pF_q(|z|^2)}\sum_{n=0}^{\infty}\frac{(|z|^2)^n}{{}_p\rho_q(n)}A(n) \\ &= \frac{1}{{}_pF_q(|z|^2)}\sum_{\ell=0}^{\infty}c_\ell\sum_{n=0}^{\infty}\frac{n^\ell(|z|^2)^n}{{}_p\rho_q(n)} \\ &= \frac{1}{{}_pF_q(|z|^2)}\sum_{\ell=0}^{\infty}c_\ell\Theta^\ell_{|z|^2}\,{}_pF_q(|z|^2) \\ &= \frac{A(\Theta_{|z|^2})\,{}_pF_q(|z|^2)}{{}_pF_q(|z|^2)}.\end{aligned} \tag{107}$$

Consequently, the expectation value in the matrix-labelled state decomposes into the expectation values associated with the two individual coherent states:

$$\begin{aligned}\langle\mathcal{A}\rangle_{z,\sigma} &= u_0\big(\mathcal{P}_{zu_0}\big)^2\langle\mathcal{A}\rangle_z+u_1\big(\mathcal{P}_{\sigma u_1}\big)^2\langle\mathcal{A}\rangle_\sigma \\ &= u_0\langle\mathcal{A}\rangle_z+u_1\langle\mathcal{A}\rangle_\sigma.\end{aligned} \tag{108}$$

5 Mixed States

We return to a quantum system in thermodynamic equilibrium with its environment. Its mixed state is described by the canonical density operator

$$\rho=\frac{1}{Z(\beta)}\sum_{n=0}^{\infty}e^{-\beta E_n}|n\rangle\langle n|. \tag{109}$$

In the coherent-state representation, the density operator is associated with quasiprobability distributions, including the Husimi Q-function and the Glauber–Sudarshan P-representation, the latter appearing in its diagonal coherent-state expansion.

The Husimi Q-function is defined through the expectation value of the density operator in the coherent-state representation:

$$\begin{aligned}Q(|zu_0+\sigma u_1|^2) &\equiv \langle zu_0+\sigma u_1|\rho|zu_0+\sigma u_1\rangle \\ &= \frac{1}{Z(\beta)}\sum_{n=0}^{\infty}e^{-\beta E_n}\langle zu_0+\sigma u_1|n\rangle \\ &\qquad\times\langle n|zu_0+\sigma u_1\rangle \\ &= \frac{1}{Z(\beta)}\left[u_0\,{}_pF_q(|z|^2)+u_1\,{}_pF_q(|\sigma|^2)\right]^{-1} \\ &\quad\times\sum_{n=0}^{\infty}e^{-\beta E_n}\frac{|zu_0+\sigma u_1|^{2n}}{{}_p\rho_q(n)} \\ &= \frac{1}{Z(\beta)}\left[u_0\,{}_pF_q(|z|^2)+u_1\,{}_pF_q(|\sigma|^2)\right]^{-1} \\ &\quad\times\sum_{n=0}^{\infty}e^{-\beta E_n}\left[u_0\frac{(|z|^2)^n}{{}_p\rho_q(n)}+u_1\frac{(|\sigma|^2)^n}{{}_p\rho_q(n)}\right].\end{aligned} \tag{110}$$

Finally, we obtain

$$\begin{aligned}Q(|zu_0+\sigma u_1|^2) &= u_0\big(\mathcal{P}_{zu_0}\big)^2Q(|z|^2) \\ &\quad+u_1\big(\mathcal{P}_{\sigma u_1}\big)^2Q(|\sigma|^2).\end{aligned} \tag{111}$$

Thus, the matrix-valued Husimi function decomposes into the Husimi functions associated with the individual variables:

$$Q(|z|^2)=\frac{1}{Z(\beta)\,{}_pF_q(|z|^2)}\sum_{n=0}^{\infty}e^{-\beta E_n}\frac{(|z|^2)^n}{{}_p\rho_q(n)}. \tag{112}$$

An analogous expression holds for σ. This is the usual Husimi function in the coherent-state representation labelled by a single complex variable. The Husimi function is nonnegative.

On the one hand, the density operator has the expansion over Fock-state projectors

$$\rho = \sum_{n=0}^{\infty} \frac{e^{-\beta E_n}}{Z(\beta)} |n\rangle\langle n|. \tag{113}$$

On the other hand, the quasiprobability distribution $P_{p,q}$ appears as a weight in the diagonal coherent-state representation:

$$\begin{aligned}\rho &= \int \mathrm{d}\mu(Z)\, P_{p,q}(|Z|^2)\, |Z\rangle\langle Z| \\ &= \sum_{n=0}^{\infty} \frac{|n\rangle\langle n|}{{}_p\rho_q(n)} \int \mathrm{d}\mu(Z)\, P_{p,q}(|Z|^2) \frac{(|Z|^2)^n}{{}_pF_q(|Z|^2)}.\end{aligned} \tag{114}$$

Equating the coefficients gives the moment condition

$$\int \mathrm{d}\mu(Z)\, P_{p,q}(|Z|^2) \frac{(|Z|^2)^n}{{}_pF_q(|Z|^2)} = \frac{e^{-\beta E_n}}{Z(\beta)} {}_p\rho_q(n). \tag{115}$$

Using the explicit integration measure, this becomes

$$\begin{aligned}&\int \frac{\mathrm{d}^2 Z}{\pi}\, G^{q+1,0}_{p,q+1}(|Z|^2 \mid \cdots) P_{p,q}(|Z|^2)(|Z|^2)^n \\ &\quad = \frac{e^{-\beta E_n}\, {}_p\rho_q(n)}{\Gamma(a/b)\, Z(\beta)}.\end{aligned} \tag{116}$$

Returning to the variables z and σ, we write

$$\begin{aligned}\rho = \int \mathrm{d}\mu(zu_0 + \sigma u_1)\, P_{p,q}\big(|z|^2 u_0 + |\sigma|^2 u_1\big) \\ \times |zu_0 + \sigma u_1\rangle\langle zu_0 + \sigma u_1|.\end{aligned} \tag{117}$$

This expression must agree with the expansion of the density operator in terms of the Fock-state projectors $|n\rangle\langle n|$.

$$\begin{aligned}&\frac{1}{Z(\beta)} \sum_{n=0}^{\infty} e^{-\beta E_n} |n\rangle\langle n| \\ &\quad = \int \mathrm{d}\mu(zu_0 + \sigma u_1)\, P_{p,q}\big(|z|^2 u_0 + |\sigma|^2 u_1\big) \\ &\qquad \times |zu_0 + \sigma u_1\rangle\langle zu_0 + \sigma u_1|.\end{aligned} \tag{118}$$

Writing the coherent-state projector in the following form,

$$\begin{aligned}&|zu_0 + \sigma u_1\rangle\langle zu_0 + \sigma u_1| \\ &\quad = \big[{}_pF_q\big(|zu_0 + \sigma u_1|^2\big)\big]^{-1} \\ &\qquad \times \sum_{n=0}^{\infty} \frac{\big(|z|^2 u_0 + |\sigma|^2 u_1\big)^n}{{}_p\rho_q(n)} |n\rangle\langle n|,\end{aligned} \tag{119}$$

we obtain

$$\begin{aligned}&\left[\frac{1}{Z(\beta)} \sum_{n=0}^{\infty} e^{-\beta E_n}\right] |n\rangle\langle n| \\ &\quad = \left[\int \mathrm{d}\mu(zu_0 + \sigma u_1)\right. \\ &\qquad \times \frac{P_{p,q}\big(|z|^2 u_0 + |\sigma|^2 u_1\big)}{{}_pF_q(|zu_0 + \sigma u_1|^2)} \\ &\qquad \left. \times \frac{\big(|z|^2 u_0 + |\sigma|^2 u_1\big)^n}{{}_p\rho_q(n)}\right] |n\rangle\langle n|.\end{aligned} \tag{120}$$

Equality of the corresponding coefficients requires

$$\begin{aligned}&\int \mathrm{d}\mu(zu_0 + \sigma u_1) \\ &\quad \times \frac{P_{p,q}\big(|z|^2 u_0 + |\sigma|^2 u_1\big)}{{}_pF_q(|zu_0 + \sigma u_1|^2)} \big(|z|^2 u_0 + |\sigma|^2 u_1\big)^n \\ &\qquad = \frac{e^{-\beta E_n}}{Z(\beta)} {}_p\rho_q(n).\end{aligned} \tag{121}$$

Substituting the expressions for the integration measure and the coherent states gives

$$\begin{aligned}\rho = \Gamma(a/b) \int &\left[\frac{\mathrm{d}\varphi_z}{2\pi} \mathrm{d}(|z|^2) u_0 + \frac{\mathrm{d}\varphi_\sigma}{2\pi} \mathrm{d}(|\sigma|^2) u_1\right] \\ &\times \big[u_0\, {}_pF_q(|z|^2) + u_1\, {}_pF_q(|\sigma|^2)\big] \\ &\times P_{p,q}\big(|z|^2 u_0 + |\sigma|^2 u_1\big) \\ &\times \Big[u_0 G^{q+1,0}_{p,q+1}(|z|^2 \mid \cdots) \\ &\qquad + u_1 G^{q+1,0}_{p,q+1}(|\sigma|^2 \mid \cdots)\Big] \\ &\times \Big[u_0 \big(\mathcal{P}_{zu_0}\big)^2 |z\rangle\langle z| \\ &\qquad + u_1 \big(\mathcal{P}_{\sigma u_1}\big)^2 |\sigma\rangle\langle\sigma|\Big].\end{aligned} \tag{122}$$

The properties of the projectors also allow the function $P_{p,q}$ to be decomposed into two functions, each depending on a single variable:

$$\begin{aligned}&P_{p,q}\big(|z|^2 u_0 + |\sigma|^2 u_1\big) \\ &\quad = u_0 P_{p,q}(|z|^2) + u_1 P_{p,q}(|\sigma|^2).\end{aligned} \tag{123}$$

Multiplying the factors and using $u_i u_j = 0$ for $i \neq j$, we obtain

$$\begin{aligned}\rho = u_0 \Gamma(a/b) &\int_0^{\infty} \mathrm{d}(|z|^2)\, {}_pF_q(|z|^2) P_{p,q}(|z|^2) \\ &\times G^{q+1,0}_{p,q+1}\left(|z|^2 \,\middle|\, \begin{matrix} -;\, \{a_i - 1\}_{i=1}^p \\ 0, \{b_j - 1\}_{j=1}^q;\, - \end{matrix}\right) |z\rangle\langle z| \\ &+ u_1 \Gamma(a/b) \int_0^{\infty} \mathrm{d}(|\sigma|^2)\, {}_pF_q(|\sigma|^2) P_{p,q}(|\sigma|^2) \\ &\times G^{q+1,0}_{p,q+1}\left(|\sigma|^2 \,\middle|\, \begin{matrix} -;\, \{a_i - 1\}_{i=1}^p \\ 0, \{b_j - 1\}_{j=1}^q;\, - \end{matrix}\right) |\sigma\rangle\langle\sigma|.\end{aligned} \tag{124}$$

The density operator is therefore expressed as the sum of two components.

$$\rho = u_0\rho_z + u_1\rho_\sigma. \tag{125}$$

The variables z and σ enter symmetrically, so the corresponding expressions have the same form. We therefore evaluate the component associated with z:

$$\begin{aligned}\rho_z = \Gamma(a/b)\int_0^\infty \mathrm{d}(|z|^2)\,{}_pF_q(|z|^2)P_{p,q}(|z|^2)\\ \times G^{q+1,0}_{p,q+1}(|z|^2\mid\cdots)\int_0^{2\pi}\frac{\mathrm{d}\varphi_z}{2\pi}\,|z\rangle\langle z|.\end{aligned} \tag{126}$$

After angular integration, this gives

$$\begin{aligned}\rho_z = \Gamma(a/b)\sum_{n=0}^\infty \frac{|n\rangle\langle n|}{{}_p\rho_q(n)}\int_0^\infty \mathrm{d}(|z|^2)\,P_{p,q}(|z|^2)\\ \times G^{q+1,0}_{p,q+1}(|z|^2\mid\cdots)(|z|^2)^n.\end{aligned} \tag{127}$$

As in Eq. (121), comparison with the canonical density operator yields

$$\begin{aligned}\int_0^\infty \mathrm{d}(|z|^2)\,P_{p,q}(|z|^2)G^{q+1,0}_{p,q+1}(|z|^2\mid\cdots)(|z|^2)^n\\ = \frac{e^{-\beta E_n}\,{}_p\rho_q(n)}{\Gamma(a/b)\,Z(\beta)}.\end{aligned} \tag{128}$$

This is a Stieltjes moment problem [20] for the unknown function $P_{p,q}(|z|^2)G^{q+1,0}_{p,q+1}(|z|^2\mid\cdots)$. The structure function ${}_p\rho_q(n)$ is expressed in terms of Euler gamma functions; see Eq. (36).

Solving the moment problem requires specifying the dependence of the energy eigenvalues on the quantum number, $E_n = f(n)$. Nonlinear spectra generally require additional techniques. For quadratic spectra, such as that of the Morse oscillator, specific procedures are available [21].

Consider a system with a linear energy spectrum, $E_n = \hbar\omega n + E_0$. In this case,

$$\begin{aligned}Z(\beta) &= \sum_{n=0}^\infty e^{-\beta E_n}\\ &= e^{-\beta E_0}\sum_{n=0}^\infty \left(e^{-\beta\hbar\omega}\right)^n\\ &= \frac{e^{-\beta E_0}}{1-e^{-\beta\hbar\omega}},\\ \frac{e^{-\beta E_n}}{Z(\beta)} &= \left(1-e^{-\beta\hbar\omega}\right)\left(e^{-\beta\hbar\omega}\right)^n.\end{aligned} \tag{129}$$

The moment equation therefore becomes

$$\begin{aligned}\int_0^\infty \mathrm{d}(|z|^2)\,P_{p,q}(|z|^2)G^{q+1,0}_{p,q+1}(|z|^2\mid\cdots)(|z|^2)^n\\ = \left(1-e^{-\beta\hbar\omega}\right)\left(e^{-\beta\hbar\omega}\right)^n\\ \times\Gamma(n+1)\frac{\prod_{j=1}^q\Gamma(b_j+n)}{\prod_{i=1}^p\Gamma(a_i+n)}.\end{aligned} \tag{130}$$

Setting $n = s-1$ gives the Mellin-transform expression. The corresponding solution can then be sought through the inverse Mellin transform [22]:

$$\begin{aligned}\int_0^\infty \mathrm{d}(|z|^2)\,P_{p,q}(|z|^2)G^{q+1,0}_{p,q+1}(|z|^2\mid\cdots)(|z|^2)^{s-1}\\ = \frac{e^{\beta\hbar\omega}-1}{(e^{\beta\hbar\omega})^s}\Gamma(s)\frac{\prod_{j=1}^q\Gamma(b_j-1+s)}{\prod_{i=1}^p\Gamma(a_i-1+s)}.\end{aligned} \tag{131}$$

The resulting P-quasiprobability distribution for a system with a linear energy spectrum is [23]

$$\begin{aligned}P_{p,q}(|z|^2) &= \left(e^{\beta\hbar\omega}-1\right)\\ &\times\frac{G^{q+1,0}_{p,q+1}\left(e^{\beta\hbar\omega}|z|^2\,\middle|\,\begin{matrix}-;\{a_i-1\}_{i=1}^p\\ 0,\{b_j-1\}_{j=1}^q;-\end{matrix}\right)}{G^{q+1,0}_{p,q+1}\left(|z|^2\,\middle|\,\begin{matrix}-;\{a_i-1\}_{i=1}^p\\ 0,\{b_j-1\}_{j=1}^q;-\end{matrix}\right)}\\ &\equiv\left(e^{\beta\hbar\omega}-1\right)\frac{G^{q+1,0}_{p,q+1}\left(e^{\beta\hbar\omega}|z|^2\mid\cdots\right)}{G^{q+1,0}_{p,q+1}(|z|^2\mid\cdots)}.\end{aligned} \tag{132}$$

An analogous expression is obtained for the variable σ.

The thermal expectation value of an observable $\mathcal{A}$ in the mixed state described by ρ is

$$\langle\mathcal{A}\rangle = \mathrm{Tr}(\rho\mathcal{A})u_0\,\mathrm{Tr}(\rho_z\mathcal{A}) + u_1\,\mathrm{Tr}(\rho_\sigma\mathcal{A}). \tag{133}$$

We now apply these results to the one-dimensional harmonic oscillator restricted to its first two energy levels:

$$\begin{aligned}E_n &= \hbar\omega\,e(n) = \hbar\omega\left(n+\frac{1}{2}\right),\\ E_0 &= \frac{1}{2}\hbar\omega,\qquad E_1 = \frac{3}{2}\hbar\omega.\end{aligned} \tag{134}$$

The dimensionless energy expression can be represented, for the structure-function construction, by

$$\begin{aligned}e(n) &= n\frac{\prod_{j=1}^q(b_j-1+n)}{\prod_{i=1}^p(a_i-1+n)},\\ p&=1,\qquad q=1,\qquad a_1=1,\qquad b_1=\frac{3}{2}.\end{aligned} \tag{135}$$

The corresponding structure function is

$$
\begin{aligned}
{}_p\rho_q(n) &= \Gamma(a/b)\,\Gamma(n+1)\frac{\prod_{j=1}^{q}\Gamma(b_j+n)}{\prod_{i=1}^{p}\Gamma(a_i+n)},\\
{}_1\rho_1(n) &= \frac{2}{\sqrt{\pi}}\,\Gamma\!\left(\frac{3}{2}+n\right),\\
{}_1\rho_1(0) &= 1, \qquad {}_1\rho_1(1) = \frac{3}{2}.
\end{aligned}
\tag{136}
$$

Starting from the coherent-state expansion in the variable Z, we write

$$
\begin{aligned}
|Z\rangle &= \frac{1}{\sqrt{{}_1F_1(|Z|^2)}}\sum_{n=0}^{1}\frac{Z^n}{\sqrt{{}_1\rho_1(n)}}|n\rangle,\\
{}_1F_1(|Z|^2) &\longrightarrow \sum_{n=0}^{1}\frac{(1)_n}{\left(\frac{3}{2}\right)_n}\frac{(|Z|^2)^n}{n!}\\
&= 1+\frac{2}{3}|Z|^2 \equiv \mathcal{N}_2(|Z|^2).
\end{aligned}
\tag{137}
$$

Here, the normalization is the first-degree polynomial $\mathcal{N}_2(|Z|^2)$ obtained by truncating the hypergeometric series. It is not the full confluent hypergeometric function.

The normalized two-level state is therefore

$$
\begin{aligned}
|Z\rangle &= \frac{1}{\sqrt{\mathcal{N}_2(|Z|^2)}}|0\rangle + \frac{Z}{\sqrt{{}_1\rho_1(1)}\sqrt{\mathcal{N}_2(|Z|^2)}}|1\rangle\\
&= \frac{1}{\sqrt{1+\frac{2}{3}|Z|^2}}\begin{pmatrix}1\\0\end{pmatrix} + \frac{\sqrt{\frac{2}{3}}\,Z}{\sqrt{1+\frac{2}{3}|Z|^2}}\begin{pmatrix}0\\1\end{pmatrix}.
\end{aligned}
\tag{138}
$$

The diagonal part of the coherent-state projector is

$$
\begin{aligned}
\int_0^{2\pi}\frac{\mathrm{d}\varphi_Z}{2\pi}\,|Z\rangle\langle Z| &= \frac{1}{1+\frac{2}{3}|Z|^2}u_0\\
&\quad + \frac{\frac{2}{3}|Z|^2}{1+\frac{2}{3}|Z|^2}u_1.
\end{aligned}
\tag{139}
$$

The density operator and the partition function for the thermal state of this two-level system are

$$
\begin{aligned}
\rho &= \sum_{n=0}^{1}\frac{e^{-\beta\hbar\omega(n+\frac{1}{2})}}{Z(\beta)}|n\rangle\langle n|\\
&= \frac{e^{-\frac{1}{2}\beta\hbar\omega}}{Z(\beta)}u_0 + \frac{e^{-\frac{3}{2}\beta\hbar\omega}}{Z(\beta)}u_1,\\
Z(\beta) &= e^{-\beta E_0}+e^{-\beta E_1}\\
&= e^{-\frac{1}{2}\beta\hbar\omega}+e^{-\frac{3}{2}\beta\hbar\omega}.
\end{aligned}
\tag{140}
$$

We now examine the integral representation of the density operator for this truncated system:

$$
\begin{aligned}
\rho &= \int \mathrm{d}\mu(Z)\,P_{1,1}(|Z|^2)|Z\rangle\langle Z|\\
&= \frac{1}{Z(\beta)}\sum_{n=0}^{1}e^{-\beta\hbar\omega(n+\frac{1}{2})}u_n.
\end{aligned}
\tag{141}
$$

Applying the formal expression in Eq. (132) gives

$$
\begin{aligned}
P_{1,1}(|Z|^2) &= \left(e^{\beta\hbar\omega}-1\right)\\
&\quad\times\frac{G_{1,2}^{2,0}\left(e^{\beta\hbar\omega}|Z|^2\,\middle|\,\begin{matrix}-;\,0\\0,\frac{1}{2};\,-\end{matrix}\right)}{G_{1,2}^{2,0}\left(|Z|^2\,\middle|\,\begin{matrix}-;\,0\\0,\frac{1}{2};\,-\end{matrix}\right)}\\
&= \left(e^{\beta\hbar\omega}-1\right)\frac{G_{0,1}^{1,0}\left(e^{\beta\hbar\omega}|Z|^2\,\middle|\,\frac{1}{2}\right)}{G_{0,1}^{1,0}\left(|Z|^2\,\middle|\,\frac{1}{2}\right)}\\
&= \left(e^{\beta\hbar\omega}-1\right)e^{\frac{1}{2}\beta\hbar\omega}e^{-(e^{\beta\hbar\omega}-1)|Z|^2}\\
&= \left(e^{\frac{3}{2}\beta\hbar\omega}-e^{\frac{1}{2}\beta\hbar\omega}\right)e^{-(e^{\beta\hbar\omega}-1)|Z|^2}.
\end{aligned}
\tag{142}
$$

Here, we used the index-reduction property of Meijer's G-function. According to Eq. (85), the measure is

$$
\begin{aligned}
\mathrm{d}\mu(Z) &= \frac{2}{\sqrt{\pi}}\frac{\mathrm{d}^2Z}{\pi}G_{0,1}^{1,0}\left(|Z|^2\,\middle|\,\frac{1}{2}\right){}_1F_1(|Z|^2)\\
&= \frac{2}{\sqrt{\pi}}\frac{\mathrm{d}^2Z}{\pi}e^{-|Z|^2}|Z|\,{}_1F_1(|Z|^2).
\end{aligned}
\tag{143}
$$

After angular integration, substitution of the coherent-state projector gives the coefficient comparison

$$
\begin{aligned}
\rho &= \sum_{n=0}^{1}\frac{u_n}{{}_1\rho_1(n)}\int \mathrm{d}\mu(Z)\,P_{1,1}(|Z|^2)\frac{(|Z|^2)^n}{\mathcal{N}_2(|Z|^2)}\\
&= \frac{1}{Z(\beta)}\sum_{n=0}^{1}e^{-\beta\hbar\omega(n+\frac{1}{2})}u_n.
\end{aligned}
\tag{144}
$$

The corresponding moment conditions are

$$
\begin{aligned}
&\int \mathrm{d}\mu(Z)\,P_{1,1}(|Z|^2)\frac{(|Z|^2)^n}{\mathcal{N}_2(|Z|^2)}\\
&\quad= \frac{e^{-\beta\hbar\omega(n+\frac{1}{2})}}{Z(\beta)}{}_1\rho_1(n)\\
&\quad= \frac{2}{\sqrt{\pi}}\frac{e^{-\frac{1}{2}\beta\hbar\omega}}{Z(\beta)}\frac{\Gamma\left(\frac{3}{2}+n\right)}{\left(e^{\beta\hbar\omega}\right)^n}, \qquad n=0,1.
\end{aligned}
\tag{145}
$$

Substituting this result into the density-operator expression gives

$$\begin{aligned}\rho &= \frac{2}{\sqrt{\pi}}\,\frac{e^{-\frac{1}{2}\beta\hbar\omega}}{Z(\beta)}\sum_{n=0}^{1}\frac{|n\rangle\langle n|}{{}_1\rho_1(n)}\\ &\quad\times\left(e^{-\beta\hbar\omega}\right)^n\Gamma\left(\frac{3}{2}+n\right)\\ &= \frac{2}{\sqrt{\pi}}\,\frac{e^{-\frac{1}{2}\beta\hbar\omega}}{Z(\beta)}\left[\Gamma\left(\frac{3}{2}\right)u_0+\frac{2}{3}e^{-\beta\hbar\omega}\Gamma\left(\frac{5}{2}\right)u_1\right]\\ &= \frac{1}{1+e^{-\beta\hbar\omega}}u_0+\frac{1}{1+e^{\beta\hbar\omega}}u_1\\ &= \frac{1}{2+e^{\beta\hbar\omega}+e^{-\beta\hbar\omega}}\begin{pmatrix}1+e^{\beta\hbar\omega} & 0\\ 0 & 1+e^{-\beta\hbar\omega}\end{pmatrix}.\end{aligned} \tag{146}$$

The trace of this density operator is $\operatorname{Tr}\rho = 1$, as required.

This calculation compares two representations of the density operator: its expansion over the orthogonal Fock-state projectors and its integral representation in terms of coherent-state projectors.

From another perspective, the density operator of a two-level system, here obtained by retaining two energy levels of a harmonic oscillator, can describe a qubit in thermal equilibrium with its environment at $T = (\beta k_{\mathrm{B}})^{-1}$. Such a system may be referred to as a thermal qubit.

6 Some Applications

Coherent states provide a connection between classical and quantum mechanics and help describe the classical limit of quantum systems, following Schrödinger's original motivation in 1926. Their applications include quantum optics, condensed-matter physics, chemistry, radar theory, and mathematical physics. They are also connected to quantum information and quantum computing through their relation to qubits [24–26].

Because coherent states can represent logical qubits in quantum computing [27], we examine how the coherent states with matrix arguments considered here relate to qubit states.

As shown previously, a coherent state with a complex label Z has the expansion

$$|Z\rangle = \frac{1}{\sqrt{{}_pF_q(|Z|^2)}}\sum_{n=0}^{n_{\max}}\frac{Z^n}{\sqrt{{}_p\rho_q(n)}}|n\rangle. \tag{147}$$

For a two-level system, obtained by retaining only the first two terms of the series and renormalizing, this becomes

$$\begin{aligned}|Z\rangle &= \frac{1}{\sqrt{\mathcal{N}_2(|Z|^2)}}\sum_{n=0}^{1}\frac{Z^n}{\sqrt{{}_p\rho_q(n)}}|n\rangle\\ &= \frac{1}{\sqrt{1+\frac{|Z|^2}{{}_p\rho_q(1)}}}|0\rangle+\frac{Z/\sqrt{{}_p\rho_q(1)}}{\sqrt{1+\frac{|Z|^2}{{}_p\rho_q(1)}}}|1\rangle,\end{aligned} \tag{148}$$

where $\mathcal{N}_2(|Z|^2) = 1+|Z|^2/{}_p\rho_q(1)$. where the two-level normalization is $\mathcal{N}_2(|Z|^2) = 1+|Z|^2/{}_p\rho_q(1)$, with

$${}_p\rho_q(1) = \frac{\prod_{j=1}^{q} b_j}{\prod_{i=1}^{p} a_i}. \tag{149}$$

The squared moduli of the coefficients of the ground state $|0\rangle$ and the excited state $|1\rangle$ sum to unity. Thus, the normalized two-level state has the form of a qubit.

Returning to matrix notation, we write

$$Z = zu_0+\sigma u_1 = \begin{pmatrix}z & 0\\ 0 & \sigma\end{pmatrix}. \tag{150}$$

The corresponding matrix-labelled state is

$$\begin{aligned}&\left|\begin{pmatrix}z & 0\\ 0 & \sigma\end{pmatrix}\right\rangle\\ &\quad=\left[I+\frac{1}{{}_p\rho_q(1)}\begin{pmatrix}|z|^2 & 0\\ 0 & |\sigma|^2\end{pmatrix}\right]^{-1/2}|0\rangle\\ &\qquad+\frac{1}{\sqrt{{}_p\rho_q(1)}}\begin{pmatrix}z & 0\\ 0 & \sigma\end{pmatrix}\\ &\qquad\times\left[I+\frac{1}{{}_p\rho_q(1)}\begin{pmatrix}|z|^2 & 0\\ 0 & |\sigma|^2\end{pmatrix}\right]^{-1/2}|1\rangle.\end{aligned} \tag{151}$$

This construction provides a matrix-labelled family of two-level states. Different physical implementations of qubits include photonic, thermal, superconducting, and spin systems. In the present mathematical setting, the matrix argument depends on two complex parameters.

The continuous range of these parameters produces infinitely many states. This does not, by itself, imply an infinite capacity for storing or retrieving classical information.

Another important concept in quantum information and quantum computing is the von Neumann entropy [28]. For a mixed state described by a density operator ρ, it is defined by

$$S = -\operatorname{Tr}(\rho\log\rho). \tag{152}$$

We now evaluate this expression when the density operator is a 2×2 diagonal matrix:

$$\begin{aligned}\rho &= \sum_{n=0}^{1} \frac{e^{-\beta E_n}}{Z(\beta)} |n\rangle\langle n| = \begin{pmatrix} \dfrac{e^{-\beta E_0}}{Z(\beta)} & 0 \\ 0 & \dfrac{e^{-\beta E_1}}{Z(\beta)} \end{pmatrix} \\ &= \begin{pmatrix} w_0 & 0 \\ 0 & w_1 \end{pmatrix} = w_0 u_0 + w_1 u_1, \\ w_n &= \frac{e^{-\beta E_n}}{Z(\beta)}, \qquad n = 0, 1, \\ Z(\beta) &= \sum_{n=0}^{1} e^{-\beta E_n} = e^{-\beta E_0} + e^{-\beta E_1}.\end{aligned} \tag{153}$$

Recall the trace formula for a product of operators:

$$\begin{aligned}\mathrm{Tr}(AB) &= \sum_n \langle n|AB|n\rangle \\ &= \sum_n \sum_m \langle n|A|m\rangle\langle m|B|n\rangle.\end{aligned} \tag{154}$$

For the diagonal density operator, $\langle n|f(\rho)|m\rangle = f(w_n)\delta_{nm}$. Using these relations, we obtain the entropy expression below.

$$\mathrm{Tr}(\rho \log \rho) = \sum_{n=0}^{1} \sum_{m=0}^{1} \langle n|\rho|m\rangle\langle m| \log \rho|n\rangle. \tag{155}$$

If $\|\rho - I\| < 1$, the logarithm can be computed using its power-series expansion; see, for example, Gradshteyn and Ryzhik [29]:

$$\log \rho = \log[I + (\rho - I)] = -\sum_{k=1}^{\infty} \frac{(-1)^k}{k} (\rho - I)^k. \tag{156}$$

Since ρ is diagonal in the Fock basis, its diagonal matrix elements satisfy

$$\begin{aligned}\langle n| \log \rho|n\rangle &= \langle n| \log[I + (\rho - I)] \, |n\rangle \\ &= -\sum_{k=1}^{\infty} \frac{(-1)^k}{k} \langle n|(\rho - I)^k|n\rangle \\ &= -\sum_{k=1}^{\infty} \frac{(-1)^k}{k} (w_n - 1)^k \\ &= \log w_n.\end{aligned} \tag{157}$$

The von Neumann entropy therefore becomes

$$\begin{aligned}S &= -\,\mathrm{Tr}(\rho \log \rho) \\ &= -\sum_{n=0}^{1} w_n \langle n| \log \rho|n\rangle \\ &= \sum_{n=0}^{1} w_n \sum_{k=1}^{\infty} \frac{(-1)^k}{k} (w_n - 1)^k \\ &= -\left(w_0 \log w_0 + w_1 \log w_1\right).\end{aligned} \tag{158}$$

Thus, the entropy of a mixed state of a two-level system is

$$S = -\left(w_0 \log w_0 + w_1 \log w_1\right). \tag{159}$$

Since $0 \le w_0, w_1 \le 1$ and $w_0 + w_1 = 1$, the entropy is nonnegative, with the convention $0 \log 0 = 0$. This expression coincides with the Shannon entropy of the eigenvalue distribution.

The same result follows directly from matrix calculus and the cyclic property of the trace, $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$; see the Appendix.

7 Concluding Remarks

The coherent states examined in this paper are labelled by a linear combination of two singular diagonal projectors with complex coefficients. This construction is related to coherent states with shifted arguments [13], while using the algebraic properties of the projectors.

The main simplification follows from the complementary projector relations $u_0 + u_1 = I$ and $u_0 u_1 = u_1 u_0 = 0$. For functions defined on the corresponding spectra, these relations give

$$\mathcal{F}(x u_0 + y u_1) = \mathcal{F}(x) u_0 + \mathcal{F}(y) u_1. \tag{160}$$

Moreover, expressions involving the complex variables z and σ can be written as

$$\mathcal{F}(z, \sigma) = u_0 \mathcal{P}_{z u_0} \mathcal{F}(z) + u_1 \mathcal{P}_{\sigma u_1} \mathcal{F}(\sigma). \tag{161}$$

Expressions involving their squared moduli take the form

$$\begin{aligned}\mathcal{F}(|z|^2, |\sigma|^2) = u_0 \big(\mathcal{P}_{z u_0}\big)^2 \mathcal{F}(|z|^2) \\ + u_1 \big(\mathcal{P}_{\sigma u_1}\big)^2 \mathcal{F}(|\sigma|^2).\end{aligned} \tag{162}$$

These expressions separate the contributions associated with the two complex variables. With the matrix normalization used here, the factors satisfy $\mathcal{P}_{z u_0} = u_0$ and $\mathcal{P}_{\sigma u_1} = u_1$, so the decomposition reduces to the two complementary projector components.

This structure allows the contributions of the two complex labels to be treated separately, unlike a shifted scalar argument that combines them into a single complex number. We considered the construction for pure states and examined its extension to thermal states described by a density operator. As applications, we discussed its relation to two-level states and the von Neumann entropy.

For a two-level system in thermal equilibrium with its environment, the density operator depends on the equilibrium temperature. The corresponding normalized two-level state provides a connection with the description of a thermal qubit.

The matrix-labelling construction gives a two-level scalar parametrization, but its interpretation as a physical qubit requires a specified encoding. Any physical implementation must also address isolation from the environment, resistance to noise and decoherence, controllability, and measurement.

Although the present construction uses two complex variables, the projector-based procedure can be extended to a finite number N of complex variables by replacing the 2×2 diagonal matrices with $N \times N$ diagonal matrices.

Appendix A: Von Neumann Entropy in Matrix Form

We derive the von Neumann entropy formula using matrix notation.

$$
\begin{aligned}
S &= -\operatorname{Tr}(\rho \log \rho) \\
&= -\sum_{n=0}^{1} \langle n|\rho \log \rho|n\rangle \\
&= -\sum_{n,m=0}^{1} \langle n|\rho|m\rangle\langle m|\log \rho|n\rangle \\
&= -\sum_{n=0}^{1} \left\langle n \left| \begin{pmatrix} w_0 & 0 \\ 0 & w_1 \end{pmatrix} \right| n \right\rangle \\
&\quad \times \left\langle n \left| \log \begin{pmatrix} w_0 & 0 \\ 0 & w_1 \end{pmatrix} \right| n \right\rangle \\
&= -\sum_{n=0}^{1} \left\langle n \left| \begin{pmatrix} w_0 & 0 \\ 0 & w_1 \end{pmatrix} \right| n \right\rangle \\
&\quad \times \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \left\langle n \left| \left[\begin{pmatrix} w_0 & 0 \\ 0 & w_1 \end{pmatrix} - I \right]^k \right| n \right\rangle .
\end{aligned}
\tag{A.1}
$$

The diagonal matrix elements are obtained from

$$
\begin{aligned}
\langle 0|\rho|0\rangle &= \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} w_0 & 0 \\ 0 & w_1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = w_0, \\
\langle 1|\rho|1\rangle &= \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} w_0 & 0 \\ 0 & w_1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = w_1.
\end{aligned}
\tag{A.2}
$$

Moreover,

$$
\begin{aligned}
(\rho - I)^k &= \left[\begin{pmatrix} w_0 & 0 \\ 0 & w_1 \end{pmatrix} - I \right]^k \\
&= \begin{pmatrix} w_0 - 1 & 0 \\ 0 & w_1 - 1 \end{pmatrix}^k \\
&= \begin{pmatrix} (w_0 - 1)^k & 0 \\ 0 & (w_1 - 1)^k \end{pmatrix} .
\end{aligned}
\tag{A.3}
$$

Consequently,

$$
\begin{aligned}
&\langle 0|(\rho - I)^k|0\rangle \\
&\quad = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} (w_0 - 1)^k & 0 \\ 0 & (w_1 - 1)^k \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&\quad = (w_0 - 1)^k, \\
&\langle 1|(\rho - I)^k|1\rangle \\
&\quad = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} (w_0 - 1)^k & 0 \\ 0 & (w_1 - 1)^k \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
&\quad = (w_1 - 1)^k .
\end{aligned}
\tag{A.4}
$$

For $0 < w_0, w_1 < 1$, the logarithmic series converge. The entropy therefore becomes

$$
\begin{aligned}
S &= -\langle 0|\rho|0\rangle \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \langle 0|(\rho - I)^k|0\rangle \\
&\quad - \langle 1|\rho|1\rangle \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \langle 1|(\rho - I)^k|1\rangle \\
&= -w_0 \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} (w_0 - 1)^k \\
&\quad - w_1 \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} (w_1 - 1)^k \\
&= -\left(w_0 \log w_0 + w_1 \log w_1 \right) .
\end{aligned}
\tag{A.5}
$$

The result extends continuously to zero eigenvalues using the convention $0 \log 0 = 0$. □

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