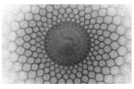
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Symmetry Reductions of Boundary Layer Equations of Second Grade Fluids

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Abstract Boundary layer flow of a second grade fluid is considered. Symmetry analysis has been applied to the equations. The arbitrary functions are the outer velocity appearing in the equations and the two functions in the boundary conditions. The specific forms of the outer velocity for which the principal Lie algebra extends is discussed first. For generalized boundary conditions with three arbitrary functions including the outer velocity, a further group classification is presented. The boundary conditions include classical no slip boundary layer flow, stretching/shrinking sheet flow, Falkner-Skan flow, flow with suction/injection, stagnation point flow and prescribed wall velocities. Group classification results are summarized in tables to guide the researchers. While the main aim in this study is to present a complete group classification for the outer velocity and the generalized boundary conditions, specific boundary conditions are also treated as an example. Flow due to stretching of the permeable plate with suction/injection is considered. With the aid of symmetries, the partial boundary value problem is reduced to an ordinary boundary value problem. It is shown that the reduced boundary value problem possesses exact solutions. While the impermeable wall and permeable wall with suction leads to one solution within the exponential family, the injection case leads to two solutions within the exponential family. This work can be used as a reference work in determining the existence of similarity transformations under various flow conditions for second grade fluids.

keywords: Non-Newtonian Fluids, Boundary Layer Equations, Symmetries of Differential Equations, Stretching Sheet Flow, Suction/Injection.

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1 Introduction

The treatment of differential equations in search of exact analytical solutions dates back to the 19th century, with the seminal work of Sophus Lie [1]. Especially for nonlinear equations, the symmetry method proposed by Lie remains the most unified and systematic approach. Many exact solutions obtained by ad hoc methods can be realized to be the consequences of the so-called symmetries of the differential equations. So far, the method has found applications in many mathematical physics problems classified in handbooks [2–4]. The theory is well established and discussed in many reference books [5–7]. To handle the involved algebra in the calculations, symbolic computational programs were written [4, 8]. Pakdemirli [9] presented the symmetry method for solutions of differential equations in a simplified form with minimal theory.

To cover many specific cases, researchers often introduce arbitrary parameters and/or functions into the mathematical models and/or boundary/initial conditions [9]. When the functions/parameters remain in their most general forms, this corresponds to the minimal symmetries, called the principal Lie algebra. The symmetries may extend for some specific forms of the functions/parameters. Determining the minimal symmetries and the higher algebras corresponding to specific forms is the task of group classification [9]. If both the equation and the conditions contain several arbitrary functions/parameters, then a group classification with respect to those in the equation must be performed first, followed by those in the conditions. The present work applies this joint classification approach to the second-grade boundary-layer model with an arbitrary outer velocity and prescribed wall velocities.

This study considers the boundary-layer flow of a second-grade fluid. Coleman and Noll [10] introduced the stress-constitutive relationship for such fluids. Dunn and Fosdick [11] derived the thermodynamic compatibility conditions for the model. Rajagopal et al. [12] discussed theoretical arguments for the existence of a boundary layer in second-grade fluids. Pakdemirli and Şuhubi [13] derived the boundary-layer equations for second-grade fluids using special curvilinear coordinates. They used streamlines and the velocity potentials orthogonal to them as the independent coordinates [14]. This coordinate system helps keep the equations independent of the objects immersed in the fluid. Pakdemirli and Şuhubi [15] calculated the symmetries using exterior differential forms, an alternative to Lie group theory, and presented a reduction for stagnation-point flow. The same curvilinear coordinate system has been employed for third-grade fluids [16], for a special third-grade fluid [17] and for power-law fluids [18].

One mathematical problem in treating second-grade fluids is the paucity of boundary conditions. For the model to be mathematically well posed for a two-dimensional steady-state flow, four boundary conditions are required, but the physical conditions provide only three. Beard and Walters [19] suggested a technique that uses regular perturbation expansions. Because the equation is singular, the regular expansion reduces the order of the equation, eliminating the need for the extra condition. The method is questionable in perturbation theory, where singular problems should be treated by constructing inner and outer expansions (regular solution) and matching them [20, 21]. Garg and Rajagopal [22] instead used physical reasoning for the extra condition and suggested that the solutions be bounded or smooth at infinity. In this work, we follow the second approach of Garg and Rajagopal [22]. Ariel [23] discussed the possibility of obtaining numerical solutions without considering the extra boundary condition at infinity.

Researchers have extensively studied various flow geometries and types to solve boundary-layer flow for second-grade fluid models and their modifications. Examples include stretching sheet flow, flow with suction/injection, stagnation point flow, wedge flow, magnetohydrodynamic (MHD) flow, and nanofluid flow. These models also incorporate heat transfer equations. Abbas et al. [24] investigated the MHD flow with velocity slip, thermal radiation and chemical reaction. Dey and Borah [25] treated the MHD nanofluid flow over a stretching sheet. Reddy et al. [26] investigated MHD flow of a second-grade nanofluid. Hosseinzadeh et al. [27] studied the MHD flow on a curved stretching surface. Nadeem et al. [28] investigated steady-state flow through a horizontal cylinder. Sajid et al. [29] also treated unsteady equations and heat transfer over an expanding boundary. The unsteady flow behaviour was also considered for time-dependent boundary conditions [30]. The moving-plate problem is another flow geometry discussed [31]. Vajravelu and Roper [32] provided an exact exponential solution for stretching-sheet motion with an impermeable wall. A closely related fluid model is the modified second-grade fluid, for which the flow due to plate expansion was analysed [33]. To incorporate shear-thinning and shear-thickening effects into second-grade fluids, power-law second-grade fluids were employed as another modified model [34]. The unsteady case of the model was also considered [35]. When the second-grade effects are eliminated, the model is a special third-grade fluid [36]. The same fluid model was used to investigate flow over a contracting boundary with a magnetic field and thermal radiation [37]. Multiple boundary layers for the second- and third-grade models were extensively treated [38].

One problem in treating stretching sheet flows is the uniqueness of solutions. A proof of uniqueness of solutions is given for stretching sheet motion of second-grade fluids with no suction or injection [39]. For the special non-Newtonian fluid model, Swain et al. [40] reported two solutions. Our results for the stretching sheet solutions with injection have also produced two distinct decaying solutions. However, in the no suction/injection and suction cases, we do not encounter two solutions, which we discuss in the relevant section.

The above-referenced papers all deal with special boundary value problems. In general, they use Lie group theory or its special transformations such as the scaling transformation to obtain an ordinary differential system. The resulting boundary value problems are then solved by exact analytical methods (if they exist), approximate analytical methods (perturbation theory, homotopy analysis method, Taylor series approximations, etc.) or numerical methods (Runge–Kutta with shooting, finite difference methods, etc.). Unlike previous literature, this work does not solve a specific boundary value problem; instead, it analyses the conditions for the success of similarity transformations. To achieve this task, we incorporate an arbitrary outer velocity into the equations and two arbitrary functions into the boundary conditions. The generalised conditions therefore cover a wide range of flow geometries, such as classical no-slip boundary layer flow, stretching sheet flow, stagnation point flow, wedge flow, prescribed wall velocities, suction/injection flow, etc. When all functions remain in their arbitrary forms, no symmetries survive, and constructing similarity solutions becomes impossible. However, for special forms

of the outer velocity function and boundary-condition functions, some symmetries survive, enabling similarity transformations. First, we present the group classification with respect to the outer velocity appearing in the equations, and we summarise the results in a table. Next, we outline the group classification with respect to boundary conditions that include the same outer velocity and summarise the results in tables. Using the given tables and the derived classification equations, we can systematically determine whether similarity solutions exist, rather than relying on trial-and-error methods.

Finally, a sample expanding sheet problem has been treated. The permeable sheet allows for suction or injection. The similarity transformation for the problem was determined. The reduced differential equation belongs to the class of stiff equations, which are hard to solve with initial-value methods. However, for the problem considered, exact analytical solutions exist and are presented. We find that only one solution exists within the exponential family for the cases of impermeable walls and permeable walls with suction. In the case of injection, however, two solutions within the exponential family exist, from which the upper-branch solutions are selected.

2 Boundary layer equations and symmetries

The stress-constitutive relation for an incompressible second-grade fluid is [10]

$$\mathbf{t} = -p\mathbf{I} + \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2, \tag{1}$$

where p is the spherical pressure, μ is the Newtonian viscosity, and α_1 and α_2 are the material constants for the second-grade fluid. The Rivlin–Ericksen tensors are defined as

$$\begin{aligned} \mathbf{A}_1 &= \mathbf{L} + \mathbf{L}^{\mathsf{T}}, \\ \mathbf{A}_2 &= \dot{\mathbf{A}}_1 + \mathbf{A}_1\mathbf{L} + \mathbf{L}^{\mathsf{T}}\mathbf{A}_1, \\ \mathbf{L} &= \operatorname{grad}\mathbf{v}, \end{aligned} \tag{2}$$

where $\mathbf{v}$ is the velocity vector and the dot denotes the total time derivative. For stable solutions that are compatible with thermodynamics, the viscosity and material constants obey [11]

$$\alpha_1 + \alpha_2 = 0, \qquad \mu \geq 0, \qquad \alpha_1 \geq 0. \tag{3}$$

The dimensionless steady-state two-dimensional boundary layer equations are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{4}$$

$$\begin{aligned} u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} &= U\frac{dU}{dx} + \frac{\partial^2 u}{\partial y^2} \\ &\quad + k_1\left(v\frac{\partial^3 u}{\partial y^3} + u\frac{\partial^3 u}{\partial x\,\partial y^2}\right. \\ &\quad \left. + \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial y^2} - \frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x\,\partial y}\right), \end{aligned} \tag{5}$$

where u is the x-component of velocity and v is the y-component inside the boundary layer, and $U(x)$ is the velocity distribution outside the boundary layer, where the flow is inviscid. The parameter k_1 is the dimensionless second-grade material constant. A systematic derivation of the equations using the method of matched asymptotic expansions for a special curvilinear coordinate system was presented earlier [13]. Take $u = W_\varphi$, $v = W_\psi$, $q_B = 1$, and $Q_B = 0$ in the equations to retrieve the above equations in Cartesian coordinates. Alternatively, $m = 0$ can be taken in the equations given by Aksoy et al. [33] and Pakdemirli et al. [34].

The associated conditions are taken fairly general to cover many different flow geometries, as already discussed in the introduction,

$$\begin{aligned} &u(x,0) = A(x), \qquad v(x,0) = B(x), \\ &u(x,\infty) = U(x), \qquad \frac{\partial u}{\partial y}(x,\infty) = 0. \end{aligned} \tag{6}$$

We apply symmetry analysis to Eqs. (4) and (5) [5, 9]. The group transformations are

$$x^* = x + \epsilon\xi_1(x,y,u,v), \tag{7}$$
$$y^* = y + \epsilon\xi_2(x,y,u,v), \tag{8}$$
$$u^* = u + \epsilon\eta^1(x,y,u,v), \tag{9}$$
$$v^* = v + \epsilon\eta^2(x,y,u,v), \tag{10}$$

where ϵ is the group transformation parameter. The generator for the transformation is

$$X = \xi_1\frac{\partial}{\partial x} + \xi_2\frac{\partial}{\partial y} + \eta^1\frac{\partial}{\partial u} + \eta^2\frac{\partial}{\partial v}. \tag{11}$$

The symmetry classification below is carried out for a fixed nonzero second-grade coefficient $k_1 > 0$. The Newtonian limit $k_1 = 0$ is used in Section 5 to check the explicit solution formulas; its enlarged symmetry algebra is outside the scope of the present classification.

Using the straightforward algebra outlined in Bluman and Kumei [5] and Pakdemirli [9], the infinitesimals are eventually found to be

$$\xi_1 = ax + b, \tag{12}$$
$$\xi_2 = c(x), \tag{13}$$
$$\eta^1 = au, \tag{14}$$
$$\eta^2 = c'(x)u. \tag{15}$$

The condition imposed on $U(x)$ is

$$(ax+b)\frac{d}{dx}(UU') - aUU' = 0. \tag{16}$$

The classification of the outer velocity in the equation is based on the above equation. In its most general case, the Lie algebra consists of two finite-parameter transformations and one infinite-parameter transformation $c(x)$. The base generators are

$$\begin{aligned} X_1 &= x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u}, \qquad X_2 = \frac{\partial}{\partial x}, \\ X_\infty &= c(x)\frac{\partial}{\partial y} + c'(x)u\frac{\partial}{\partial v}, \end{aligned} \tag{17}$$

where X_1 represents scalings in the x and u variables and X_2 represents translations in the x-coordinate. Results are compatible with the previous work [15, 33].

3 Group classification (equations)

In Eqs. (4) and (5), group classification is performed with respect to $U(x)$ based on Eq. (16). If the outer velocity is in its most general arbitrary form, then satisfying Eq. (16) requires $a = b = 0$, leaving only the infinite-parameter base symmetry

$$\begin{aligned} \xi_1 &= 0, \qquad \xi_2 = c(x), \\ \eta^1 &= 0, \qquad \eta^2 = c'(x)u. \end{aligned} \tag{18}$$

The above symmetries are the minimal base symmetries when the boundary conditions are disregarded.

To obtain the symmetry extensions, we solve the classifying relation. From Eq. (16), we have

$$\frac{d(UU')}{UU'} = \frac{a\,dx}{ax+b}, \tag{19}$$

with a solution

$$UU' = c_1(ax+b). \tag{20}$$

Upon separation,

$$U\,dU = c_1(ax+b)\,dx, \tag{21}$$

and integrating gives

$$U(x) = \sqrt{C(x+x_0)^2 + D}, \qquad C \neq 0. \tag{22}$$

For this specific form, the infinitesimals are given by Eqs. (12)–(15) with the condition $b = ax_0$. Other special cases follow from the classifying relation.

If $a = 0$, Eq. (16) gives $b\,\frac{d}{dx}(UU') = 0$. The solution for $b \neq 0$ is

$$U(x) = \sqrt{c_1 x + c_2}. \tag{23}$$

The symmetries for this case are

$$\begin{aligned} \xi_1 &= b, \qquad \xi_2 = c(x), \\ \eta^1 &= 0, \qquad \eta^2 = c'(x)u. \end{aligned} \tag{24}$$

If $U = U_0$ is constant or $U = 0$, Eq. (16) is identically satisfied, and the symmetries are in the general form given by Eqs. (12)–(15).

For the linear profile $U = U_0 x$ with $U_0 \neq 0$, Eq. (16) gives $bU_0^2 = 0$. Thus $b = 0$, and the remaining symmetries are

$$\begin{aligned} \xi_1 &= ax, \qquad \xi_2 = c(x), \\ \eta^1 &= au, \qquad \eta^2 = c'(x)u. \end{aligned} \tag{25}$$

If $b = 0$ and $a \neq 0$, Eq. (16) reduces to $x\,\frac{d}{dx}(UU') - UU' = 0$, with a solution

$$U(x) = \sqrt{c_1 x^2 + c_2}. \tag{26}$$

The symmetries for this case are again given by Eq. (25).

Finally, for $U = K(x + x_0)$ with $K \neq 0$, Eq. (16) is satisfied when $b = ax_0$. Table 1 summarises the results.

4 Group classification (boundary conditions)

The conditions are selected in a fairly general form,

$$\begin{aligned} u(x,0) &= A(x), \qquad v(x,0) = B(x), \\ u(x,\infty) &= U(x), \qquad \frac{\partial u}{\partial y}(x,\infty) = 0. \end{aligned} \tag{27}$$

The general form of the conditions includes no-slip flow, stretching-sheet flow, suction or injection, parallel-plate flow, wedge flow, and prescribed wall velocities. For nonlinear partial differential systems, both the equations and the conditions must admit the corresponding Lie group of transformations [5, 9]. Applying the generator obtained from the equations to the boundary conditions may reduce or eliminate the symmetries. The arbitrary functions in the conditions allow us to identify the specific forms for which symmetries survive. For linear partial differential systems, similarity solutions may exist even when the conditions are not invariant [5]. We apply the generator to the finite boundaries and to the boundary conditions.

i) Boundary $y = 0$.

Applying the generator X in Eq. (11) to the boundary $y = 0$ gives

$$\begin{aligned} &X(y)|_{y=0} = 0 \\ &\quad \Longrightarrow \quad \xi_2|_{y=0} = 0 \quad \Longrightarrow \quad c(x) = 0. \end{aligned} \tag{28}$$

Table 1 Group classification (equations).

Outer velocity	Infinitesimals	Parameter restrictions
$U(x)$ arbitrary	$\xi_1 = 0,\ \xi_2 = c(x),\ \eta^1 = 0,\ \eta^2 = c'(x)u$	$a = b = 0$
$U = 0$ or $U = U_0$	$\xi_1 = ax + b,\ \xi_2 = c(x),\ \eta^1 = au,\ \eta^2 = c'(x)u$	No restrictions
$U = \sqrt{c_1 x + c_2}$	$\xi_1 = b,\ \xi_2 = c(x),\ \eta^1 = 0,\ \eta^2 = c'(x)u$	$a = 0$
$U = \sqrt{c_1 x^2 + c_2}$	$\xi_1 = ax,\ \xi_2 = c(x),\ \eta^1 = au,\ \eta^2 = c'(x)u$	$b = 0$
$U = \sqrt{C(x + x_0)^2 + D}$		
$C \neq 0$	$\xi_1 = ax + b,\ \xi_2 = c(x),\ \eta^1 = au,\ \eta^2 = c'(x)u$	$b = ax_0$
$U = K(x + x_0)$		
$K \neq 0$	$\xi_1 = ax + b,\ \xi_2 = c(x),\ \eta^1 = au,\ \eta^2 = c'(x)u$	$b = ax_0$

Therefore, when conditions are imposed at the boundary $y = 0$, the infinite-parameter Lie group of transformations cannot be used for similarity reductions. The boundary at infinity, $y = \infty$, imposes no additional restriction on the generator [5].

ii) Condition $u(x, 0) = A(x)$.

Applying the generator to the first boundary condition gives

$$\begin{aligned} &X\big(u - A(x)\big)\big|_{y=0,\, u=A(x)} = 0 \\ &\quad\Longrightarrow\quad \eta^1\big|_{y=0,\, u=A(x)} = \xi_1|_{y=0}\, A'(x). \end{aligned} \tag{29}$$

Substituting Eqs. (12) and (14) gives

$$aA = (ax + b)A'. \tag{30}$$

If $A(x)$ is arbitrary, Eq. (30) requires $a = b = 0$, and no symmetries survive. If $A = 0$, the classifying relation is identically satisfied and imposes no further restriction. For a nonzero constant $A = A_0$, the relation requires $a = 0$, leaving only the translational symmetry associated with b. For $A = A_0 x$ with $A_0 \neq 0$, as in the stretching-sheet problem, the relation requires $b = 0$, leaving only the scaling symmetry associated with a. More generally, a nonconstant affine profile $A = K_A(x + x_A)$ satisfies the relation when $b = ax_A$. Table 2 summarises the results.

iii) Condition $v(x, 0) = B(x)$.

Applying the generator to the second boundary condition gives

$$\begin{aligned} &X\big(v - B(x)\big)\big|_{y=0,\, u=A(x),\, v=B(x)} = 0 \\ &\quad\Longrightarrow\quad \eta^2\big|_{y=0,\, u=A(x),\, v=B(x)} = \xi_1|_{y=0}\, B'(x). \end{aligned} \tag{31}$$

Since $c(x) = 0$ by Eq. (28), we have $\eta^2 = 0$. The classifying relation is therefore

$$0 = (ax + b)B'. \tag{32}$$

If $B(x)$ is arbitrary, Eq. (32) requires $a = b = 0$, and no nontrivial symmetry survives. If $B = 0$ or $B = B_0$ is constant, the relation is identically satisfied. Thus an impermeable wall or a wall with constant suction or injection imposes no further restriction on the symmetries. Table 3 summarises the results.

iv) Condition $u(x, \infty) = U(x)$.

Applying the generator to the third boundary condition gives

$$\begin{aligned} &X\big(u - U(x)\big)\big|_{u=U(x)} = 0 \\ &\quad\Longrightarrow\quad \eta^1\big|_{u=U(x)} = \xi_1 U'(x). \end{aligned} \tag{33}$$

Substituting Eqs. (12) and (14) yields

$$aU = (ax + b)U'. \tag{34}$$

The outer velocity must satisfy the classifying relations from both the boundary layer equations and the third boundary condition, Eqs. (16) and (34). Table 4 gives the restrictions imposed by the third boundary condition.

v) Condition $u_y(x, \infty) = 0$.

Applying the first prolongation of the generator to the fourth boundary condition gives

$$X^{(1)}(u_y)\Big|_{u_y=0} = 0 \Longrightarrow\quad \eta_2^1\big|_{u_y=0} = 0. \tag{35}$$

The prolonged infinitesimal is obtained from the recursion formulas [5, 9]. For the symmetries considered here, it is

$$\eta_2^1 = au_y. \tag{36}$$

Since η_2^1 vanishes at $u_y = 0$, Eq. (35) imposes no further restriction.

Table 5 collects the classifying relations.

For a specific boundary value problem governed by Eqs. (4)–(6), the admissible similarity transformations can be determined by combining the restrictions in Tables 1–4 or by solving the classifying relations in Table 5 jointly. The constants defining each prescribed profile are held fixed when determining the admitted symmetry algebra. All classifying relations must be satisfied

Table 2 Group classification (first boundary condition).

First boundary condition	Infinitesimals	Parameter restrictions
$A(x)$ arbitrary	$\xi_1 = 0,\ \xi_2 = 0,\ \eta^1 = 0,\ \eta^2 = 0$	$a = b = c(x) = 0$
$A = 0$	$\xi_1 = ax + b,\ \xi_2 = 0,\ \eta^1 = au,\ \eta^2 = 0$	$c(x) = 0$
$A = A_0,\ A_0 \neq 0$	$\xi_1 = b,\ \xi_2 = 0,\ \eta^1 = 0,\ \eta^2 = 0$	$a = c(x) = 0$
$A = A_0 x,\ A_0 \neq 0$	$\xi_1 = ax,\ \xi_2 = 0,\ \eta^1 = au,\ \eta^2 = 0$	$b = c(x) = 0$
$A = K_A(x + x_A)$		
$K_A \neq 0$	$\xi_1 = ax + b,\ \xi_2 = 0,\ \eta^1 = au,\ \eta^2 = 0$	$c(x) = 0,\ b = ax_A$

Table 3 Group classification (second boundary condition).

Second boundary condition	Infinitesimals	Parameter restrictions
$B(x)$ arbitrary	$\xi_1 = 0,\ \xi_2 = 0,\ \eta^1 = 0,\ \eta^2 = 0$	$a = b = c(x) = 0$
$B = 0$	$\xi_1 = ax + b,\ \xi_2 = 0,\ \eta^1 = au,\ \eta^2 = 0$	$c(x) = 0$
$B = B_0$	$\xi_1 = ax + b,\ \xi_2 = 0,\ \eta^1 = au,\ \eta^2 = 0$	$c(x) = 0$

Table 4 Group classification (third boundary condition).

Third boundary condition	Infinitesimals	Parameter restrictions
$U(x)$ arbitrary	$\xi_1 = 0,\ \xi_2 = 0,\ \eta^1 = 0,\ \eta^2 = 0$	$a = b = c(x) = 0$
$U = 0$	$\xi_1 = ax + b,\ \xi_2 = 0,\ \eta^1 = au,\ \eta^2 = 0$	$c(x) = 0$
$U = U_0,\ U_0 \neq 0$	$\xi_1 = b,\ \xi_2 = 0,\ \eta^1 = 0,\ \eta^2 = 0$	$a = c(x) = 0$
$U = U_0 x,\ U_0 \neq 0$	$\xi_1 = ax,\ \xi_2 = 0,\ \eta^1 = au,\ \eta^2 = 0$	$b = c(x) = 0$
$U = K_U(x + x_U)$		
$K_U \neq 0$	$\xi_1 = ax + b,\ \xi_2 = 0,\ \eta^1 = au,\ \eta^2 = 0$	$c(x) = 0,\ b = ax_U$

Table 5 Classifying relations.

Origin	Relation
Boundary layer equations	$(ax + b)\frac{d}{dx}(UU') - aUU' = 0$
First boundary condition	$aA = (ax + b)A'$
Second boundary condition	$0 = (ax + b)B'$
Third boundary condition	$aU = (ax + b)U'$

by the same generator. In particular, nonconstant affine wall and outer velocities admit a common scaling about a shifted origin only when their shifts coincide.

5 Stretching sheet with suction/injection

As a sample boundary value problem, we consider a stretching sheet with constant suction or injection. The fluid moves with the stretching sheet, and the outer velocity at infinity is zero. The boundary conditions are

$$\begin{aligned} &u(x,0) = A_0 x, \qquad v(x,0) = B_0, \\ &u(x,\infty) = 0, \qquad \frac{\partial u}{\partial y}(x,\infty) = 0. \end{aligned} \tag{37}$$

We assume $A_0 > 0$ for the stretching sheet. For $U = 0$, $A = A_0 x$, and $B = B_0$, the restrictions in Tables 1–4, or equivalently the relations in Table 5, give $b = c(x) = 0$. The infinitesimals are

$$\begin{aligned} &\xi_1 = ax, \qquad \xi_2 = 0, \\ &\eta^1 = au, \qquad \eta^2 = 0. \end{aligned} \tag{38}$$

For $a \neq 0$, the characteristic equations for the similarity transformation, after rescaling the group parameter, are

$$\begin{aligned} &\frac{dx}{ds} = x, \qquad \frac{dy}{ds} = 0, \\ &\frac{du}{ds} = u, \qquad \frac{dv}{ds} = 0. \end{aligned} \tag{39}$$

Since y is invariant, the independent similarity variable is

$$\xi = y. \tag{40}$$

The first and third characteristic equations give u/x as an invariant where $x \neq 0$. Thus the velocity in the x-direction is

$$u = xf(\xi). \tag{41}$$

Similarly, v is invariant, so the velocity in the y-direction is

$$v = g(\xi). \tag{42}$$

Substituting Eqs. (40)–(42) into Eqs. (4) and (5) gives the ordinary differential system

$$f + g' = 0, \tag{43}$$

$$f^2 + gf' = f'' + k_1\big(gf''' + 2ff'' - (f')^2\big). \tag{44}$$

The reduced boundary conditions are

$$\begin{aligned} f(0) &= A_0, \qquad g(0) = B_0, \\ f(\infty) &= 0, \qquad f'(\infty) = 0. \end{aligned} \tag{45}$$

The reduced equations are stiff, which makes Runge–Kutta shooting difficult. Shooting requires choosing two initial values so that the conditions at infinity are satisfied. Finite-difference methods are often used for similar problems. Here, the following exponential ansatz satisfies the boundary conditions and yields exact solutions for suitable values of μ and B_1:

$$\begin{aligned} f(\xi) &= A_0 e^{-\mu\xi}, \\ g(\xi) &= B_0 + B_1\big(1 - e^{-\mu\xi}\big), \qquad \mu > 0. \end{aligned} \tag{46}$$

Substitution into Eq. (43) gives

$$B_1 = -\frac{A_0}{\mu}. \tag{47}$$

Using this relation in Eq. (44) yields

$$p(\mu) = k_1 B_0 \mu^3 - (1 + k_1 A_0)\mu^2 - B_0\mu + A_0 = 0. \tag{48}$$

Only positive real roots give decaying exponential profiles that satisfy the conditions at infinity.

For a Newtonian fluid, $k_1 = 0$, and Eq. (48) has a single positive root,

$$\mu = \frac{-B_0 + \sqrt{B_0^2 + 4A_0}}{2} \qquad \text{(Newtonian)}. \tag{49}$$

In the absence of suction or injection, $B_0 = 0$, and the single positive root is

$$\mu = \sqrt{\frac{A_0}{1 + k_1 A_0}} \qquad \text{(no suction/injection)}. \tag{50}$$

For $A_0 > 0$ and $k_1 > 0$, Eq. (48) has exactly one positive root when $B_0 \le 0$ and exactly two positive roots when $B_0 > 0$. Thus the exponential family (46) contains one decaying solution for an impermeable wall or for suction, and two distinct decaying solutions for injection. For $k_1 = 0$, there is one positive root for every B_0. These counts apply to the exponential family.

Table 6 Numerical results for the parameter μ.

A_0	B_0	k_1	μ	Number of positive roots
1	1	1	0.5550, 2.2470	Two
1.5	1	1	0.6728, 2.6640	Two
2	1	1	0.7459, 3.1149	Two
1	−1	1	0.8019	One
1	0	1	0.7071	One
1	1	0	0.6180	One
1	1	2	0.5000, 1.6180	Two

For suction, Descartes' rule of signs gives exactly one positive root. The case $B_0 = 0$ follows from Eq. (50). For injection, $p(0) = A_0 > 0$, $p\big(k_1^{-1/2}\big) = -\frac{1}{k_1} < 0$, and $\lim_{\mu\to+\infty} p(\mu) = +\infty$. There is therefore one positive root on each side of $k_1^{-1/2}$, and Descartes' rule of signs excludes additional positive roots.

The roots of Eq. (48) are determined using the following root-finding algorithm with fourth-order convergence [41].

$$\begin{aligned} \mu_{n+1} = \mu_n &- \frac{p(\mu_n)}{p'(\mu_n)} - \frac{p(\mu_n)^2 p''(\mu_n)}{2[p'(\mu_n)]^3} \\ &+ p(\mu_n)^3 \frac{p'(\mu_n)p'''(\mu_n) - 3[p''(\mu_n)]^2}{6[p'(\mu_n)]^5}. \end{aligned} \tag{51}$$

Table 6 lists the positive roots of the cubic polynomial used to generate the figures.

Returning to the original variables, the analytical solutions are

$$\begin{aligned} u &= xf(y) = A_0 x e^{-\mu y}, \\ v &= g(y) = B_0 - \frac{A_0}{\mu}\big(1 - e^{-\mu y}\big), \qquad \mu > 0. \end{aligned} \tag{52}$$

Figures 1 and 2 show the profiles f and g, associated with the x- and y-components of velocity, respectively, for various stretching rates. As the stretching rate increases, the tangential velocity increases near the wall (Fig. 1). Farther from the wall, the profiles cross in a narrow region and then approach the zero outer velocity. The normal velocity decreases as the stretching rate increases (Fig. 2).

For $B_0 = 1$, each stretching rate shown admits two decaying exponential solutions, as listed in Table 6. The figures show the branch associated with the smaller positive root of Eq. (48), which gives slower decay of the tangential velocity. The temporal stability of the two branches remains undetermined.

Figures 3 and 4 show the effect of suction and injection. The values $B_0 = -1$, $B_0 = 1$, and $B_0 = 0$ correspond to suction, injection, and an impermeable wall, respectively.

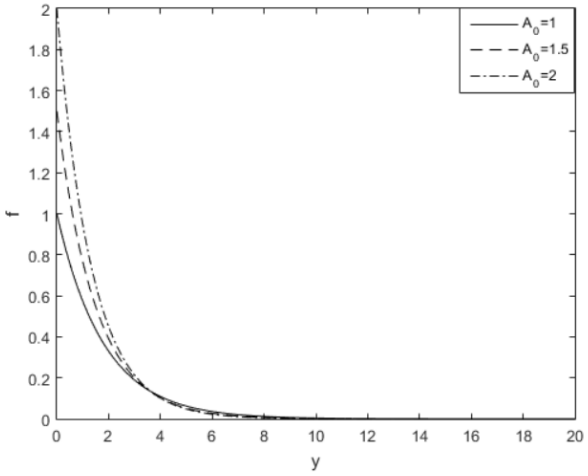


Fig. 1 Profiles of f for various stretching-sheet rates A_0, with $B_0 = 1$ and $k_1 = 1$.

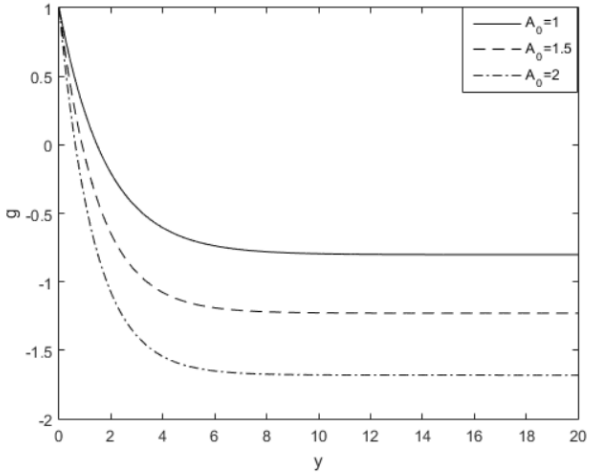


Fig. 2 Profiles of g for various stretching-sheet rates A_0, with $B_0 = 1$ and $k_1 = 1$.

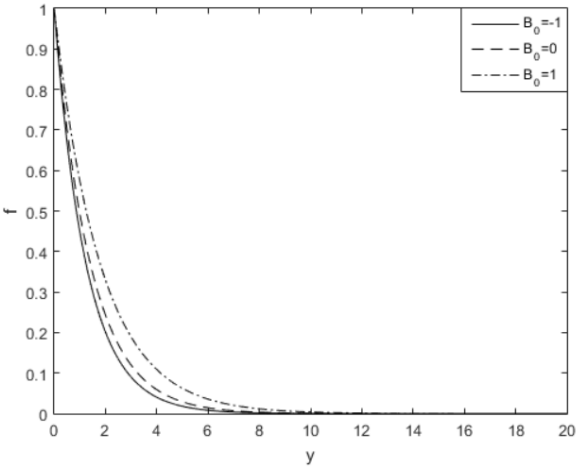


Fig. 3 Profiles of f for various suction and injection velocities B_0, with $A_0 = 1$ and $k_1 = 1$.

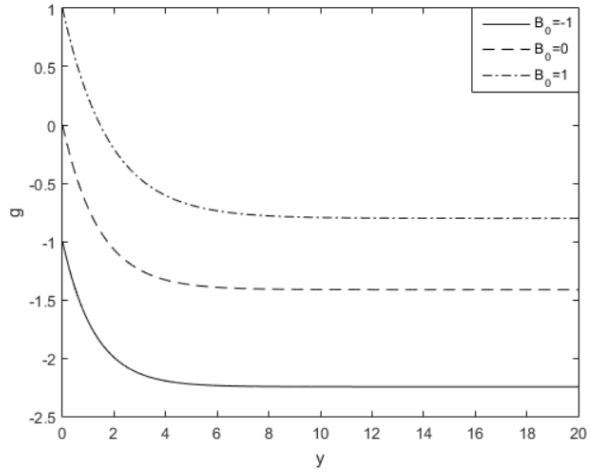


Fig. 4 Profiles of g for various suction and injection velocities B_0, with $A_0 = 1$ and $k_1 = 1$.

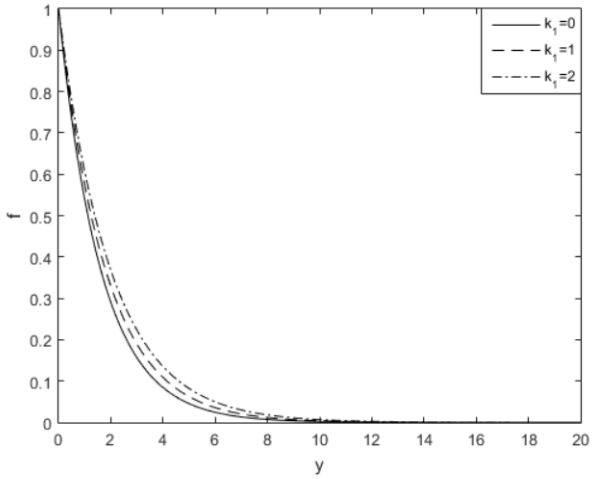


Fig. 5 Profiles of f for various second-grade coefficients k_1, with $A_0 = 1$ and $B_0 = 1$.

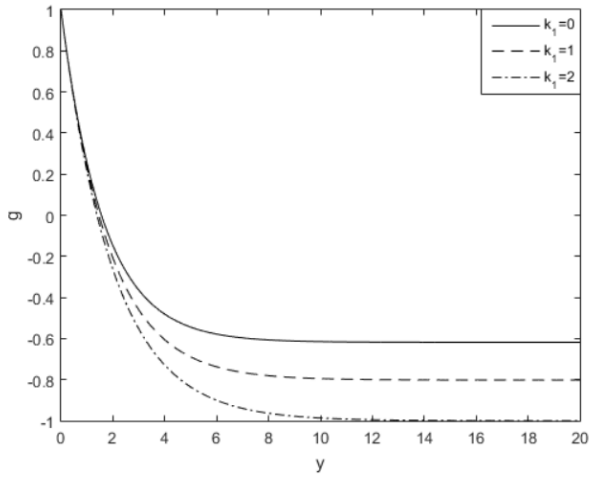


Fig. 6 Profiles of g for various second-grade coefficients k_1, with $A_0 = 1$ and $B_0 = 1$.

As the wall condition changes from suction to injection, the tangential velocity increases slightly within the boundary layer (Fig. 3). The normal velocity approaches a negative value far from the wall, even in the injection case (Fig. 4). Among the cases shown, only injection admits two decaying exponential solutions. The profiles shown for injection correspond to the smaller positive root μ of Eq. (48).

Figures 5 and 6 show the effect of the dimensionless second-grade coefficient k_1. The value $k_1 = 0$ corresponds to the Newtonian limit. For the second-grade fluid, we consider positive values of k_1, consistent with the thermodynamic compatibility conditions in Eq. (3).

As k_1 increases, the tangential velocity decays more slowly and the boundary layer thickens (Fig. 5). The normal velocity decreases with increasing k_1 away from the wall (Fig. 6).

6 Concluding remarks

This study considers the flow of a second-grade fluid. The boundary layer restrictions are considered. Fairly general boundary conditions are used to cover a range of flow conditions. We use a symmetry group approach for both the equations and the boundary conditions. This work systematically classifies, for the present second-grade boundary-layer model, the outer-velocity and prescribed wall-velocity profiles compatible with Lie point symmetry reductions. The tables systematically outline the conditions under which similarity-type solutions exist. Using these classifications, we treat the stretching sheet problem with suction/injection. The scaling symmetry reduces the original system to an ordinary differential system. We present exact analytical solutions for the reduced system. We discuss the effects of stretching-sheet rates, suction or injection, and the non-Newtonian coefficient. We find that the boundary value problem has two solutions within the exponential family for injection. For suction or no suction/injection, however, there is only one solution within the exponential family. This study focuses on the classification problem, which reveals the existence of similarity transformations for special forms of the outer velocity and boundary conditions. Detailed analysis of specific boundary value problems and physical interpretations of the results is beyond the scope and left for further research.

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