Table of Contents

TTMP vol 1(4), 2024
doi: 10.30511/TTMP.2024.2037384.1037

Transactions in Theoretical and Mathematical Physics

Explorations on elementary mathematics and physics: Some relations on Bessel functions

Borzoo Nazari[1]

[1]Department of Geodesy, College of Engineering, University of Tehran, Tehran, Iran

Received: 02 August 2024 / Accepted: 27 September 2024 / Published: 04 October 2024

Abstract Relations and identities on special functions are traditionally emerging from investigations on mathematical analysis. However, some special functions such as Bessel appear in various physical problems, a way in which some of their properties can be analyzed. There are plenty of identities on special functions which have been derived in such a way, especially some elementary problems in basic physics that play interesting roles in this. A uniformly charged electrostatic disk is such a problem that it appears unexpectedly in some cases. We study it more here and find some relations on Bessel functions.

1 Introduction

Special functions have an important place in all investigations of applied physics and mathematics. Topological properties, zeros, convergence, asymptotic expansions and many other properties are still open problems of some of these functions. Identities on special functions have also been studied a lot [1, 2]. Naturally, investigations of specific functions are performed in mathematical analysis. However, historically, many mathematical theories have arisen in theoretical physics. Today, in terms of formalism, it is difficult to distinguish between theoretical physics and some branches of new mathematics. For this reason, specific investigations in theoretical physics have become an approach to discovering and creating some mathematical theories, and the scientific community has noticed such closeness more than before. If we want to find simple examples to express such a relationship, we can find many examples of a genetic relationship between theoretical physics and today's mathematics even in preliminary problems. There are many elementary examples and the author presented an example of it in the previous conferences [3]. In this article, following our results in recent conferences [3, 4], we report the main results presented there on the problems of the electric potential of uniformly charged rings and discs. The idea is that we solve the problem with two methods, one of which is inspired by physics (direct method) and the other is the standard method of solving a partial differential equation. By equating the answers in two methods, we reach interesting identities. This operation can show us what mathematical physics is practically.

[1]e-mail: borzoo.nazari@ut.ac.ir

2 Uniformly charged ring

In this section, we first find the electric potential of the ring using the direct method of integration over elements. In subsection B, we show that the Method of Separation of variables (MSV) is also capable of giving the correct answer.

2.1 Direct method

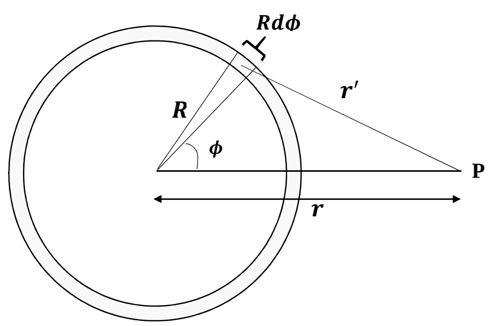

Figure 1: line element to find electric potential at the plane of a uniformly charged ring

Here we first calculate the electric potential of the uniformly charged ring at a typical point P with the aid of direct integration over elements. The point P might be inside or outside of the ring. Our setting is according to Fig.1. Thus, we see that

$$dV = \frac{kdq}{r'} = k\frac{\lambda_0 R d\phi}{r'}, \tag{1}$$

$$r' = (R^2 + r^2 - 2Rr\cos\phi)^{\frac{1}{2}}, \tag{2}$$

where λ_0 is charge density and $k = \frac{1}{4\pi\varepsilon_0}$.

2.1.1 The case $t = \frac{R}{r} < 1$

In this case, the point P is outside the ring and it founds that

$$\begin{aligned} V_{out} &= \frac{\lambda_0 R}{4\pi\varepsilon_0}\int_0^{2\pi}\frac{d\phi}{(R^2+r^2-2Rr\cos\phi)^{\frac{1}{2}}} \\ &= \frac{\lambda_0}{4\pi\varepsilon_0}\int_0^{2\pi}\frac{td\phi}{(t^2+1-2t\cos\phi)^{\frac{1}{2}}} \\ &= \frac{\lambda_0}{4\pi\varepsilon_0}\int_0^{2\pi} t\sum_{n=0}^{\infty} t^n P_n(\cos\phi)d\phi \\ &= \frac{\lambda_0}{4\pi\varepsilon_0}\sum_{n=0}^{\infty} t^{n+1}\int_0^{2\pi} P_n(\cos\phi)d\phi, \end{aligned} \tag{3}$$

where we have used the following generating function for Legendre polynomials

$$\frac{1}{(1+t^2-2t\cos\phi)^{\frac{1}{2}}} = \sum_{n=0}^{\infty} t^n P_n(\cos\phi)d\phi. \tag{4}$$

Using the identity (7.221(3) in [2])

$$\int_0^{2\pi} P_{2m}(cos\phi)d\phi = 2\pi\binom{-\frac{1}{2}}{m}^2, \tag{5}$$

the potential is found to be

$$V_{out}(r) = \frac{\lambda_0}{2\varepsilon_0}\sum_{m=0}^{\infty}\binom{-\frac{1}{2}}{m}^2 (\frac{R}{r})^{2m+1}. \tag{6}$$

Note that

$$\binom{x}{m} = \frac{x(x-1)...(x-(m-1))}{m!}, \qquad x\in\mathbb{R}. \tag{7}$$

Therefore

$$\begin{aligned} \binom{-\frac{1}{2}}{m} &= \frac{-\frac{1}{2}\cdot-\frac{3}{2}\cdot\dots\cdot(-\frac{1}{2}-m+1)}{m!} \\ &= (-1)^m\frac{1}{2^m}\frac{1\cdot 3\cdot 5\cdot\dots\cdot(2m-1)}{m!} \\ &= \frac{(-1)^m}{2^m m!}\cdot\frac{1\cdot 2\cdot 3\cdot 4\cdot\dots\cdot(2m)}{2\cdot 4\cdot\dots\cdot 2m} \\ &= \frac{(-1)^m(2m)!}{2^{2m}(m!)^2}. \end{aligned} \tag{8}$$

2.1.2 The case $r < R$

The process as previous section shows that

$$\begin{aligned} V_{in} &= \frac{\lambda_0 R}{4\pi\varepsilon_0}\int_0^{2\pi}\frac{d\phi}{R\left(1+\frac{r^2}{R^2}-2\frac{r}{R}\cos\phi\right)^{\frac{1}{2}}} \\ &= \frac{\lambda_0}{4\pi\varepsilon_0}\int_0^{2\pi} d\phi\sum_{n=0}^{\infty}\left(\frac{r}{R}\right)^n P_n(\cos\phi) \\ &= \frac{\lambda_0}{2\varepsilon_0}\sum_{m=0}^{\infty}\binom{-\frac{1}{2}}{m}^2\left(\frac{r}{R}\right)^{2m}. \end{aligned} \tag{9}$$

where we have used the relation [2]

$$0 = \int_0^{2\pi} P_{2m+1}(\cos\phi)d\phi, \tag{10}$$

and Eq. (5) again.

2.2 Method of Separation of variables(MSV)

We saw in section II that the MSV did not give the correct answer for the electric potential in the plan of a *uniformly* charged ring. Now we show that the MSV gives the right answer. The idea consists of finding a solution to the three dimensional Laplace equation using the MSV and reducing the final answer to two dimensions by putting $\theta \to \pi/2$. Our setting is according to Fig. 2.

2.2.1 The case $r > R$

Using the MSV, it can be shown [?] that the 3-dimensional Laplace equation ends up with the following solution in the region $r > R$ for the ring depicted in Fig. 2.

$$V_{out}(r,\theta,\phi) = \sum_{m=0}^{\infty} A_m r^{-m-1} P_m(\cos\theta). \tag{11}$$

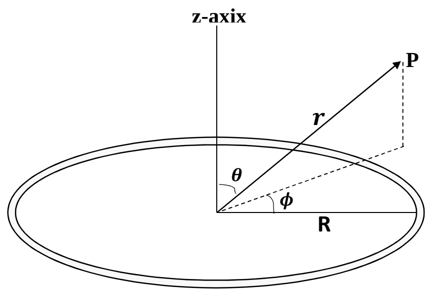

Figure 2: finding electric potential of a uniformly charged disk in a three dimensional coordinate system

Note that we have azimuthal symmetry around $z-axis$ hence (11) is independent of ϕ. Therefor, at a point on the $z-axis$, i.e. $\theta = 0 \; or \; r = z$, we have

$$V_{out}(r,0) = \sum_{m=0}^{\infty} A_m r^{-m-1} P_m(0)$$
$$\sum_{m=0}^{\infty} A_m z^{-m-1} \equiv V_{out}(z). \tag{12}$$

in which we have used $P_m(0) = 1$. On the other hand, we see from Figure 2 that the electric potential at a some point on the $z-axis$ is simply

$$V_{out}(z) = k\frac{Q}{(z^2+R^2)^{\frac{1}{2}}} = \frac{1}{4\pi\varepsilon_0}\frac{2\pi R\lambda_0}{(z^2+R^2)^{\frac{1}{2}}}$$
$$= \frac{\lambda_0 R}{2\varepsilon_0}\frac{1}{z}\left(1+\left(\frac{R}{z}\right)^2\right)^{-\frac{1}{2}}. \tag{13}$$

Using extended binomial theorem we expand Eq. (13) as follows:

$$(a+b)^x = \sum_{n=0}^{\infty}\binom{x}{n} a^n b^{x-n},$$
$$\rightarrow (1+(\frac{R}{z})^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}((\frac{R}{z})^2)^n(1)^{-\frac{1}{2}-n}$$
$$= \sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}(\frac{R}{z})^{2n}. \tag{14}$$

So we find

$$V(z)_{out} = \frac{\lambda_0}{2\varepsilon_0}\sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}(\frac{R}{z})^{2n+1}. \tag{15}$$

A comparison between Eq. (12) and Eq.(15) shows

$$A_{2n} = \binom{-\frac{1}{2}}{n}\frac{\lambda_0}{2\varepsilon_0}R^{2n+1}, \qquad A_{2n+1} = 0. \tag{16}$$

Thus, Eq. (11) reads

$$V_{out}(r,\theta) = \frac{\lambda_0}{2\varepsilon_0}\sum_{n=0}^{\infty}.\binom{-\frac{1}{2}}{n}\frac{R^{2n+1}}{r^{2n+1}}P_{2n}(\cos\theta). \tag{17}$$

Finally, the electric potential in the plane of a uniformly charged ring in the region $r > R$ is found by setting $\theta = \frac{\pi}{2}$ in Eq. (17) and the result is

$$V_{out}(r,\frac{\pi}{2}) = V_{out}(r) = \frac{\lambda_0}{2\varepsilon_0}\sum_{n=0}^{\infty},\binom{-\frac{1}{2}}{n}^2\frac{R^{2n+1}}{r^{2n+1}}, \tag{18}$$

in which use is made of $P_{2n}(0) = 1$. This is exactly what we found in Eq. (6) using the direct method.

2.2.2 The case $r < R$

A process similar to previous section can be implemented here. For the region $R < r$, the MSV will results in [?]

$$V_{in,ring,MSV}(r,\theta) = \sum_{n=0}^{\infty} B_n r^n P_n(\cos\theta), \tag{19}$$

To find B_n we seek again the potential at $z-axix$. From Eq. (18) it should be

$$V_{in}(r,0) = \sum_{n=0}^{\infty} B_n r^n P_n(1) = \sum_{m=0}^{\infty} B_{2m} z^{2m} = V_{in}(z). \tag{20}$$

By rearranging Eq. (12) we have

$$V_{in}(z) = \frac{\lambda_0 R}{2\varepsilon_0}\frac{1}{\sqrt{R^2+z^2}} = \frac{\lambda_0}{2\varepsilon_0}\left(1+\left(\frac{z}{R}\right)^2\right)^{-\frac{1}{2}}$$
$$= \frac{\lambda_0}{2\varepsilon_0}\sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}\left(\frac{z}{R}\right)^{2n}, \tag{21}$$

in which we have used the extended binomial expansion formulae again. So, by comparing Eqs. (19) and (20) we see that

$$B_{2m} = \frac{\lambda_0}{2\varepsilon_0}\binom{-\frac{1}{2}}{m}\frac{1}{R^{2m}}, \tag{22}$$

and

$$V_{in}(r,\theta)=\frac{\lambda_0}{2\varepsilon_0}\sum_{m=0}^{\infty}\binom{-\frac{1}{2}}{m}(\frac{r}{R})^{2m}P_{2m}(cos\theta). \qquad (23)$$

Finally, the electric potential in the plane of a uniformly charged ring in the region $r<R$ is found by setting $\theta=\frac{\pi}{2}$ in Eq. (22) and the result is

$$V_{in}(r)=\frac{\lambda_0}{2\varepsilon_0}\sum_{m=0}^{\infty}\binom{-\frac{1}{2}}{m}^2(\frac{r}{R})^{2m}. \qquad (24)$$

This is the previous result found in (9) using the direct method.

3 Uniformly charged disk

Consider a uniformly charged disk with the following set up

3.1 Direct method using Coulomb's law

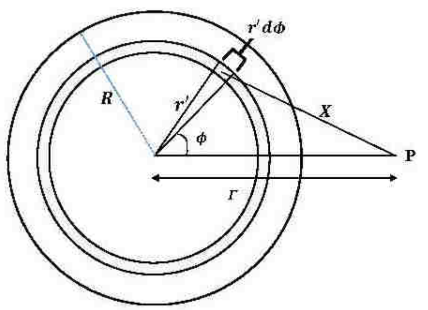

Figure 3: Line element to find electric potential at the plane of a uniformly charged disk

In this section, we first find the electric potential of the ring using the direct method of integration over elements, i.e. using Coulomb's law. In the next subsection, we show that the method of separation of variables is used to solve the same problem. Note that we will use the cylindrical coordinates from now on. The resulting electric potential at the arbitrary point P is as follows:

$$V_{out}(r)=\frac{1}{4\pi\varepsilon_0}\int\frac{dq}{|\mathbf{X}|}$$
$$=\frac{\sigma_0}{4\pi\varepsilon_0}\int\frac{r'dr'd\phi'}{(r'^2+r^2-2rr'\cos\phi')^{\frac{1}{2}}}, \qquad (25)$$

where (r',ϕ') describes a surface element on the disk. With the Taylor expansion of the integral and through several steps of easy integration, the potential of a point on the disk at a distance $r<R$ from the center of the disk is equal to [7,8]:

$$V_{out}(r)=\frac{\sigma_0 R}{2\varepsilon_0}\sum_{m=0}^{\infty}\frac{1}{2m+2}\binom{-\frac{1}{2}}{m}^2(\frac{R}{r})^{2m+1}, \qquad (26)$$

3.2 Solution to Laplace equation

In this section, we first find the solution to Laplace's equation at an arbitrary point outside the disk (and outside its plane) with the help of cylindrical coordinates (ρ,ϕ,z) and then find the solution by setting $z=0$, i.e. we convert the solution to two-dimensional polar coordinates. In this way, we find the potential value on the disc plane at $r<R$. Since there is an electric charge in the region $r<R$ on the disk, we should solve Poisson's equation instead of Laplace's. We are not worried that because the potential is a continuous function and at the boundary of the electric charge (here on the disk) the solution value of Poisson's and Laplace's equation should be the same. It should be noted that using this property is only possible in two dimensions where we have surface charge. The potential at an arbitrary point in cylindrical coordinates is assumed as $V(\rho,\phi,z)$. The Poisson's equation in cylindrical coordinates is given by

$$\frac{\partial^2}{\partial^2\rho}V+\frac{\partial}{\partial\rho}V+\frac{1}{\rho^2}\frac{\partial^2}{\partial^2\phi^2}V+\frac{\partial^2}{\partial^2 z}V=0. \qquad (27)$$

Since the solution should be independent of ϕ we assume the separation of the variables of the form

$$V(r,\theta)=R(\rho)Z(z). \qquad (28)$$

Substituting Eq. (28) back into Eq. (27) we arrive to

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right)+k^2R(\rho)=0, \qquad (29a)$$

$$\frac{d^2}{d\rho^2}Z(z)-k^2Z(z)=0, \qquad (29b)$$

where k is some constant. After solving Eqs. (29a) and (29b) we find

$$Z(z)=e^{-kz} \quad \text{or} \quad e^{kz}, \qquad (30a)$$

$$R(\rho)=J_0(k\rho) \quad \text{or} \quad R_0(k\rho), \qquad (30a)$$

in which $J_0(k\rho)$ and $R_0(k\rho)$ are Bessel functions of the first and second kind. We know that our problem is internal, hence, e^{kz} and $R_0(k\rho)$ should be discarded. The 0th-order Bessel function $Y_0(k\rho)$ is not present in the answer because the answer will be divergent at $\rho = 0$. Thus, the final solution is given by

$$V(\rho,z) = \int_0^\infty B(k)J_0(k\rho)e^{-kz}dk. \tag{31}$$

We do not have any constraints or boundary conditions on the parameter k and this is why we have integrated it in Eq. (31). Note that Eq. (7) actually states that the potential $V(\rho,z)$ is a Laplace transform:

$$V(\rho,z) = \mathscr{L}(B(k)J_0(\rho k))|_{s=z}. \tag{32}$$

To find $B(k)$, it is sufficient to try a particular case of answers in Eq. (7) or Eq. (8). If we consider the field point to be on the z axis, we know from basic physics that the potential is given by

$$V(0,z) = \frac{\sigma_0}{2\varepsilon_0}(\sqrt{a^2+z^2} - z). \tag{33}$$

On the other hand, from Eq. (31) we have

$$V(0,z) = \int_0^\infty B(k)e^{-kz}dk, \tag{34}$$

in which we have used $J_0(0) = 1$. Again, Eq. (34) shows that $V(0,z)$ is also the Laplace transform of $B(k)$ with respect to the variable k, i.e.

$$B(k) = \mathscr{L}^{-1}(\frac{\sigma_0}{2\varepsilon_0}(\sqrt{a^2+s^2} - s)). \tag{35}$$

From relation 12.13(116) in [2] we have

$$\mathscr{L}\left(\frac{J_k(ax)}{x}\right) = \frac{1}{ka^k}\left[\sqrt{a^2+s^2} - s\right]^k,$$
$$k > -1, \quad \mathrm{Re}\, s > |\mathrm{Im}\, a|. \tag{36}$$

from which and Eq. (35) we see that

$$B(k) = \frac{\sigma_0 a}{2\varepsilon_0}\frac{J_1(ak)}{k}. \tag{37}$$

Therefore, the potential relation, Eq. (31) will be as follows:

$$V(\rho,z) = \frac{\sigma_0 a}{2\varepsilon_0}\int_0^\infty \frac{J_1(ak)}{k}J_0(k\rho)e^{-kz}dk. \tag{38}$$

This is also a Laplace transform of the form

$$V(\rho,z) = \frac{\sigma_0 a}{2\varepsilon_0}\mathscr{L}(\frac{J_1(ak)}{k}J_0(k\rho)). \tag{39}$$

Now we investigate some well-known cases.

3.2.1 Electric potential at the center of the disk

The potential at the center of the disc is as follows [7]:

$$V(0,0) = \frac{\sigma_0 a}{2\varepsilon_0}. \tag{40}$$

By Eqs. (37) and (35) we find

$$\int_0^\infty \frac{J_1(u)}{u}du = 1. \tag{41}$$

3.2.2 Electric potential at the edge of the disk

The potential due to a uniformly charged disk at its edge with a simple integration is as follows [5]:

$$V(0,0) = \frac{\sigma_0 a}{\pi\varepsilon_0}. \tag{42}$$

In addition, by putting $\rho = z = 0$ in Eq. (40) we find

$$V(a,o) = \frac{\sigma_0 a}{2\varepsilon_0}\int_0^\infty \frac{J_1(ak)}{k}J_0(k\rho)dk. \tag{43}$$

Using Eqs. (42) and (43) we find

$$\int_0^\infty \frac{J_1(ak)}{k}J_0(ak)dk = \frac{2}{\pi}. \tag{44}$$

3.2.3 Electric potential at $z = 0$ and $\rho < a$ on the disk

In this case, using Eq. (38) we have

$$V(\rho,0) = \frac{\sigma_0 a}{2\varepsilon_0}\int_0^\infty \frac{J_1(ak)}{k}J_0(k\rho)dk. \tag{45}$$

Eqs. (45) and (26) should be equal. Thus we find

$$\int_0^\infty \frac{J_1(ak)}{k}J_0(k\rho)dk$$
$$= -\Sigma_{m=0}^\infty \frac{1}{2m-1}\binom{-\frac{1}{2}}{m}^2(\tfrac{\rho}{a})^{2m}. \tag{46}$$

4 Conclusion

In this article, the usual methods for special mathematical physics functions were used preliminarily, and the relationships that may be difficult to obtain with pure mathematical methods were obtained with less calculation. Eqs. (41), (44) and (46) are the result of solving a problem by two methods and equating the answers. To find similar relations [7] for Bessel functions of the second type, a problem in the region $r > a$ can be used. In general, the charged disc has been the source of many interesting relationships in mathematics and physics, some of which we have presented in the relevant literature and the references section.

References

1. J.D. Jackson, *Classical Electrodynamics* (3rd edition, John Wiley and Sons, 1999)
2. I.S. Gradshteyn, I.M. Ryzhik, D. Zwillinger, V. Moll, *Table of Integrals, Series, and Products* (8th edition, Academic Press, Elsevier Inc., 2014)
3. B. Nazari, 8th Iranian Conference on Mathematical Physics, 2022.
4. B. Nazari, Khayyam 7th Annual Conference, 2024.
5. G.B. Arfken, H.J. Weber, F.E. Harris, *Mathematical Methods for Physicists: A Comprehensive Guide* (7th edition, Academic Press, 2012)
6. H. Lass, L. Blitzer, Celestial Mechanics, **30** (1983); O. Ciftja, I. Hysi, Applied Mathematics Letters, **24** (2011)
7. K.N. Boyadzhiev, *Identities for Squared Central Binomial Coefficients* (Ohio Northern University, 2021), arXiv:2111.08571
8. M. Nayfeh, M.K. Brussel, *Electricity and Magnetism* (Dover Publications, Illustrated edition, 2015)

TTMP vol 1(4), 2024
doi: 10.30511/TTMP.2024.2037649.1038

Transactions in Theoretical and Mathematical Physics

Boson propagator under rigid rotation

Mode expansion approach

E. Siri[1,a], **N. Sadooghi**[1,b]

[1]Department of Physics, Sharif University of Technology, P.O. Box 11155-9161, Tehran, Iran

Received: 05 August 2024 / Accepted: 02 October 2024 / Published: 09 October 2024

Abstract To explore how rigid rotation affects the thermodynamic properties of free relativistic bosons, we employ the standard imaginary time formalism of thermal field theory to calculate the free propagator of complex scalar fields under rotation. We introduce the corresponding partition function and explicitly compute it by expanding the modes in cylindrical coordinates. The resulting propagator is in full agreement with similar findings in the existing literature.

Keywords Finite temperature field theory · Bosonic propagator · Rigid rotation · Imaginary time formalism · Partition function

1 Introduction

Recently, there has been significant interest in studying the effects of rigid rotation on the thermodynamic properties of relativistic particles [1–9]. In [5], a system of spin zero massless bosons bounded on a one-dimensional ring is studied in the presence of an imaginary angular frequency. It is shown that the system's thermodynamics exhibits a fractal dependency on the rotation frequency, and ninionic statistics emerge. In [8], an interacting relativistic bosonic gas subjected to a rigid rotation is studied at finite temperature. Various thermodynamic quantities, including pressure, energy, entropy, and angular momentum densities, as well as heat capacity, moment of inertia, and the speed of sound are computed analytically and numerically. It is shown that certain thermodynamic instabilities appear at high temperatures and large coupling constants. They are manifested as zero and negative values of the moment of inertia and heat capacity, as well as superluminal sound velocities. Zero moment of inertia, which was previously observed in [6] in a rotating and hot spin one gluon gas, leads to the phenomenon of supervorticity in relativistic a Bose gas under rigid rotation and is believed to be the reason for the negative Barnett effect [7]. We notice that the study of bosonic and fermionic systems under rotation may offer potential applications in diverse fields such as heavy ion collision experiments [1–3,9], condensed matter physics [10] and astrophysics of boson stars [11–13].

The standard method for determining thermodynamic quantities involves calculating the partition function, which yields the pressure of relativistic gases. Other thermodynamic quantities can then be derived from the pressure. The primary component of the partition function is the free propagator. This paper aims to determine the partition function and the free propagator of relativistic bosons under rigid rotation.

The propagator for free fermions under rigid rotation was first determined in [14] by Ayala et al. using the generalized Fock-Schwinger method. Subsequently, in [4], the propagator for the Yukawa model was derived, incorporating both bosons and fermions. The general form of the boson propagator was also calculated in [8] using a method similar to the approaches in [14] and [4]. It is the aim of this paper, to compute the free propagator of a rotating Bose gas, by making use of an alternative method, which is referred to as ‘‘mode expansion approach”.

The paper is organized as follows: In Section 2, we use the corresponding metric to rigid rotation and determine the Lagrangian density of free complex scalar fields in the rigidly rotating medium. In Section 3, we define the canonical partition function of this model and the corresponding Hamiltonian density. Section 4 includes the mode expansion of the boson field, including a Fourier sum in cylindrical coordinates. In Section 5, we obtain the free propagator of rigidly rotating complex scalar fields by using the aforementioned mode expansion. Finally, Section 6 is devoted to our conclusions.

[a]e-mail: e.siri@physics.sharif.ir
[b]e-mail: sadooghi@physics.sharif.ir

2 Free complex field in the presence of rigid rotation

We use the Lagrangian density Eq. (1) to describe the non-interacting relativistic bosons,

$$\mathcal{L} = g^{\mu\nu}\partial_\mu \phi^* \partial_\nu \phi - m^2 \phi^* \phi, \tag{1}$$

where

$$g_{\mu\nu} = \begin{pmatrix} 1-\left(x^2+y^2\right)\Omega^2 & y\Omega & -x\Omega & 0 \\ y\Omega & -1 & 0 & 0 \\ -x\Omega & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{2}$$

is a rigid rotation metric [1]. Considering this metric means that the bosons rotate around the z-axis with a uniform angular velocity Ω. By defining angular momentum

$$L_z = i(y\partial_x \phi - x\partial_y \phi), \tag{3}$$

and plugging Eq. (2) into Eq. (1), the Lagrangian density is expressed as

$$\mathcal{L} = (\partial_0 - i\Omega L_z)\phi^* (\partial_0 - i\Omega L_z)\phi - |\nabla\phi|^2 - m^2|\phi|^2. \tag{4}$$

Also the conjugate momentum of each fields can be obtained form Lagrangian density and are given by

$$\pi = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi^*)} = \partial_0 \phi - i\Omega L_z \phi, \tag{5a}$$

$$\pi^* = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} = \partial_0 \phi^* - i\Omega L_z \phi^*. \tag{5b}$$

3 Partition function

In the canonical ensemble, the partition function is given by

$$\mathcal{Z} = \int \mathcal{D}\pi^* \mathcal{D}\pi \int \mathcal{D}\phi^* \mathcal{D}\phi \times \exp\left[\int_X \mathcal{J}(\pi^*, \pi^*; \phi^*, \phi)\right], \tag{6}$$

where

$$\int_X \equiv \int_0^\beta d\tau \int d^3x, \qquad \text{and} \qquad \beta \equiv 1/T. \tag{7}$$

Here, $\tau = it$ is imaginary time [16]. In Eq. (6), the integrand is defined as

$$\mathcal{J}(\pi^*, \pi^*; \phi^*, \phi) \equiv (\pi^* \partial_0 \phi + \pi \partial_0 \phi^* - \mathcal{H}), \tag{8}$$

where

$$\mathcal{H} = \partial_0 \phi^* \pi + \partial_0 \phi \pi^* - \mathcal{L}, \tag{9}$$

denotes the Hamiltonian density. Plugging Eq. (4) into Eq. (9) and using Eq. (5a) and Eq. (5b), yields

$$\mathcal{H} = |\pi|^2 + i\Omega(\pi L_z \phi^* + \pi^* L_z \phi) + |\nabla\phi|^2 + m^2|\phi|^2. \tag{10}$$

To integrate over π and π^* in Eq. (6), we introduce following shifted momenta:

$$\tilde{\pi} = \pi - \partial_0 \phi + i\Omega L_z \phi, \tag{11a}$$

$$\tilde{\pi}^* = \pi^* - \partial_0 \phi^* + i\Omega L_z \phi^*. \tag{11b}$$

Using Eq. (10) and after some standard algebraic steps, the integrand Eq. (8) reads

$$\mathcal{J} = -\tilde{\pi}^* \tilde{\pi} + \mathcal{L}', \tag{12}$$

where $\mathcal{L}'$ is given by

$$\mathcal{L}' = |(\partial_0 - i\Omega L_z)\phi|^2 - |\nabla\phi|^2 - m^2|\phi|^2. \tag{13}$$

4 Mode expansion in cylindrical coordinates

We can describe a complex scalar field $\phi(x)$ with a periodic boundary condition $\phi(\tau = 0) = \phi(\tau = \beta)$ by using a Fourier sum in cylindrical coordinates (τ, r, φ, z),

$$\phi(x) = \sqrt{\beta V} \sum\!\!\!\!\!\!\int_{n,\ell,k} e^{i(\omega_n \tau + \ell\varphi + k_z z)} \times J_\ell(k_\perp r)\tilde{\phi}_{n,\ell}(k), \tag{14a}$$

$$\phi^*(x) = \sqrt{\beta V} \sum\!\!\!\!\!\!\int_{n',\ell',k'} e^{-i(\omega_{n'}\tau+\ell'\varphi+k_z' z)} \times J_{\ell'}(k_\perp' r)\tilde{\phi}^*_{n',\ell'}(k'). \tag{14b}$$

Here, r is the radial coordinate, φ the azimuthal angle, and z the height of the cylinder. Moreover, $\omega_n \equiv 2\pi nT$ are Matsubara frequencies and $J_\ell(k_\perp r)$ refers to Bessel function of the first kind. Following notation is introduced,

$$\sum\!\!\!\!\!\!\int_{n,\ell,k} \equiv \sum_{n=-\infty}^{+\infty}\sum_{\ell=-\infty}^{+\infty}\int \frac{dk_\perp k_\perp dk_z}{(2\pi)^2}, \tag{15a}$$

$$\sum\!\!\!\!\!\!\int_{n',\ell',k'} \equiv \sum_{n'=-\infty}^{+\infty}\sum_{\ell'=-\infty}^{+\infty}\int \frac{dk_\perp' k_\perp' dk_z'}{(2\pi)^2}. \tag{15b}$$

The current series expansion differs from the conventional expansion introduced in references such as [15, 16] in that it includes solutions derived from the Klein-Gordon equation in cylindrical coordinates. In this coordinate system, these solutions involve plane waves in three directions and Bessel functions along the radial direction [8].

5 Free propagator under rigid rotation

According to above arguments, the partition function of free complex scalar fields is given by

$$\mathcal{Z} = \int \mathcal{D}\pi^*\mathcal{D}\pi \int \mathcal{D}\phi^*\mathcal{D}\phi \exp\left[\int_X (\mathcal{L}' - \tilde{\pi}^*\tilde{\pi})\right]. \tag{16}$$

Performing the Gaussian integration over shifted momenta, we arrive at

$$\mathcal{Z} = N\int \mathcal{D}\phi^*\mathcal{D}\phi \exp\left[\int_X \mathcal{L}'\right]. \tag{17}$$

The result of the Gaussian momentum integral is included in the constant factor N. In cylindrical coordinates, Eq. (7) turns out to be

$$\int_X = \int_0^\beta d\tau \int_0^\infty dr\, r \int_0^{2\pi} d\varphi \int_{-\infty}^{\infty} dz. \tag{18}$$

In what follows, we compute

$$\int_X \mathcal{L}' = \int_X |(\partial_0 - i\Omega L_z)\phi|^2 - |\nabla\phi|^2 - m^2|\phi|^2, \tag{19}$$

by using the mode expansion of Eq. (14a) and Eq. (14b). Utilizing the completeness relations presented in Appendix A, it can be readily demonstrated that

$$I = \int_X e^{i(\omega_n-\omega_{n'})\tau} e^{i(\ell-\ell')\varphi} e^{i(k_z-k_z')z} J_\ell(k_\perp r) J_\ell(k_\perp' r) = \beta(2\pi)^2 \hat{\delta}^{n,n'}_{\ell,\ell'}(k_z,k_\perp;k_z',k_\perp'), \tag{20}$$

where $\hat{\delta}^{n,n'}_{\ell,\ell'}$ is defined as

$$\hat{\delta}^{n,n'}_{\ell,\ell'}(k_z,k_\perp;k_z',k_\perp') \equiv \frac{1}{k_\perp}\delta(k_z-k_z')\delta(k_\perp-k_\perp')\delta_{n,n'}\delta_{\ell,\ell'}. \tag{21}$$

The first contribution of Eq. (19) is first given by

$$\mathcal{I}_1 \equiv \int_X |(\partial_0 - i\Omega L_z)\phi|^2 = \int_X (\partial_0 - i\Omega L_z)\phi^*(\partial_0 - i\Omega L_z)\phi, \tag{22}$$

where in the imaginary time formalism $\partial_0 = i\partial_\tau$. Plugging the mode expansion of ϕ and ϕ^* into Eq. (22), using

$$(i\partial_\tau - i\Omega L_z)\phi^* = \sqrt{\beta V} \sum\!\!\!\!\!\!\int_{n',\ell',k'} \left(\omega_{n'} + i\ell'\Omega\right) e^{-i(\omega_{n'}\tau+\ell'\varphi+k_z'z)} \times J_{\ell'}(k_\perp' r)\tilde{\phi}^*_{n',\ell'}(k'), \tag{23}$$

and

$$(i\partial_\tau - i\Omega L_z)\phi = \sqrt{\beta V} \sum\!\!\!\!\!\!\int_{n,\ell,k} (-\omega_n + i\ell\Omega)\, e^{i(\omega_n\tau+\ell\varphi+k_z z)} \times J_\ell(k_\perp r)\tilde{\phi}_{n,\ell}(k), \tag{24}$$

as well as Eq. (20), and performing the integration over k' and summation over n', ℓ', we have

$$\mathcal{I}_1 = -V \sum\!\!\!\!\!\!\int_{n,\ell,k} \tilde{\phi}^*_{n,\ell}(k)\left(\beta^2(\omega_n + i\ell\Omega)^2\right)\tilde{\phi}_{n,\ell}(k). \tag{25}$$

The gradient contribution of Eq. (19) is

$$\mathcal{I}_2 \equiv \int_X |\nabla\phi|^2 = \int_X \nabla\phi^*\nabla\phi, \tag{26}$$

where the gradient operator in the cylindrical coordinates is given by

$$\nabla\phi = \frac{\partial\phi}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial\phi}{\partial\varphi}\hat{\varphi} + \frac{\partial\phi}{\partial z}\hat{z}. \tag{27}$$

Using the orthonormality of the unit vectors in cylinder coordinates, Eq. (26) is equal to

$$\mathcal{I}_2 = \int_X \Big(\frac{\partial\phi^*}{\partial r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial\phi^*}{\partial\varphi}\frac{\partial\phi}{\partial\varphi} + \frac{\partial\phi^*}{\partial z}\frac{\partial\phi}{\partial z}\Big). \tag{28}$$

The radial derivative is

$$\frac{\partial\phi^*}{\partial r} = \sqrt{\beta V}\,\sum\!\!\!\!\!\!\int_{n',\ell',k'} e^{-i(\omega_{n'}\tau+\ell'\varphi+k_z'z)} \times \frac{\partial J_{\ell'}(k_\perp' r)}{\partial r}\tilde{\phi}^*_{n',\ell'}(k'), \tag{29a}$$

$$\frac{\partial\phi}{\partial r} = \sqrt{\beta V}\,\sum\!\!\!\!\!\!\int_{n,\ell,k} e^{i(\omega_n\tau+\ell\varphi+k_z z)} \times \frac{\partial J_\ell(k_\perp r)}{\partial r}\tilde{\phi}_{n,\ell}(k). \tag{29b}$$

Using the relations of Bessel functions from Appendix B, the product of radial derivatives is given by

$$\begin{aligned}\frac{\partial\phi^*}{\partial r}\frac{\partial\phi}{\partial r} &= \beta V\,\sum\!\!\!\!\!\!\int_{n',\ell',k'}\sum\!\!\!\!\!\!\int_{n,\ell,k} e^{-i(\omega_{n'}\tau+\ell'\varphi+k_z'z)}e^{i(\omega_n\tau+\ell\varphi+k_z z)}\\ &\times\frac{k_\perp k_\perp'}{4}\Big[2J_{\ell-1}(k_\perp r)J_{\ell'-1}(k_\perp' r)+2J_{\ell+1}(k_\perp r)J_{\ell'+1}(k_\perp' r)\\ &-\frac{4\ell\ell'}{k_\perp k_\perp' r^2}J_\ell(k_\perp r)J_{\ell'}(k_\perp' r)\Big]\tilde{\phi}^*_{n',\ell'}(k')\tilde{\phi}_{n,\ell}(k).\end{aligned} \tag{30}$$

The azimuthal derivative is computed as follows:

$$\frac{\partial\phi^*}{\partial\varphi} = \sqrt{\beta V}\,\sum\!\!\!\!\!\!\int_{n',\ell',k'} \ell'\,e^{-i(\omega_{n'}\tau+\ell'\varphi+k_z'z)} \times J_{\ell'}(k_\perp' r)\tilde{\phi}^*_{n',\ell'}(k'), \tag{31a}$$

$$\frac{\partial\phi}{\partial\varphi} = \sqrt{\beta V}\,\sum\!\!\!\!\!\!\int_{n,\ell,k} \ell\,e^{i(\omega_n\tau+\ell\varphi+k_z z)} \times J_\ell(k_\perp r)\tilde{\phi}_{n,\ell}(k). \tag{31b}$$

The product of azimuthal derivative is thus given by

$$\frac{\partial\phi^*}{\partial\varphi}\frac{\partial\phi}{\partial\varphi} = \beta V\,\sum\!\!\!\!\!\!\int_{n',\ell',k'}\sum\!\!\!\!\!\!\int_{n,\ell,k}\ell\ell'\,e^{-i(\omega_{n'}\tau+\ell'\varphi+k_z'z)} \times e^{i(\omega_n\tau+\ell\varphi+k_z z)}J_\ell(k_\perp r)J_{\ell'}(k_\perp' r)\tilde{\phi}^*_{n',\ell'}(k')\tilde{\phi}_{n,\ell}(k). \tag{32}$$

Finally, the product of derivative in the z direction is given by

$$\frac{\partial\phi^*}{\partial z}\frac{\partial\phi}{\partial z} = \beta V\,\sum\!\!\!\!\!\!\int_{n',\ell',k'}\sum\!\!\!\!\!\!\int_{n,\ell,k}k_z k_z'\,e^{-i(\omega_{n'}\tau+\ell'\varphi+k_z'z)} \times e^{i(\omega_n\tau+\ell\varphi+k_z z)}J_\ell(k_\perp r)J_{\ell'}(k_\perp' r)\tilde{\phi}^*_{n',\ell'}(k')\tilde{\phi}_{n,\ell}(k). \tag{33}$$

Similarly, by using Eq. (20) and performing integration over k' and summation over n',ℓ', we arrive at

$$\mathcal{I}_2 = V\,\sum\!\!\!\!\!\!\int_{n,\ell,k}\tilde{\phi}^*_{n,\ell}(k)\left(\beta^2\left(k_\perp^2+k_z^2\right)\right)\tilde{\phi}_{n,\ell}(k). \tag{34}$$

The final result for the last contribution is given by

$$\mathcal{I}_3 = V\,\sum\!\!\!\!\!\!\int_{n,\ell,k}\tilde{\phi}^*_{n,\ell}(k)\left(\beta^2 m^2\right)\tilde{\phi}_{n,\ell}(k). \tag{35}$$

Combining Eqs (25), (34), and (35), we obtain

$$\int_X \mathcal{L}' = -V\,\sum\!\!\!\!\!\!\int_{n,\ell,k}\tilde{\phi}^*_{n,\ell}(k)\left(\beta^2 D^{-1}_{\ell,0}(k)\right)\tilde{\phi}_{n,\ell}(k), \tag{36}$$

where $D^{-1}_{\ell,0}(k)$ is the inverse free propagator in the presence of rigid rotation,

$$D^{-1}_{\ell,0}(k) \equiv (\omega_n + i\ell\Omega)^2 + \omega^2, \tag{37}$$

and, $\omega^2 = k_\perp^2 + k_z^2 + m^2$.

6 Conclusion

In this paper, we have derived Eq. (37) representing the propagator of a free boson at finite temperature under the influence of a rigid rotation. Our approach utilized a new method based on mode expansion in the cylindrical coordinate system, deviating from the standard plane wave approach used in references such as [15] and [16]. The previously calculated propagator for a rigidly rotating Bose gas at finite temperature, as seen in [4] and [8], was determined using the generalized Fock-Schwinger method. The result in Eq. (37) is consistent with the findings in [8], except for the sign of ℓ, which is related to the symmetric nature of this propagator under such a change (specifically, see Eq. (2.25) in [8]).

It is worth noting that the Fock-Schwinger method emphasizes geometric and path integral aspects and uses the Schwinger proper time technique, while the mode expansion approach focuses on the Fourier transform and operator formalism. In this work, a Bessel-Fourier expansion is employed due to the special cylindrical geometry. Each of these methods has its own advantages. For example, the Fock-Schwinger method is more advantageous in curved space-time scenarios, while mode expansion is generally more straightforward for perturbative calculations in flat space-time. Importantly, both methods yield the same result, as demonstrated in the present study.

7 Acknowledgments

This work is presented by E. S. at the 8th Iranian Conference on Mathematical Physics (ICMP) in Qom, Iran, on 7-8 July 2024. The authors express their gratitude to the conference organizers for the invitation and hospitality.

Appendix A: Useful completeness relations

To arrive at Eq. (20), we used completeness relations [8]

$$\int_0^\beta d\tau e^{i(\omega_n-\omega_{n'})\tau} = \beta\delta_{n,n'}, \tag{A.1a}$$

$$\int_0^{2\pi} d\varphi e^{i(\ell-\ell')\varphi} = (2\pi)\delta_{\ell,\ell'}, \tag{A.1b}$$

$$\int_{-\infty}^{\infty} dz e^{i(k_z-k_z')z} = (2\pi)\delta(k_z-k_z'), \tag{A.1c}$$

$$\int_0^\infty dr\, r J_\ell(k_\perp r) J_\ell(k_\perp' r) = \frac{1}{k_\perp}\delta(k_\perp-k_\perp'). \tag{A.1d}$$

Appendix B: Bessel function relations

To derive the propagator in Fourier space, we used, in particular, the derivative of the ℓ-th order Bessel function

$$\frac{\partial J_\ell(k_\perp r)}{\partial r} = \frac{k_\perp}{2}\left(J_{\ell-1}(k_\perp r) - J_{\ell+1}(k_\perp r)\right), \tag{B.2}$$

and another useful recursive relation [17]

$$J_{\ell+1}(k_\perp r) = \frac{2\ell}{k_\perp r} J_\ell(k_\perp r) - J_{\ell-1}(k_\perp r). \tag{B.3}$$

References

1. H.L. Chen, K. Fukushima, X.G. Huang, K. Mameda, *Phys. Rev. D*, **93**, 10 (2016)
2. Y. Jiang, J. Liao, *Phys. Rev. Lett.*, **117**, 19 (2016)
3. N. Sadooghi, S.M.A. Tabatabaee Mehr, F. Taghinavaz, *Phys. Rev. D*, **104**, 11 (2021)
4. I.I. Gaspar, L.A. Hernández, R. Zamora, *Phys. Rev. D*, **108**, 9 (2023)
5. V.E. Ambruş, M.N. Chernodub, *Phys. Rev. D*, **108**, 8 (2023)
6. V.V. Braguta, M.N. Chernodub, A.A. Roenko, D.A. Sychev, *Phys. Lett. B*, **852** (2024)
7. V.V. Braguta, M.N. Chernodub, I.E. Kudrov, A.A. Roenko, D.A. Sychev, *Phys. Rev. D*, **110** (2024)
8. E. Siri, N. Sadooghi, *Phys. Rev. D*, **110** (2024)
9. H. Mortazavi Ghalati, N. Sadooghi, *Phys. Rev. D*, **108** (2023)
10. J. Lončar, B. Igrec, D. Babić, *Symmetry*, **14** (2022)
11. F.E. Schunck, E.W. Mielke, *Phys. Lett. A*, **249** (1998)
12. F.E. Schunck, E.W. Mielke, *Class. Quant. Grav.*, **20** (2003)
13. J.L. Wright, PhD Thesis, University of Glasgow (2022)
14. A. Ayala, L.A. Hernández, K. Raya, R. Zamora, *Phys. Rev. D*, **103**, 7 (2021) [erratum: *Phys. Rev. D*, **104**, 3 (2021)]
15. J.I. Kapusta, C.G. Gale, *Finite-temperature field theory: Principles and Applications* (Cambridge University Press, 2011)
16. M. Laine, A. Vuorinen, *Basics of Thermal Field Theory*, (Lect. Notes Phys., **925**, Springer, 2016)
17. M. Abramowitz, I.A. Stegun, *Handbook of Mathematical Functions*, (US Dept. of Commerce, 1972)

TTMP vol 1(4), 2024
doi: 10.30511/TTMP.2024.2036457.1033

Transactions in Theoretical and Mathematical Physics

Linear independence of field equations in the Brans-Dicke theory

E. Ahmadi-Azar[1,a], **K. Atazadeh**[1,b], **A. Eghbali**[1,c]

[1]Department of Physics, Azarbaijan Shahid Madani University, Tabriz, Iran

Received: 23 July 2024 / Accepted: 11 October 2024 / Published:/ 16 October 2024

Abstract In solving the Brans-Dicke (BD) equations in the BD theory of gravity, their linear independence is important. This is due to the fact that in solving these equations in cosmology if the number of unknown quantities is equal to the number of independent equations, then the unknowns can be uniquely determined. In the BD theory, the tensor field $g_{\mu\nu}$ and the BD scalar field φ are not two separate fields, but they are coupled together. The reason behind this is a corollary proposed by V. B. Johri and D. Kalyani in cosmology, which states that the cosmic scale factor of the universe, a, and the BD scalar field φ are related by a power law. Therefore, when the principle of least action is used to derive the BD equations, the variations $\delta g^{\mu\nu}$ and $\delta\varphi$ should not be considered as two independent dynamical variables. So, there is a constraint on $\delta g^{\mu\nu}$ and $\delta\varphi$ that causes the number of independent BD equations to decrease by one unit, in such a way that in the equations that have been known as BD equations, one of them is redundant. In this paper, we prove this issue, that is, we show that one of these equations, which we choose as the modified Klein-Gordon equation, is not an independent equation, but a result establishing other BD equations, the law of conservation of energy-momentum of matter and Bianchi's identity. Therefore, we should not look at the modified Klein-Gordon equation as an independent field equation in the BD theory, but rather it is included in the other BD equations and should not be mentioned separately as one of the BD equations once again.

1 Introduction

In solving the BD equations in the BD theory [1, 2], their linear independence is important. This is due to the fact that the number of independent equations must be equal to the number of unknowns in their solutions. Actually, in BD theory, BD equations form a system of coupled nonlinear second-order differential equations. One of these equations is the modified Klein-Gordon equation and the other equations are generalized Einstein's field equations (EFEs). In the spatially flat $(k=0)$ FLRW cosmological model [3–9] for a universe that is a perfect fluid with the equation of state

$$p = w\rho, \tag{1}$$

where $-1 \leq w \leq 1$, what we seek from solving the BD equations is that to find four quantities: the cosmic scale factor a, the BD scalar field φ, the energy density of the universe ρ and its pressure p. To determine these quantities as functions of cosmic time t, we must have four differential-algebraic equations. Then we can uniquely determine the unknown quantities a, φ, ρ and p as functions of the cosmic time t from solutions of this system of differential-algebraic equations.

The Johri-Kalyani's corollary requires that quantities a and φ are not independent of each other, but they are related by a power law as [6–10]

$$\varphi a^n = \mathscr{C}, \tag{2}$$

where n is an adjustable parameter and $\mathscr{C}$ is a constant. This law first introduced into cosmology by Dehnen and Obregón [11] as a hypothesis to solve the BD equations. Later, following Dehnen and Obregón, authors [7–9, 12, 13] used this assumption to solve the BD equations. Finally, in 1994 Johri and Kalyani [6] proved that this relation should not be seen as an assumption but rather a result of the constancy of the deceleration parameter of the universe. Accordingly, in this article we have called this power law as Johri-Kalyani's corollary. Thus,

[a]e.ahmadi.azar@azaruniv.ac.ir
[b]atazadeh@azaruniv.ac.ir
[c]eghbali978@gmail.com

in the FLRW cosmological model, in addition to the equation of state (1), we have also the power law (2) as an algebraic equation between the unknowns of the problem. So, to determine the unknowns of the problem, we only need two differential equations that together with these two algebraic equations form a system with four equations for four unknowns a, φ, ρ and p. Based on this, the BD equations must contain two coupled independent second-order differential equations.

As a result, in order to solve the FLRW cosmological model, the BD equations must be three differential equations. One of them must depend on the other two equations. In other words, from the set of BD equations, one of them is repeated. Our goal in this paper is that to show in the BD theory, one of these equations, which we choose here, the modified Klein-Gordon equation [14, 15], is not an independent equation, but a result of the establishment of other BD equations. For this purpose, we first derive the BD equations by using the principle of least action for the BD-action.

2 BD theory

The BD-action in Jordan's frame is given by [14–20]

$$I_{BD} = \frac{1}{16\pi}\int_{\mathscr{M}} d^4x\sqrt{-g}\Big(\varphi\mathscr{R} - \frac{\omega}{\varphi}\, g^{\mu\nu}\, \nabla_\mu\varphi\, \nabla_\nu\varphi - V(\varphi) + 16\pi\, \mathscr{L}_M\Big) \tag{3}$$

where φ is the BD scalar field, ω is the adjustable BD coupling parameter, $V(\varphi)$ is the potential energy of the field φ and $\mathscr{R}$ is the Ricci scalar of the space-time manifold $\mathscr{M}$ with the local coordinates $x^\mu = (x^0, x^1, x^2, x^3)$. Also, $g = \det g_{\mu\nu}$ and ∇_μ denotes the covariant derivative operator in the space-time and finally, $\mathscr{L}_M := \mathscr{L}_M(g_{\mu\nu}, \partial_\rho g_{\mu\nu})$ is the Lagrangian density of the matter which is minimally coupled to the BD scalar field φ. Moreover, the BD scalar field φ inversely proportional to the effective gravitational constant G_{eff} by the relation [7, 8, 15, 19, 20]

$$\varphi = \frac{1}{G_{eff}}\frac{4+2\omega}{3+2\omega}, \tag{4}$$

where G_{eff} is equal to the Newton's gravitational constant $G = 6.67\times10^{-8}\ cm^3\ g^{-1}\ s^{-2}$ in the limit when ω tends to infinity.

It should be noted that Eq. (3) is the original action of the BD theory. In 2011, the generalization of this action was presented by S. Nojiri and S. D. Odintsov in [21, 22] as follows:

$$I_{NO} = \frac{1}{16\pi}\int_{\mathscr{M}} d^4x\sqrt{-g}\Big(e^{\alpha(\varphi)}\mathscr{R} - \frac{\omega(\varphi)}{\varphi}\, g^{\mu\nu}\, \nabla_\mu\varphi\, \nabla_\nu\varphi - V(\varphi) + 16\pi\, \mathscr{L}_M\Big), \tag{5}$$

where $e^{\alpha(\varphi)}$ and $\omega(\varphi)$ are some appropriate functions of the BD scalar field φ. The above action, Eq. (6), was applied to dark energy problem [23]. But in the present paper we focus on the original BD action, Eq. (3).

Similar to the theory of general relativity (GR), the definition of energy-momentum tensor (EMT) of the matter

$$T_M^{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\partial}{\partial g_{\mu\nu}}\Big(\sqrt{-g}\mathscr{L}_M\Big), \tag{6}$$

and its conservation

$$\nabla_\mu T_M^{\mu\nu} = 0, \tag{7}$$

also hold in the BD theory [7–9, 14, 15, 19, 24].

By varying the BD action, Eq. (3) with respect to the dynamical variables, BD scalar field φ and metric tensor $g^{\mu\nu}$, and by using the least action principle for the BD action, i.e. $\delta I_{BD} = 0$, and this fact that the variations of $\delta g^{\mu\nu}$ and $\delta\varphi$ are arbitrary, then we get the following equations, respectively [9, 14, 15, 20]

$$\frac{2\omega}{\varphi}\Box\varphi + \mathscr{R} - \frac{\omega}{\varphi^2}\nabla^\mu\varphi\nabla_\mu\varphi - \frac{dV}{d\varphi} = 0, \tag{8}$$

$$\mathscr{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathscr{R} = \frac{8\pi}{\varphi}T_{M\mu\nu} + \frac{1}{\varphi}\big(\nabla_\mu\nabla_\nu\varphi - g_{\mu\nu}\Box\varphi\big) + \frac{\omega}{\varphi^2}\big(\nabla_\mu\varphi\nabla_\nu\varphi - \frac{1}{2}g_{\mu\nu}\nabla^\rho\varphi\nabla_\rho\varphi\big) - \frac{V}{2\varphi}g_{\mu\nu}, \tag{9}$$

where $\mathscr{R}_{\mu\nu}$ is the Ricci tensor and $\Box$ is the covariant d'Alemberian operator of the metric tensor $g_{\mu\nu}$, which is defined by

$$\Box := \nabla^\rho\nabla_\rho = \frac{1}{\sqrt{-g}}\partial_\mu\big(\sqrt{-g}g^{\mu\nu}\partial_\nu\big). \tag{10}$$

We note Eq. (9) is a generalization of the Einstein's field equations $\mathscr{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathscr{R} + \Lambda g_{\mu\nu} = 8\pi G T_{M_{\mu\nu}}$. By performing contraction on Eq. (9), we obtain

$$\mathscr{R} = -\frac{8\pi T_{M_\lambda}^{\lambda}}{\varphi} + \frac{\omega}{\varphi^2}\nabla^\mu\varphi\nabla_\mu\varphi + \frac{3\Box\varphi}{\varphi} + \frac{2V}{\varphi}, \tag{11}$$

where $T_{M_\lambda}^{\lambda} := g^{\mu\nu}T_{M_{\mu\nu}}$ is the trace of the energy-momentum tensor of the ordinary matter fields. By substituting Eq. (11) into Eq. (8) one gets

$$\Box\varphi = \frac{1}{3+2\omega}\big(8\pi T_{M_\lambda}^{\lambda} + \varphi\frac{dV}{d\varphi} - 2V\big). \tag{12}$$

This equation together with Eq. (9) form a system of coupled second-order non-linear differential equations which are called the general form of the BD equations in the BD theory of gravity. Notice that Eq. (12) is known as the modified Klein-Gordon equation. Furthermore, Eq. (9) is sometimes called BD field equations.

3 Linear independence of the field equations in the BD theory

The purpose of this section is to show that in the BD theory, the Klein-Gordon equation, Eq. (12) is not an independent equation; but it can be derived from the BD field equations, Eq. (9). For this purpose, we take the covariant derivative from both sides of Eq. (9) with respect to the general coordinate x^ν, giving us

$$\nabla_\nu\Big(\mathscr{R}^{\mu\nu}-\frac{1}{2}g^{\mu\nu}\mathscr{R}\Big)=\nabla_\nu\Big[\frac{8\pi}{\varphi}T_M^{\mu\nu}+\frac{1}{\varphi}(\nabla^\mu\nabla^\nu\varphi-g^{\mu\nu}\Box\varphi)+\frac{\omega}{\varphi^2}(\nabla^\mu\varphi\nabla^\nu\varphi-\frac{1}{2}g^{\mu\nu}\nabla^\rho\varphi\nabla_\rho\varphi)-\frac{V}{2\varphi}g^{\mu\nu}\Big]. \quad (13)$$

Then, using the Bianchi's identity $\nabla_\nu G^{\mu\nu}=0$, ($G_{\mu\nu}=\mathscr{R}_{\mu\nu}-\frac{1}{2}\mathscr{R}g_{\mu\nu}$ is Einstein's tensor), the left side of Eq. (13) becomes zero. On the right side, according to the EMT conservation law of the matter, Eq. (7), the divergence of the EMT is zero. Finally, we obtain

$$\begin{aligned}&-\frac{8\pi}{\varphi^2}T_M^{\mu\nu}\nabla_\nu\varphi-\frac{2\omega}{\varphi^3}\Big(\nabla^\mu\varphi\nabla^\nu\varphi-\frac{1}{2}g^{\mu\nu}\nabla^\rho\varphi\nabla_\rho\varphi\Big)\nabla_\nu\varphi\\&+\frac{\omega}{\varphi^2}\Big[(\nabla_\nu\nabla^\mu\varphi)\nabla^\nu\varphi+(\nabla_\nu\nabla^\nu\varphi)\nabla^\mu\varphi\\&-\frac{1}{2}g^{\mu\nu}(\nabla_\nu\nabla^\rho\varphi)\nabla_\rho\varphi-\frac{1}{2}g^{\mu\nu}\nabla^\rho\varphi(\nabla_\nu\nabla_\rho\varphi)\Big]\\&-\frac{1}{\varphi^2}(\nabla^\mu\nabla^\nu\varphi-g^{\mu\nu}\Box\varphi)\nabla_\nu\varphi\\&+\frac{1}{\varphi}\Big[\nabla_\nu(\nabla^\mu\nabla^\nu\varphi)-g^{\mu\nu}\nabla_\nu(\Box\varphi)\Big]\\&+\Big(\frac{V}{2\varphi^2}-\frac{1}{2\varphi}\frac{dV}{d\varphi}\Big)g^{\mu\nu}\nabla_\nu\varphi=0. \end{aligned}\quad (14)$$

By doing some tensor calculations, Eq. (14) can be simplified as follows:

$$\begin{aligned}&-\frac{8\pi}{\varphi^2}T_M^{\mu\nu}\nabla_\nu\varphi-\frac{\omega}{\varphi^3}(\nabla^\rho\varphi\nabla_\rho\varphi)\nabla^\mu\varphi\\&+\frac{\omega}{\varphi^2}(\Box\varphi)\nabla^\mu\varphi-\frac{1}{\varphi^2}(\nabla^\mu\nabla^\nu\varphi)\nabla_\nu\varphi\\&+\frac{1}{\varphi^2}(\Box\varphi)\nabla^\mu\varphi+\frac{1}{\varphi}\Big[\Box(\nabla^\mu\varphi)-\nabla^\mu(\Box\varphi)\Big]\\&+(\frac{V}{2\varphi^2}-\frac{1}{2\varphi}\frac{dV}{d\varphi})\nabla^\mu\varphi=0.\end{aligned}\quad (15)$$

In order to calculate expression $\Box(\nabla^\mu\varphi)-\nabla^\mu(\Box\varphi)$ in the above equation, let us consider the following tensor identity [18, 24]

$$\nabla_\mu\nabla_\nu V^\lambda-\nabla_\nu\nabla_\mu V^\lambda=V^\sigma\mathscr{R}^\lambda{}_{\sigma\mu\nu}, \quad (16)$$

where V^λ is an arbitrary four-vector and $\mathscr{R}^\lambda{}_{\sigma\mu\nu}$ is the Riemann curvature tensor. By putting $\lambda=\nu$ in the above identity, we then get

$$\begin{aligned}\nabla_\mu\nabla_\nu V^\nu-\nabla_\nu\nabla_\mu V^\nu&=V^\sigma\mathscr{R}^\nu{}_{\sigma\mu\nu},\\&=-V^\sigma\mathscr{R}^\nu{}_{\sigma\nu\mu}\\&=-V^\sigma\mathscr{R}_{\sigma\mu}\\&=-V_\sigma\mathscr{R}^\sigma_\mu,\end{aligned}\quad (17)$$

where $\mathscr{R}_{\mu\nu}:=\mathscr{R}^\lambda{}_{\mu\lambda\nu}$ is the Ricci tensor. In Eq. (17), we define co-vector $V_\sigma=\nabla_\sigma\varphi$, where φ is the BD scalar field. Accordingly, we get the following relation

$$\nabla_\mu(\nabla_\nu\nabla^\nu\varphi)-\nabla_\nu(\nabla_\mu\nabla^\nu\varphi)=-\nabla_\sigma\varphi\mathscr{R}^\sigma_\mu, \quad (18)$$

which can be written as follows:

$$\nabla_\nu(\Box\varphi)-\Box(\nabla_\nu\varphi)=-(\nabla_\mu\varphi)\mathscr{R}^\mu_\nu. \quad (19)$$

For the next use, it is useful to write the above equation in the following form

$$\nabla^\nu(\Box\varphi)-\Box(\nabla^\nu\varphi)=-(\nabla_\mu\varphi)\mathscr{R}^{\mu\nu}. \quad (20)$$

From the contraction of Eq. (9) one can obtain the Ricci scalar $\mathscr{R}$, giving us

$$\mathscr{R}^\mu_\nu-\frac{1}{2}\delta^\mu_\nu\mathscr{R}=\frac{8\pi}{\varphi}T^\mu_{M\nu}+\frac{1}{\varphi}(\nabla^\mu\nabla_\nu\varphi-\delta^\mu_\nu\Box\varphi)+\frac{\omega}{\varphi^2}(\nabla^\mu\varphi\nabla_\nu\varphi-\frac{1}{2}\delta^\mu_\nu\nabla^\rho\varphi\nabla_\rho\varphi)-\frac{V}{2\varphi}\delta^\mu_\nu. \quad (21)$$

By putting $\mu=\nu$ in the above equation and then by using the fact that $\mathscr{R}^\mu_\mu=\mathscr{R}$ and $\delta^\mu_\mu=4$ we arrive at Eq. (11).

Now, in Eq. (9), instead of the Ricci scalar $\mathscr{R}$, we put its value from Eq. (11) and then calculate the tensor $\mathscr{R}^{\mu\nu}$ from the resulting equation, giving us

$$\mathscr{R}^{\mu\nu} - \frac{1}{2}g^{\mu\nu}\Big(-\frac{8\pi}{\varphi}T^{\lambda}_{M\,\lambda} + \frac{3}{\varphi}\Box\varphi + \frac{\omega}{\varphi^2}\nabla^{\lambda}\varphi\nabla_{\lambda}\varphi + \frac{2V}{\varphi}\Big)$$
$$= \frac{8\pi}{\varphi}T^{\mu\nu}_{M} + \frac{\omega}{\varphi^2}\big(\nabla^{\mu}\varphi\nabla^{\nu}\varphi - \frac{1}{2}g^{\mu\nu}\nabla^{\rho}\varphi\nabla_{\rho}\varphi\big)$$
$$+\frac{1}{\varphi}\big(\nabla^{\mu}\nabla^{\nu}\varphi - g^{\mu\nu}\Box\varphi\big) - \frac{V}{2\varphi}g^{\mu\nu}. \qquad (22)$$

After some simplification, we then obtain

$$\mathscr{R}^{\mu\nu} = -\frac{4\pi}{\varphi}T^{\lambda}_{M\,\lambda}g^{\mu\nu} + \frac{8\pi}{\varphi}T^{\mu\nu}_{M} + \frac{\omega}{\varphi^2}\nabla^{\mu}\varphi\nabla^{\nu}\varphi$$
$$+\frac{1}{2\varphi}\Box\varphi g^{\mu\nu} + \frac{V}{2\varphi}g^{\mu\nu} + \frac{1}{\varphi}\nabla^{\mu}\nabla^{\nu}\varphi. \qquad (23)$$

By putting the tensor $\mathscr{R}^{\mu\nu}$ from the above equation into Eq. (20), we get the following equation

$$\nabla^{\mu}(\Box\varphi) - \Box(\nabla^{\mu}\varphi)$$
$$= -\Big[-\frac{4\pi}{\varphi}T^{\lambda}_{M\,\lambda}g^{\mu\nu} + \frac{8\pi}{\varphi}T^{\mu\nu}_{M}$$
$$+\frac{\omega}{\varphi^2}\nabla^{\mu}\varphi\nabla^{\nu}\varphi + \frac{1}{2\varphi}\Box\varphi g^{\mu\nu}$$
$$+\frac{V}{2\varphi}g^{\mu\nu} + \frac{1}{\varphi}\nabla^{\mu}\nabla^{\nu}\varphi\Big]\nabla_{\nu}\varphi. \qquad (24)$$

From the combination of Eqs (15) and (24), one may get the following equation

$$-\frac{8\pi}{\varphi^2}T^{\mu\nu}_{M}\nabla_{\nu}\varphi - \frac{\omega}{\varphi^3}(\nabla^{\rho}\varphi\nabla_{\rho}\varphi)\nabla^{\mu}\varphi$$
$$+\frac{\omega}{\varphi^2}(\Box\varphi)\nabla^{\mu}\varphi - \frac{1}{\varphi^2}(\nabla^{\mu}\nabla^{\nu}\varphi)\nabla_{\nu}\varphi$$
$$+\frac{1}{\varphi^2}(\Box\varphi)\nabla^{\mu}\varphi + \frac{1}{\varphi}\Big[-\frac{4\pi}{\varphi}T^{\lambda}_{M\,\lambda}g^{\mu\nu} + \frac{8\pi}{\varphi}T^{\mu\nu}_{M}$$
$$+\frac{\omega}{\varphi^2}\nabla^{\mu}\varphi\nabla^{\nu}\varphi + \frac{1}{2\varphi}\Box\varphi g^{\mu\nu} + \frac{V}{2\varphi}g^{\mu\nu} + \frac{1}{\varphi}\nabla^{\mu}\nabla^{\nu}\varphi\Big]\nabla_{\nu}\varphi$$
$$+\Big(\frac{V}{2\varphi^2} - \frac{1}{2\varphi}\frac{dV}{d\varphi}\Big)\nabla^{\mu}\varphi = 0. \qquad (25)$$

After performing some tensor calculations we find that

$$\frac{1}{2\varphi^2}\Big[-8\pi T^{\lambda}_{M\,\lambda} + (2\omega+3)\Box\varphi + 2V - \varphi\frac{dV}{d\varphi}\Big]\nabla^{\mu}\varphi = 0. \qquad (26)$$

Clearly, from the above equation we have

$$-8\pi T^{\lambda}_{M\,\lambda} + (2\omega+3)\Box\varphi + 2V - \varphi\frac{dV}{d\varphi} = 0. \qquad (27)$$

From solving Eq. (27) for $\Box\varphi$, we get the modified Klein-Gordon equation, Eq. (12). In this way, we reached the desired result.

4 Conclusion

In this paper, we proved that the modified Klein-Gordon equation, Eq. (12) in the BD theory of gravity is not an independent equation of the BD field equation, Eq. (9), but it results from the establishment of the BD field equation, Eq. (9). Therefore, when we want to introduce the BD theory, it is enough to consider only the BD field equation, Eq. (9) as the basic equation of this theory. For this reason, if we consider Eqs (1), (2) and (9) without the modified Klein-Gordon equation, Eq. (12) as a system of equations in the "problem of the FLRW cosmological model" to determine the four unknowns $a(t)$, $\varphi(t)$, $\rho(t)$ and $p(t)$, we have not made any mistake. Our reason for doing this is that the modified Klein-Gordon equation, Eq. (12) in the BD theory is not a fundamental equation and only Eq. (9) forms the fundamental equation of the theory. The BD field equation, Eq. (9) provides us with two independent equations that are not enough to determine the four unknowns. Hence, we need two other equations that together with Eq. (9) form a system of four independent equations with four unknowns. As seen in this paper, we have chosen the equation of state, Eq. (1) and the power law, Eq. (2), which are compatible with the physics of the problem, as the desired equations. It is worth noting that although Eq. (12) is not one of the basic equations of the BD theory, it can be added to the system of algebraic-differential Eqs (1), (2) and (9), like $\nabla_{\nu}G^{\mu\nu} = 0$ (Bianchi's identity) and $\nabla_{\nu}T^{\mu\nu} = 0$ (conservation of the energy-momentum tensor of the matter field) without creating an obstacle in solving the FLRW cosmological model.

References

1. C. Brans and R. H. Dicke, Phys. Rev. (series I) **124**, 925 (1961)
2. C. H. Brans, Phys. Rev. (series I), **125**, 2194 (1962)
3. J. M. Cerveró and P. G. Estévez, Gen. Relativ. Grav., **15**, 351 (1983)
4. P. Chvanis, Phys. Rev. D, **92**, 103004 (2015)
5. N. Banerjee and A. Beesham, Int. J. Mod. Phys. D, **6** , 119 (1997)
6. V. B. Johri and D. Kalyani, Gen. Relativ. Grav., **26**, 1217 (1994)
7. E. Ahmadi-Azar and N. Riazi, Astrophys. Space Sci., **226**, 1 (1995)
8. I. Ahmadi-Azar, Iranian J. Phys. Research, **3**, 177 (2002)
9. E. Ahmadi-Azar, K. Atazadeh and A. Eghbali, Int. J. Geom. Methods Mod. Phys., **21**, (2024)
10. S. M. M. Rasouli and P. V. Moniz, Class. Quant. Grav., **33**, (2016)

11. H. Dehnen and O. Obregón, Astrophys. Space Sci., **14**, (1971)
12. L. O. Pimentel, Astrophys. Space Sci., **112**, (1985)
13. G. P. Singh, A. Y. Kale and J. Tripathi, Rom. J. Phys. **58**, (2013)
14. S. Capozziello and M. De Laurentis, Phys. Rep., **509**, 167 (2011)
15. S. Capozziello and V. Faraoni, *Beyond Einstein Gravity: A survey of gravitational theories for cosmology and astrophysics,* (Heidelberg: Springer, 2011)
16. P. Jordan, Nature, **164**, 637 (1949)
17. Y. Fujii and K. Maeda, *The scalar-tensor theory of gravitation,* (Cambridge: Cambridge University Press, 2004)
18. F. Bajardi and S. Capozziello, *Noether symmetries in theories of gravity: with applications to astrophysics and cosmology,* (Cambridge: Cambridge University Press, 2023)
19. S. Weinberg, *Gravitation and cosmology: principles and applications of the general theory of relativiity,* (New York: John Wiley & Sons, 1972)
20. Ø. Grøn and S. Hervik, *Einstein's General Theory of Relativity: With Modern Applications in Cosmology,* (New York: Springer, 2007)
21. S. Nojiri and S. D. Odintsov, Phys. Rep., **505**, (2011)
22. S. Nojiri, S. D. Odintsov and V. K. Oikonomou, Phys. Rep., **692**, (2017)
23. E. Elizalde, S. Nojiri, S. D. Odintsov, Phys. Rev. D. **70**, (2004)
24. T. Frankel, *The geometry of physics: an introduction,* (Cambridge: Cambridge University Press, 2012)

TTMP vol 1(4), 2024
doi: 10.30511/TTMP.2024.2039661.1039

Transactions in Theoretical and Mathematical Physics

Einstein manifolds from a mathematical and physical point of view

Ali Taghavi[1,2,a]

[1] Department of Mathematics, Qom University of Technology, Qom, Iran
[2]School of Mathematics, Institute for Research in Fundamental Sciences (IPM), P. O. Box 19395-5746, Tehran, Iran

Received: 26 August 2024 / Accepted: 17 October 2024 / Published: 03 November 2024

Abstract In this paper, we introduce Einstein's manifold theory from a mathematical and physical viewpoint. We present some obstructions for a manifold to be equipped with the structure of an Einstein manifold. We discuss the Hamiltonian approach to Einstein manifold theory and the chirality of the Lie algebra of an Einstein Lie group. We also discuss Einstein's structure on non-compact manifolds and present some partial results on Einstein's structure on the tangent bundle.

1 Introduction

The vacuum Einstein field equation is the main motivation for considering Einstein manifolds. The Einstein field equation is

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}, \qquad (1)$$

where $G_{\mu\nu}$ is the Einstein tensor, $g_{\mu\nu}$ is the metric tensor, $T_{\mu\nu}$ is the stress-energy tensor, κ is the Einstein gravitational constant, and Λ is the cosmological constant. In the vacuum case, the equation becomes:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 0. \qquad (2)$$

In the mathematical setting, the Einstein tensor is the Ricci tensor of a certain Riemannian metric, motivating the definition of an Einstein manifold, a manifold whose Ricci tensor is a constant multiple of the metric tenso, namely $Ric = \kappa g$ where g is the metric tensor and Ric is the Ricci tensor. A particular case is when $\kappa = 0$, which defines a Ricci-flat manifold.

The scalar curvature of an Einstein manifold is constant. The theory is deeply involved with problems in Riemannian geometry that concern objects with constant Ricci curvature.

2 Preliminaries

In this section, we provide the prerequisites and notations necessary to introduce the theory of Einstein manifolds. Let (M, g) be a Riemannian manifold with the Levi-Civita connection ∇. The curvature tensor is defined as

$$R(X,Y)Z = (\nabla_X \circ \nabla_Y - \nabla_Y \circ \nabla_X - \nabla_{[X,Y]})Z. \qquad (3)$$

The Ricci tensor R is a 2-linear map Ric defined as

$$Ric(X,Z) = \mathrm{trace}(Y \mapsto R(X,Y)Z), \qquad (4)$$

which is a symmetric tensor.

Proposition 1 The scalar curvature is the trace of Ric with respect to the metric tensor g.

Proof Let (M, g) be a n dimensional Einstein manifold. The scalar curvature is $scal = g^{ij} R_{ij}$ where R_{ij} are Ricci coefficients and g^{ij} are coefficient of inverse metric g^{-1}. From the equation $Ric = \lambda g$ we get $scalar(M) = n\lambda$

[a]e-mail: etaghavi@qut.ac.ir.
[a]This research was in part supported by a grant from IPM (1403550049).

Proposition 2 Let (M, g) be a Riemannian manifold with constant sectional curvature then M is an Einstein manifold.

Proof Every manifold of constant sectional curvature is locally isometric to either Euclidean space, the hyperbolic space or the round sphere. The property of being Einstein is a local isometric property. So every manifold of constant curvature is an Einstein manifold since each of the above 3 mentioned spaces is an Einstein manifold.

Remark There are Einstein manifolds whose sectional curvature is not constant. The complex projective space $\mathbb{C}P^n$ is an example of Einstein manifolds whose sectional curvature is not constant. However, the holomorphic sectional curvature of the complex projective space is constant.

For a compact manifold M the Euler characteristic $\chi(M)$ can be defined n several equivalent forms; First definition is based on the self ntersection number of M as a submanifold of TM.

An equivalent definition is in terms of the cell structure of a manifold which admit a CW complex structure. Note that every compact manifold is homotopic equivalent to a CW complex. For any such manifold the Euler characteristic is defined as $k_0 - k_1 + k_2 - k - 3 + \ldots \pm k_n$ where k_i is the number of cells of dimension i. A more general definition is in terms of Betti numbers. Namely the Euler characteristic $\chi(M) = \sum_i (-1)^i b_i$ where b_i is the rank of the ith singular homology group $H_i(M)$. All of these equivalent definitions are indeed equivalent to the Euler characteristic of a triangulated manifold. From a dynamical point of view the definition we mentioned in terms of self-intersection number is identical to the sum of the index of singularities of a generic vector field on M

The signature of a Riemannian manifold M of dimension $4k$ is defined as follows: We define a symmetric 2-form $S : H^{2k}(M) \times H^{2k}(M) \to \mathbb{R}$ via $(\alpha, \beta) \to \int_M \alpha \wedge \beta$. This symmetric form can be represented by a symmetric matrix M. Let n_+, n_- be the number of positive and negative eigenvalues of M respectively. Then the signature is defined as $\sigma(M) = n_+ - n_-$. The signature is a bordism invariant namely if a compact manifold is the boundary of a compact orientable manifold N the $Sign(M) = 0$ so we get that the signature of $S^{4k} = 0$. Moreover, we conclude that the projective space $\mathbb{C}P^{2k}$ can not be the boundary of any manifold since the signature of all $\mathbb{C}P^{2k}$ is equal to 1.

Proposition The Euler characteristic $\chi(M)$ and the signature $\sigma(M)$ have the same parity namely $\chi(M) \equiv \sigma(M) \pmod 2$

Remark A common property of Euler characteristic and signature is that both quantities are multiplicative.

A Fredholm index interpretation of signature of manifolds. The famous Hirzebruch signature theorem presents an index theoretical interpretation for the signature of a smooth manifold. To formulate this interpretation we need to define the signature operator acting on a certain space of differential form. Let (M, g) be a Riemannian manifold of dimension $2l$ with Hodge star operator $\star$ and the Dirac operator $d + d^*$. For every given p we consider the exterior differential $d : \omega^p(M) \to \omega^{p+1}(M)$ and $d^* : \Omega^{p+1}(M) \to \omega^p(M)$. To these data we assign an involution $\tau(\alpha) = i^{p(p-1)+l}\alpha$. This involution anti commutes with the direct operator $d+d^*$. the involution τ has two eigenspaces Ω_+, Ω_- corresponds to eigenvalues ± 1 of the involution τ. This implies that the Dirac operator $d+d^*$ maps Ω_+ to Ω_- and vice versa. So we can represent the Dirac operator in the matrix form $d+d^* = \begin{pmatrix} 0 & D_+ \\ D_- & 0 \end{pmatrix}$ where $D : \Omega_+ \to \Omega_-$.

Hirzebruch Signature Theorem $\sigma(M) = Ind(D)$. A Lie algebra L is called a chiral algebra if every lie algebra automorphism is an orientation preserving linear map.

Remark We should not confuse this terminology with the similar name terminology chiral algebra in algebraic geometry and D module theory mentioned in [2] and [3].

Example of a chiral Lie algebra is $\mathbb{R}^3$ with the cross-product operation. In the next section we address the existence of an Einstein structure on Lie groups and pose the question of possible relation to chirality of the associated lie algebra. In the paper, we would focus on possible relation of chirality of Lie algebras and Einstein structure on Lie groups. But the classification of all finite dimensional chiral algebra is another problem which can be studied independently.

3 Einstein manifolds

In this section we provide some obstructions on a manifold to have an Einstein structure.

Hitchin Thorpe Inequality Every compact 4 dimensional Einstein manifold satisfies the Einstein Thorpe inequality $\chi(M) \geq \frac{3}{2}|\sigma(M)|$.

Proof The idea of proof is based on computation of the Euler characteristic and signature of manifold in terms of entries of the matrix of the curvature operator. The curvature operator T defined on $\Lambda^2 TM$ satisfies $< T(x \wedge y), z \wedge w >= R(x, y, z, w)$. Existence of Einstein metrics enable use to represent the curvature operator in the 6×6 matrix $I_2 \otimes A + J_2 \otimes B$ where I_2, J_2 are the identity and almost complex 2- matrix respectively and A, B are diagonal matrices $A = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix}$ and $B = \begin{pmatrix} b_1 & 0 & 0 \\ 0 & b_2 & 0 \\ 0 & 0 & b_3 \end{pmatrix}$. Then as a consequence of Chern weil theory we have

$$\chi(M) = \frac{1}{4\pi^2} \int_M \left(\sum a_i^2 + \sum b_i^2 \right), \tag{5}$$

and

$$\sigma(M) = \frac{1}{3\pi^2} \int_M (a_1 b_1 + a_2 b_2 + a_3 b_3). \tag{6}$$

This obviously implies that $\chi(M) \geq 3/2|\sigma(M)|$ because $\sum a_i^2 + \sum b_i^2 = \sum (a_i - b_i)^2 + 2 \sum a_i b_i$.

Remark Note that the Hitchin-Thorpe inequality is a necessary condition for the existence of an Einstein structure on a manifold. But it is not a sufficient condition condition. For a counterexample of manifolds which satisfy this inequality but does not admit any Einstein structure see [5] and [7].

Example Every compact manifold whose Euler characteristic is negative can not be an Einstein manifold.

4 Summery and discussion

Einstein manifolds can be seen as critical points of the Hilbert functionals. For every Riemannian manifold (M, g) we define the Hilbert functional $\int_M scaldv - g$ the integral of scalar curvature with respect to the volume form associated to the metric g. This functional is defined on the space of all Riemannian metrics on a given manifold. The Hilbert functional is invariant under the action of all diffeomorphisms. So this functional can be studied in the equivalent class of all Riemannian metrics up to isometric. Recall that two metrics on a manifold are equivalent if there is a diffeomorphism on the manifold which pulls back one metric to another one. Consideration of such a functional generates some researches on study some other kind of Hilbert like functionals. For example in [1] it is proved that the critical points of the functional $\int_m |Ric_g|^2 vol_g$ are flat metric provided they have non negative scalar curvature.

The Ricci flow structure of an Einstein manifold has a simple formulation. Recall that a Ricci flow is a 1 parameter of Riemannian metrics g_t which satisfy $\frac{d}{dt} g_t = -2Ricg_t$. These flows are very important framework in Perleman's approach to the Poincare conjecture see [6]. Now if (M, g) is an Einstein manifold then the Ricci flow can be produced as $g(t) = (1 - 2\lambda t)g$ for $|t| < \frac{1}{2|\lambda|}$ where λ is the Einstein constant with $Ric_g = \lambda g$.

The instantons interpretation of 4 dimensional Einstein manifolds are presented in [8]. The Einstein manifolds can be viewed as a problem in Hamiltonian dynamics. We explain how one can investigate existence of an Einstein structure via Hamiltonian vector field:

Let (M, g) be a Riemannian manifold. So TM has a natural structure of a symplectic manifold. The zero section is denoted by Z. We define a Hamiltonian on $T^0 M = TM \setminus Z$, via

$$H = \frac{Ric(V, V)}{|V|^2}. \tag{7}$$

For Einstein manifold this produce the trivial Hamiltonian dynamics(All points are singularity). But what about general case? What can be said about the critical points of this Hamiltonian?

References

1. M. Anderson, Calc. Var. **5**, (1997)
2. A. Beilinson and V. Drinfeld, *Chiral Algebras*, (American Mathematical Society Colloquium Publications, vol. 51, Providence, R.I., 2004)
3. D. Ben-Zvi, E. Frenkel, *Vertex algebras and algebraic curves*, (Second Ed. Providence, Rhode Island: American Mathematical Society, 2004)
4. D. Conti, F. Rossi,, J. . Pure and Appl. Algebra, **223**, (2019)
5. C. LeBrun, Math. Res. Lett. **3**, (1996)
6. J. Morgan, G. Tian, Clay math. Monograph **3**, (2007)
7. A. Sambusetti, *An obstruction to the existence of Einstein metrics on 4-manifolds*, (C. R. Acad. Sci. Paris, **322**, 1996)
8. J. Park, J. Shin, H. Seok Yang, Phys. Rev. D **105**, (2022)

TTMP vol 1(4), 2024
doi: 10.30511/TTMP.2024.2035805.1030

Transactions in Theoretical and Mathematical Physics

Investigation of Dense Coding with One of the Quasi-Bell States

Ardeshir Rabeie[1,a], **Ardalan Fatahizadeh**[1,b]

[1]Physics Department, Faculty of Science, Razi University, Kermanshah, Iran

Received: 16 July 2024 / Accepted: 12 November 2024 / Published: 21 November 2024

Abstract In this paper, we study the qubit-photon quantum logic utilizing a strong off-resonant coupling of a qubit and cavity. We study dense coding with one of the quasi-Bell states and propose an experimental scheme for dense coding with quasi-Bell measurement.

1 Introduction

Although dense coding is an important quantum communication protocol to communicate two classical bits of information by only transmitting a Bell state between the sender (Alice) and the receiver (Bob) [1–6], in this paper, we perform this protocol by sharing one of the quasi-Bell states. The quasi-Bell states are defined as:

$$|\phi_1\rangle = \frac{1}{\sqrt{N_1}}\left(|a\rangle|-a\rangle + |-a\rangle|a\rangle\right), \tag{1}$$

$$|\phi_2\rangle = \frac{1}{\sqrt{N_2}}\left(|a\rangle|-a\rangle - |-a\rangle|a\rangle\right), \tag{2}$$

$$|\phi_3\rangle = \frac{1}{\sqrt{N_1}}\left(|a\rangle|a\rangle + |-a\rangle|-a\rangle\right), \tag{3}$$

$$|\phi_4\rangle = \frac{1}{\sqrt{N_2}}\left(|a\rangle|a\rangle - |-a\rangle|-a\rangle\right). \tag{4}$$

where $N_1 = 2(1+e^{-4a^2})$, $N_2 = 2(1-e^{-4a^2})$, and $|\pm a\rangle$ are the standard coherent states [7–10]. The standard coherent state can be represented in terms of photon number states [11]:

$$|a\rangle = \exp\left(-\frac{|\alpha|^2}{2}\right)\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle, \tag{5}$$

where a is a complex amplitude. In quantum optics, the electromagnetic field can be decomposed into modes, where the dynamics of each mode in free space is equivalent to the dynamics of a simple harmonic oscillator, with n representing the number of photons in the given mode. The coherent state is a state for the corresponding electromagnetic field mode.

In this work, we first present an interesting method to create the quasi-Bell state $|\phi_2\rangle$, then we study the dense coding by sharing this state[1].

2 Creation of the Quasi-Bell State $|\phi_2\rangle$

The qubit-photon quantum logic via strong off-resonant coupling between a qubit and cavity is modeled by the dispersive Hamiltonian [12]:

$$\frac{H}{\hbar} = \omega_q|e\rangle\langle e| + \omega_s a^\dagger a + \chi_{qs}a^\dagger a|e\rangle\langle e|, \tag{6}$$

where $\omega_{q,s}$ are the qubit and cavity, $|e\rangle$ is the excited state of the qubit, transition frequencies, $a\,(a^\dagger)$ is the lowering (raising) ladder operator of the cavity resonator and, and χ_{qs} is the dispersive interaction between these modes. This interaction generates a state-dependent shift in either the qubit or cavity transition frequency. We use this conditional frequency shift to

[a]Email: rabeie@razi.ac.ir
[b]Email: ardalan_fatahizadeh@yahoo.com

[1]Under conditions, the quasi-Bell state $|\phi_2\rangle$ has maximum entanglement and this is very important for sharing

produce qubit-photon entanglement with two operations: conditional cavity phase shifts and conditional qubit rotations. The condi-tional cavity phase shift may be represented as

$$C_{\Phi} = e^{i\Phi' a^{\dagger} a |e\rangle\langle e|} = I \otimes |0\rangle\langle 0| + e^{i\Phi' a^{\dagger} a} \otimes |e\rangle\langle e|, \quad (7)$$

where Φ' is the conditional phase shift induced on the cavity state and $|0\rangle$ is the ground state of the qubit. This conditional phase emerges from the free evolution of the dispersive Hamiltonian for a time τ, where $\Phi = \chi_{qs}\tau$. The conditional qubit rotation is a rotation on the qubit state for the photon number of the cavity state.

Because the qubit transition frequency strongly depends on the photon number, we can choose the specific transition in the cavity Fock state. The rotation on the qubit state for an n-photon Fock state can be ideally described as follows [12]:

$$R^{n}_{\vec{n}',\theta}(\theta,\eta) = |n\rangle\langle n| \otimes R_{\vec{n}',\theta} + I \sum_{m\neq n} |m\rangle\langle m| I, \quad (8)$$

where $R_n(\theta,\eta) = e^{-i\frac{\theta}{2}(\vec{\sigma}\cdot\vec{n}')}$ is a qubit rotation around the vector $\vec{n}$ with rotation angle θ, and $\vec{\sigma} = \sigma_x\hat{i} + \sigma_y\hat{j} + \sigma_z\hat{k}$, where σ_x, σ_y, and σ_z are the Pauli matrices.

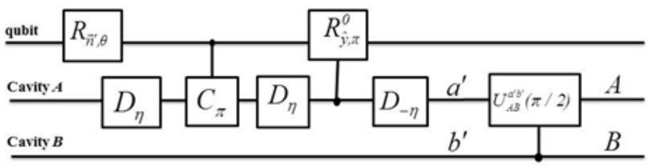

Fig. 1: A plan to create the quasi-Bell state $|\phi_2\rangle$ in several steps: qubit state preparation using a single qubit rotation $R_{\vec{n}',\theta}$; mapping the qubit to the cavity state A using conditional operations C_π and $R^0_{\vec{n}',\theta}$ along with cavity displacements D_η; and mapping the cavity state A to the cavity state B using a 50:50 beam splitter $U^{a'b'}_{\mathrm{AB}}(\pi/2)$.

We suggest an experimental setup as shown in Fig. 1 to create the quasi-Bell state $|\Phi_2\rangle$. We start with an unentangled qubit/cavity state: $\zeta_0 = \eta\otimes(|0\rangle - |e\rangle)$, (ignore the normalization), where $|\eta\rangle$ is a coherent state. Performing a conditional cavity π-phase shift on the initialized state creates an entangled qubit-cavity state A $|\zeta_1\rangle = C_\pi|\zeta_0\rangle = |\eta, 0\rangle - |-\eta, e\rangle$. With acting cavity displacements D_η on this state, we have $|\zeta_2\rangle = D_\eta|\zeta_1\rangle = |2\eta, 0\rangle - |0, e\rangle$. We here apply a qubit π-rotation conditional on the cavity vacuum state $|0\rangle$, which produces the unentangled state: $|\zeta_3\rangle \approx R^0_{\hat{y},\pi}|\zeta_2\rangle = (|2\eta\rangle - |0\rangle)\otimes|0\rangle$. With acting cavity displacements $D_{-\eta}$ on this state, we have $|\zeta_4\rangle = D_{-\eta}|\zeta_3\rangle = (|\eta\rangle - |-\eta\rangle)\otimes|0\rangle$. Finally, suppose that each mode of the cavity A and cavity B is incident on the beam splitter. After passing through the beam splitter, the quasi-Bell state becomes:

$$\begin{aligned}|\zeta_5\rangle_{AB} &= U^{a'b'}_{\mathrm{AB}}(\phi' = \pi/2)\,(|\eta\rangle_{a'} - |-\eta\rangle_{a'}) \otimes |0\rangle_{b'}\\ &= \left(|\eta/\sqrt{2}\rangle_A |-\eta/\sqrt{2}\rangle_B - |-\eta/\sqrt{2}\rangle_A |\eta/\sqrt{2}\rangle_B\right), \quad (9)\end{aligned}$$

where the 50:50 beam splitter for two field modes is represented as $U^{a'b'}_{\mathrm{AB}}(\phi') = e^{-\frac{\phi}{2}\left(a^\dagger_{a'}a_{b'} - a_{a'}a^\dagger_{b'}\right)}$ where a' and b' are the two field modes entering the beam splitter, A and B are the two field modes passing through the beam splitter, and ϕ is related to the transmissivity as $\theta = \cos^2(\phi/2)$. For $\alpha = \eta/\sqrt{2}$, the state $|\zeta_5\rangle_{\mathrm{AB}}$ reduces to the quasi-Bell state $|\phi_2\rangle_{\mathrm{AB}}$.

3 Dense Coding Process

Let us assume that the quasi-Bell state $|\phi_2\rangle_{\mathrm{AB}}$ has been shared between Alice and Bob. Alice wants to send Bob a binary number $x \in \{00, 01, 10, 11\}$. She chooses one of the following quasi-Pauli gates:

$$I = |-\alpha\rangle\langle -\alpha| + |\alpha\rangle\langle\alpha|, \quad (10)$$

$$X = |\alpha\rangle\langle -\alpha| + |-\alpha\rangle\langle\alpha|, \quad (11)$$

$$Y = -i|\alpha\rangle\langle -\alpha| + i|-\alpha\rangle\langle\alpha|, \quad (12)$$

$$Z = |\alpha\rangle\langle\alpha| - |-\alpha\rangle\langle -\alpha|. \quad (13)$$

According to x, Alice applies the transformation on mode A. Applying the transformation for mode A means she applies an identity transformation for mode B, which Bob keeps with him. By applying these conditions, the shared mode changes as follows:

$$00 \to I \otimes I \to |\phi_2\rangle_{\mathrm{AB}}, \quad (14)$$

$$01 \to X \otimes I \to |\phi_4\rangle_{\mathrm{AB}}, \quad (15)$$

$$10 \to Y \otimes I \to |\phi_3\rangle_{\mathrm{AB}}, \quad (16)$$

$$11 \to Z \otimes I \to |\phi_1\rangle_{\mathrm{AB}}. \quad (17)$$

When α is reasonably large, after performing one of the operations described above, Alice can send her mode A to Bob using a quantum network through some conventional physical medium. Now the two modes have been sent to Bob, and he can find the transmitted state using the quasi-Bell measurement [13]. Additionally, according to the contract between himself and Alice, Bob knows which pair of binary numbers $x \in \{00, 01, 10, 11\}$ has been sent by Alice.

We suggest an experimental setup, as shown in Fig. 2, to discriminate quasi-Bell states when α is reasonably large. To discriminate between the quasi-Bell states, we use two photodetectors and a 50:50 beam splitter for modes A and B. After passing through the beam splitter, the quasi-Bell states become:

$$|\Phi_1\rangle_{AB} \rightarrow \frac{1}{\sqrt{N_1}}|0\rangle_F|\text{even}\rangle_G,$$

$$|\Phi_2\rangle_{AB} \rightarrow \frac{1}{\sqrt{N_2}}|0\rangle_F|\text{odd}\rangle_G,$$

$$|\Phi_3\rangle_{AB} \rightarrow \frac{1}{\sqrt{N_1}}|\text{even}\rangle_F|0\rangle_G,$$

$$|\Phi_4\rangle_{AB} \rightarrow \frac{1}{\sqrt{N_2}}|\text{odd}\rangle_F|0\rangle_G,$$

where $|\text{even}\rangle = \frac{|\sqrt{2}\alpha\rangle+|-\sqrt{2}\alpha\rangle}{\sqrt{N_1}}$ and $|\text{odd}\rangle = \frac{|\sqrt{2}\alpha\rangle-|-\sqrt{2}\alpha\rangle}{\sqrt{N_2}}$ have been termed respectively the *even* and *odd* coherent states. Also the term Cat states have been applied to the even and odd coherent states. The even-odd terminology refers to the fact that the photon number distribution is non-zero only for even photon numbers in the case of the even coherent state and is non-zero only for odd photon numbers in the case of the odd coherent state.

If an odd number of photons is detected at detector L for mode F, then we know that the entangled state incident on the measurement setup was $|\Phi_4\rangle \rightarrow 01$.On the other hand, if an odd number of photons is detected at detector R for mode G, then the incident entangled state was: $|\Phi_2\rangle \rightarrow 00$. If a non-zero even number of photons is detected for mode F, the incident state was $|\Phi_3\rangle \rightarrow 10$ and if a non-zero even number is detected for mode G, it was $|\Phi_1\rangle \rightarrow 00$.

4 Conclusions

In this paper, we have studied the qubit-photon quantum logic utilizing a strong off-resonant coupling of a qubit and cavity. We have investigated the creation of the quasi-Bell state $|\Phi_2\rangle$. We have studied dense coding with one of the quasi-Bell states (as channels). Additionally, we have proposed an experimental scheme for dense coding with quasi-Bell measurement.

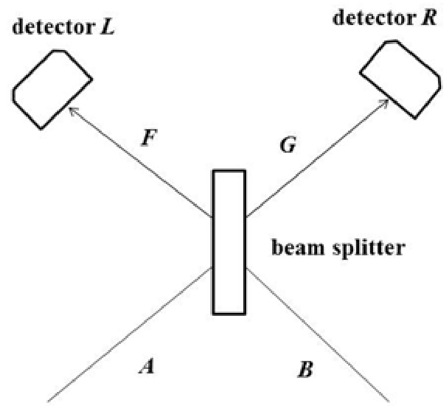

Fig. 2: Schematic diagram of the detection setup. A beam splitter separates the paths into detectors L and R, allowing for discrimination between different quasi-Bell states. Detectors L and R are positioned to observe the outputs F and G from inputs A and B.

References

1. C.H. Bennett and S.J. Wiesner, Phys. Rev. Lett., **69** (1992)
2. B.P. Williams, R.J. Sadlier, T.S. Humble, Phys. Rev. Lett., **118** (2017)
3. X.M. Hu, Y. Guo, B.H. Liu, Y.F. Huang, C.F. Li, G.C. Guo, Sc. Adv. **4**, (2018)
4. T. Schaetz, M.D. Barrett, D. Leibfried, J. Chiaverini, J. Britton, W.M. Itano, J.D. Jost, C. Langer, D.J. Wineland, Phys. Rev. Lett., **93**, (2004)
5. C. Wang, F.G. Deng, Y.S. Li, X.S. Liu, G.L. Long, *Quantum secure direct communication with high-dimension entangled states*, Phys. Rev. A, **71**, 044305 (2005)
6. M.A. Nielsen and I.L. Chuang, *Quantum Computation and Quantum Information*, 10th Anniversary Edition, Cambridge University Press (2010)
7. C. Wang, X. Zhu, W. Cui, Y. Gao, X. Li, *Dense coding with photon pairs*, Science, **352**, 1087 (2016)
8. J.P. Gazeau, *Coherent States in Quantum Physics*, Wiley-VCH Verlag GmbH (2009)
9. B. Vlastakis, G. Kirchmair, Z. Leghtas, S.E. Nigg, L. Frunzio, S.M. Girvin, M. Mirrahimi, M.H. Devoret, R.J. Schoelkopf, Science, **342** (2013)
10. K. Park and H. Jeong, Phys. Rev. A, **82** (2010)
11. D.E. Souza, V. Barranco, Optik, **244** (2021)
12. J. Yamauchi, M. Takeoka, M. Sasaki, *Dense coding over optical fibers using quasi-Bell states*, Int. Sym. Inf. Theory and Appl. (2020)
13. M.H. Devoret, R.J. Schoelkopf, Science, **339** (2013)

TTMP vol 1(4), 2024
doi: 10.30511/TTMP.2024.2027623.1025

Transactions in Theoretical and Mathematical Physics

A mathematical procedure for work of the friction force on the arbitrary path

Mehdi Jafari Matehkolaee[1,a], **Mohadese Sadat Hosaini Aeen, Aidin Fam Zagharami**

[1]Department of Physics and Energy Engineering, Amirkabir University of Technology (Tehran Polytechnic), Hafez Avenue, Tehran, Iran
[2]Department of Physics, Alzahra University, Tehran 19938-93973, Iran
[3]Seneca Polytechnic College Newnham Campus,Toronto,Canada

Received: 28 April 2024 / Accepted: 15 November 2024 / Published: 21 November 2024

Abstract In this paper, we have obtained the work of friction force on arbitrary paths. Our results indicate that the The work of the friction force depends highly on the path equation. Our explicit calculations show that the procedure and related examples are not simple.

1 Introduction

In elementary classical mechanics, the work done by friction is introduced as the change of mechanical energy of the system. Whenever the energy of the system is not conserved, we can write:

$$W' = \Delta E = \Delta U + \Delta K \tag{1}$$

In this equation, W' is the work done by friction, ΔE is the change in mechanical energy, and ΔU and ΔK are the changes in potential and kinetic energy, respectively. Obviously, Eq. (1) shows that the work done by friction consists of two parts: the change in potential energy, which is path-independent if the force is conservative, and the change in kinetic energy, which is path-dependent. If we want to calculate the work done by friction directly without using Eq. (1), then we can write the following:

$$W' = \int \overrightarrow{\mathbf{F}} \cdot \overrightarrow{d\mathbf{l}}. \tag{2}$$

In this procedure, it is not important to know the speed values at the start and end points of the path. However, the equation of the path is what is very important. This procedure with the mentioned purposes has been reported in [1]. However, we indicate that this paper [1] did not lead to the correct results.

A different article [3] has shown that the direct equation of work of friction on a particular curved path leads to Eq. (1). This paper [3] somehow concludes the equivalency between two equations (1) and (2).

In some textbook exercises, we can see that the action of dissipative forces leads to entering the domain of thermodynamics [2]. Indeed, the work done by frictional forces has often been calculated incorrectly [4]. The key to a correct treatment lies in distinguishing between the energy equation and the center of mass equation [4–6]. This subject leads to an informal concept called "Pseudowork" [7]. In any case, this concept is not considered in our work. Section 2 begins with a critique of reference [1] and continues with direct calculations of the work done by the friction force. An example is given in Section 3 to test the ultimate relationship for the work of friction.

2 Calculation of the Work of Friction

Now consider a particle sliding on a curved path with the equation $y = f(x)$, under the action of gravity. The final equation in Ref. [1] is given as:

$$W' = \frac{1}{2}\mu mg \int \frac{1 + [f'(x)]^2}{2 + [f'(x)]^2 - \mu f'(x)} dx, \tag{3}$$

where m is the mass of the particle and μ is the coefficient of friction.

We believe that Eq.(3) is not correct. At the very least, it does not give the correct result in limiting cases. For example, consider the case of an inclined plane. In this case, $f'(x) = \tan\theta$, where θ is constant. Substituting this into Eq. (3) from [1] gives:

[a]e-mail: m.matehkolaee@aut.ac.ir

$$W' = 2\mu mg \tan\theta 2 \frac{1+\tan^2\theta}{2+\tan^2\theta - \mu\tan\theta}\Delta x. \tag{4}$$

With respect to $\Delta x = (\Delta l)\cos\theta$, we can write:

$$W' = (\mu mg\cos\theta)(\Delta l)\frac{2}{1+\cos^2\theta - \mu\sin\theta\cos\theta}\Delta x. \tag{5}$$

But for an inclined plane, from basic classical mechanics we know that $N = mg\cos\theta$ and $W' = (\mu mg\cos\theta)(\Delta l)$ which is not the same as Eq. (5).

In fact, the definition of speed in the mentioned article is problematic. Specifically, the equations:

$$v = \frac{dl}{dt}, \quad \text{and} \quad v \neq \frac{dl}{dx}.$$

Newton's law, projected along the path and normal to the path give the following equations:

$$\frac{1}{g}\frac{dv}{dt} = -\sin\theta - \mu n, \tag{6}$$

$$\frac{v^2}{gR} = n - \cos\theta, \tag{7}$$

where R is the curvature radius of the path, given by $R = \frac{dl}{d\theta}, \quad n = \frac{N}{mg}$. By using

$$\frac{dv}{dt} = \frac{dv}{dl}\frac{dl}{dt} = v\frac{dv}{dl}, \tag{8}$$

and defining:

$$\psi = \frac{v^2}{g} = R(n - \cos\theta), \tag{9}$$

we obtain:

$$\frac{d\psi}{dl} = -2(\sin\theta + \mu n). \tag{10}$$

So:

$$\frac{d\psi}{d\theta} = -2R(\sin\theta + \mu n) = -2\mu\psi - 2R(\mu\cos\theta + \sin\theta). \tag{11}$$

Integrating this equation results in:

$$\psi(\theta) = \exp\left(-2\mu(\theta-\theta_1)\right)\Big\{\psi(\theta_1) - 2\int_{\theta_1}^{\theta} d\varphi \\ (\mu\cos\varphi + \sin\varphi)R(\varphi)\exp\left[2\mu(\varphi-\theta_1)\right]\Big\}. \tag{12}$$

Assuming the particle starts from rest, we conclude:

$$\psi(\theta) = -2\int_{\theta_1}^{\theta}(\mu\cos\varphi + \sin\varphi) \\ \times R(\varphi)\exp\left(2\mu(\varphi-\theta)\right)d\varphi. \tag{13}$$

Therfore

$$n(\theta) = \cos\theta - \frac{2}{R(\theta)}\int_{\theta_1}^{\theta}(\mu\cos\varphi + \sin\varphi) \\ \times R(\varphi)\exp\left[2\mu(\varphi-\theta)\right]d\varphi. \tag{14}$$

Finally, W' or the work of friction is given by:

$$W' = \mu mg\int_{\theta_1}^{\theta_2} R(\theta)n(\theta)d\theta. \tag{15}$$

3 Example

Consider a trajectory with the following equation:

$$f(x) = \sqrt{x}. \tag{16}$$

and:

$$\tan\theta = -\frac{1}{2\sqrt{x}}, \quad \text{and} \quad \int_0^1 d\xi\sqrt{1+\frac{1}{4\xi}}. \tag{17}$$

Defining the variable α as $\sinh(\alpha) = \frac{1}{2\sqrt{x}}$, we get

$$\frac{d\theta}{d\alpha} = \frac{1}{\cosh\alpha}, \quad \frac{dl}{d\alpha} = \frac{\cosh^2 h\alpha}{2}, \quad R = \frac{\cosh^3 h\alpha}{2}. \tag{18}$$

We can rewrite Eqs (14) and (15) in terms of α instead of θ:

$$\theta = q(\alpha), \tag{19}$$

where:

$$g(\alpha) = -\frac{\pi}{2} + \int_0^\alpha \frac{d\beta}{\cosh\beta} = -\frac{\pi}{2} + \tan^{-1}(\sinh\alpha), \tag{20}$$

and

$$R\left[\rho(\alpha)\right] =: \rho(\alpha), \quad n\left[\rho(\alpha)\right] =: \nu(\alpha). \tag{6}$$

These latter equations give $R(\alpha)$ and $n(\alpha)$ in terms of α instead of θ. Additionally, defining $\lambda(\alpha)$ as:

$$\lambda(\alpha) = \rho(\alpha)q'(\alpha) = \frac{\cosh^2 h\alpha}{2}, \tag{22}$$

we arrive at:

$$\nu(\alpha) = \cos\left[q(\alpha)\right] - \frac{2}{\rho(\alpha)}\int_{\alpha_1}^{\alpha}\Big\{\mu\cos\left[q(\beta)\right] + \sin\left[q(\beta)\right]\Big\} \\ \times \exp\Big\{\mu\left[q(\beta) - q(\alpha)\right]\Big\}\lambda(\beta)d\beta. \tag{23}$$

Finally, the work of friction is:

$$W' = \mu mg\int_{\alpha_1}^{\alpha} d\beta\lambda(\beta)\nu(\beta). \tag{24}$$

4 Conclusion

We have investigated, in a general form, the direct calculation of the work done by the friction force. As observed, explicit calculations of the integrals may not be straightforward. However, we invite the reader to examine the same procedure for a given path $f(x) = -x^2$, because the radius of curvature of these two functions is identical. Comparing the work of friction on these two paths and analyzing the speed of objects sliding along these paths at the end of the trajectories is highly desirable.

Acknowledgements The authors would like to thank Professor Mohammad Khorrami for critical reading and precious comments.

References

1. M.J. Matehkolaee, K. Majidian, *Eur. J. Phy. Edu.*, **4**, 2013
2. R. Resnick, D. Halliday, and K. S. Krane, *Physics (Vol. 1)*, 4th Ed. New York: Wiley, 1992
3. T.S. Aurora, *Phy. Edu.*, **29**, 1994
4. B.A. Sherwood, W.H. Bernard, *Amer.J. Phy.*, **52**, 1984
5. D.C.H. Poon, *The Phy. Teacher*, **44**, 2006
6. C. E. Mungan, *The Phy. Teacher*, **45**, 2007
7. B. A. Sherwood, *Amer.J. Phy.*, **51**, 1983