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TTMP vol 2 (1), 2025
doi: 10.30511/TTMP.2025.2045267.1044

Transactions in Theoretical and Mathematical Physics

Bound state current density of an isolated vortex line in a superconductor

Elaheh Abdolrahimi Niari[1a], **Neda Ebrahimian**[2b], **Reza Afzali**[1c]

[1] Department of Physics, K. N. Toosi University of Technology, Tehran 15875-4416, Iran
[2] Department of Physics, Faculty of Basic Sciences, Shahed University, Tehran 3319118651, Iran

Received: 08 November 2024 / Accepted: 04 January 2025 / Published: 18 January 2025

Abstract We investigate the vortex lines in a heterogeneous superconducting system. Using Bogoliubov's equations, we calculate the equations governing the vortex line structure in this system. Following that, we use the Wentzel-Kramers-Brillouin (WKB) approximation and numerically calculate the bound state current of the isolated vortex line. Finally, we study the behavior of current in terms of temperature.

1 Introduction

Vortex lines do not exist in a homogeneous microscopic system because they are unstable in a magnetic field where $H < H_{c1}$ and only form in a magnetic field lattice when $H > H_{c1}$. When analyzing the properties of an isolated vortex line, potential effects related to the order parameter should be disregarded. For temperatures close to the critical temperature T_c, only the Ginzburg-Landau (G-L) theory is applicable for calculating the properties of vortex lines. However, a microscopic theory is necessary for investigating more detailed aspects, such as the excitation spectra within vortex cores or the spatial dependence of the field, the current density, and order parameters at distances far from T_c. We will follow the approach used by Caroli et al. (1964) and later employed by Bardeen, Kummel, Jacobs, and Tewordt (1969), which uses the Bogoliubov method [1].

The Bogoliubov method is utilized in superconductors to describe the behavior of electrons in the presence of an effective potential arising from Cooper pairing and magnetic vortices. This approach aids in analyzing the energy spectrum and the distribution of electronic density of states in type-II superconductors. Moreover, it is employed to study the effects of magnetic fields and structural defects on electronic behavior within superconductors [2].

In the microscopic theory, the current density is related to the Bogoliubov wave functions, u_n, v_n, and the vector potential $\mathbf{A}$. We consider a specific gauge where the vector potential for the real gap function describes the magnetic field. The vortex line is expressed along the Z-axis in cylindrical coordinates. We compute the vortex line structure equations using the superconducting system's Hamiltonian and the wave functions in cylindrical coordinates. Two solutions are derived under boundary conditions: $r \to 0$ and $r \to \infty$. We then apply the WKB method's quasiclassical approximation to the Bogoliubov equations. We discuss the bound state of the vortex in the superconductor. Vortex-bound states play a significant role in the dynamic properties of superconductors, as they can influence local currents and energy transport. These bound states can act as energy channels within vortices, affecting the system's resistive behavior and electrical currents. Studying these bound states also provides valuable insights into quantum phase coherence and the effects of external fields on superconductors [3]. Subsequently, based on the energy of the bound state of the isolated vortex line, which depends on the magnetic quantum number, we calculate the bound state current density of the isolated vortex line as a function of temperature, and we compared some of the characteristics of vortices in superconductors with those in ultracold Fermi gases. Finally, we plot the current density as a function of temperature.

2 Vortex Line Structure Equations

We calculate the current density of the bound state of the vortex line using the WKB method and consider its temperature effects:

[a]e-mail: e.abdolrahiminiari@email.kntu.ac.ir
[b]e-mail: n.ebrahimian@shahed.ac.ir
[c]e-mail: afzali@kntu.ac.ir

$$\mathbf{J}(r) = \frac{e\hbar}{2mi} \sum_n \Big[f(E_n) u_n^* \big(\nabla - i\frac{e}{\hbar c}\mathbf{A}\big) u_n + \big(1 - f(E_n)\big) \nu_n \big(\nabla - i\frac{e}{\hbar c}\mathbf{A}\big) \nu_n^* - c.c. \Big]. \tag{1}$$

In Eq. (1), $f(E_n)$ is Dirac's Fermi function, and the vector potential is obtained from Maxwell's equations. u_n and v_n are the Bogoliubov wave functions. We choose a gauge for the calculations. In the London model, as $r \to 0$, the current density is given by $j_\theta = \frac{ne}{2mr}\hbar$. The current density varies inversely with the radius r, increasing near the cylinder's axis. This characteristic is essential for studying Type-II superconductors' circular currents with vortex structures. In these systems, Cooper pairs generate angular momentum around the cylinder's axis, which increases as the radius decreases. In the microscopic theory, the current density is related to the Bogoliubov wave functions and the vector potential. The following equations describe the effect of the gauge function $\chi(r)$ on the vector potential, the Bogoliubov wave functions, and the gap function:

$$A(r) = \nabla\chi(r), \tag{2}$$

$$u_n(r) = u_n^{(0)}(r) e^{i\frac{e}{\hbar c}\chi(r)}, \tag{3}$$

$$v_n(r) = v_n^{(0)}(r) e^{-i\frac{e}{\hbar c}\chi(r)}, \tag{4}$$

$$\Delta(r) = \Delta^{(0)}(r) e^{i\frac{2e}{\hbar c}\chi(r)}. \tag{5}$$

where $u_n^{(0)}$, $v_n^{(0)}$, and $\Delta^{(0)}$ are the solutions with $\chi = 0$. $\Delta(r)$ represents the gap function of the system. In the London gauge, the form of the vector potential in the limit as $r \to 0$ is given by: $A_\theta = \frac{\hbar c}{2er}$,
and the vector potential in the vortex is defined as:

$$A_\theta(r) = \frac{\hbar c}{2er} + A'_\theta(r). \tag{6}$$

Here, $A'_\theta \to 0$ as $r \to 0$. When r is smaller than the penetration depth λ, and A'_θ is given by:

$$A'_\theta = -\frac{1}{2} r h_0, \tag{7}$$

where h_0 is the magnetic field in the $-z$ direction on the axis of the vortex line. The vector potential is crucial for describing magnetic fields and analyzing vortices in various physical systems. In superconductors, particularly type II, the potential is instrumental in defining magnetic fields and analyzing magnetic vortices. The Ginzburg-Landau model is utilized to investigate the behavior of vortices and the influence of external magnetic fields on superconducting currents. On the other hand, the vector potential is mainly used to study the mechanical and quantum properties of vortices in ultracold Fermi gases. In these systems, vortices are characterized as quantized structures within the superfluid, and the Schrödinger equation is applied to explore their dynamics and internal features. External magnetic fields have a less significant impact in these contexts. In both situations, the vector potential facilitates the description of magnetic fields and the analysis of vortices. The effect of external magnetic fields on vortex distribution is more evident in superconductors. In contrast, in ultracold Fermi gases, the emphasis is placed on the superfluid's internal dynamics and the vortices' quantized nature [4, 5]. We perform a gauge transformation $A_\theta \to A_\theta + \frac{1}{r}\frac{\partial\chi(\theta)}{\partial\theta}$, where $\chi(\theta) = -\frac{\hbar c}{2e}\theta$. In this measurement, the vector potential vanishes in the limit of small r. Nevertheless, the form of the Bogoliubov wave functions is explicitly determined as a function of θ:

$$u_n \to u_n e^{-i\frac{\theta}{2}}, \tag{8}$$

and,

$$v_n \to v_n e^{i\frac{\theta}{2}}. \tag{9}$$

In this gauge, the gap function depends on θ:

$$\Delta(r) \to \Delta(r) e^{-i\theta}. \tag{10}$$

On the basis of the G-L theory, we expect $|\Delta(r)| \propto r$ as $r \to 0$, and $|\Delta(r)| \to \Delta_\infty$ as $r \to \infty$, where Δ_∞ is the equilibrium gap. We consider a vortex line of unit strength along the z-axis of cylindrical coordinates (r, θ, z). The Bogoliubov wave functions are written as a two-component spinor:

$$\begin{pmatrix} U_n \\ V_n \end{pmatrix} = \hat{f} e^{ik_z z} e^{i\mu\theta} e^{-\frac{1}{2}i\tau_z\theta}, \tag{11}$$

where τ_i denote the Pauli matrices, 2μ is an odd integer, and $f(r)$ is given as:

$$f = \begin{pmatrix} f_+(r) \\ f_-(r) \end{pmatrix}. \tag{12}$$

Eq. (12) represents the radial part of the wave function in cylindrical coordinates, which is related to the Bogoliubov wave functions according to Eq. (11). Here, $f_+(r)$ is associated with U_n, and $f_-(r)$ is associated with V_n and k_z is the wave vector along the z-axis in the superconductor. Bogoliubov's equations to determine the structure of the vortex line are as follows [6, 7].

$$E_n u_n(r) = [H_0' + U(r)]u_n(r) + \Delta(r)v_n(r), \tag{13a}$$

$$E_n v_n(r) = -[H_0'^* + U(r)]v_n(r) + \Delta^*(r)u_n(r), \tag{13b}$$

where E_n represents the excitation energy of the system. Note that in a magnetic field the kinetic energy operator, $H' = -\frac{\hbar^2}{2m}(\nabla - \frac{ie}{\hbar c}A)^2$, contains an imaginary cross term that changes sign on taking the complex conjugate; hence $H_0' \neq H_0'^*$. The operator H_0' is given by:

$$H_0' = -\frac{\hbar^2}{2m}(\nabla - \frac{ie}{\hbar c}A)^2 - \mu. \tag{14}$$

The Hamiltonian system in a superconductor is defined as follows:

$$H' = \frac{1}{2m}\left[\mathbf{p} - \frac{e}{c}A\tau_z\right]^2 + U(r) - \mu, \tag{15}$$

where $U(r)$ is a mean (one-electron) potential and μ is the chemical potential [1]. The Bogoliubov-de Gennes (BdG) equations are employed in the physics of superconductors to describe quasiparticle excitations. In these equations, the effective potential experienced by electrons plays a crucial role in determining the behavior and energy distribution of quasiparticles. This potential can arise from external fields, localized impurities, or structural variations within the material. The presence of these potentials affects the energy structure and wave functions of the quasiparticles, leading to changes in their excitation behavior and superconducting properties. External magnetic or electric fields may also generate vortices, altering the quasiparticles' energy distribution.

Localized potentials, such as impurities or structural defects, also influence the behavior of electrons and Cooper pairs. These potentials can create localized or bound states, leading to modifications in the superconducting energy gap. Variations in the local potential may reduce the strength of Cooper pairs, destabilizing the superconducting state. Thus, the electronic potential affects the energy distribution and plays a critical role in shaping the overall properties of the superconducting system [3–5].

Eq.(13) can be rewritten as follows:

$$E\psi = [H'\tau_z + \Delta_1\tau_x + \Delta_2\tau_y]\psi. \tag{16}$$

The complex gap function is expressed as $\Delta(r) = \Delta_1(r) + i\Delta_2(r)$, which is useful for some applications.

The wave function ψ, according to Eq. (11), is a two-component spinor used in the calculations. The expression of the Laplacian in cylindrical coordinates, the substitution of the wave function from Eq. (11), and the explanation of the gap function $\Delta(r)$ help us derive the equations of the vortex line structure as stated in Eq. (16). The general result of the differential calculations in r is given by:

$$\begin{aligned} \tau_z \frac{\hbar^2}{2m}\Big[&- \frac{d^2\hat{f}}{dr^2} - \frac{1}{r}\frac{d\hat{f}}{dr} \\ &+ \left(\mu - \frac{er\tau_z}{\hbar c}A_\theta(r)\right)^2 \frac{\hat{f}}{r^2} - k_\rho^2\hat{f}(r)\Big] \\ &+ \tau_x|\Delta(r)|\hat{f} = E\hat{f}, \end{aligned} \tag{17}$$

where k_ρ is a separation constant given by $k_z^2 + k_\rho^2 = k_F^2$ and $A_\theta(r)$ is defined by Eq. (6). The vortex line structure equations are employed for the precise description of vortices and the distribution of magnetic fields and electric currents around them. Through these analyses, the microscopic characteristics of vortices can be understood. Additionally, these equations allow physicists to model the dynamic behavior of vortices under the influence of magnetic fields and external forces.

The exact solution of Eq. (17), when $A_\theta(r) \equiv 0$, $\Delta(r) \equiv \Delta_\infty$, and $r \to \infty$ are desired, is as follows:

$$\begin{aligned} f(r) = &\frac{\text{const}}{\sqrt{2}} \begin{pmatrix} \sqrt{1 \pm \frac{(E^2-\Delta_\infty^2)^2}{E}} \\ \sqrt{1 \mp \frac{(E^2-\Delta_\infty^2)^2}{E}} \end{pmatrix} \\ &\times H_\mu^{(1)\cdot(2)}\left(k_\rho^2 \pm \frac{2m}{\hbar^2}(E^2 - \Delta_\infty^2)^2\right)^{\frac{1}{2}} r, \end{aligned} \tag{18}$$

where $H_\mu^{(1)\cdot(2)}(k_\rho r)$ are the Hankel functions of the first and second kind. For another exact solution in the vortex core when $A_\theta'(r) = 0$ and $\Delta(r) = 0$, we have

$$f_\pm \approx A_\pm J_{\mu\mp\frac{1}{2}}[(k_\rho \pm q)r], \tag{19}$$

where $q = \frac{mE}{\hbar^2 k_\rho}$ and $J_{\mu\mp\frac{1}{2}}(kr)$ represents the Bessel functions of the first kind. We apply the quasi-classical approximation to the Bogoliubov equations and write the general form of the solution of Eq. (17) using Eqs. (18) and (19) as:

$$\hat{f}(r) = \hat{g}(r) H_{\mu}^{(1)}(k_\rho r) + \text{c.c.}, \tag{20}$$

where $H_{\mu}^{(1)}(k_\rho r)$ represents the Hankel function of the fast oscillation of the radial partof the wave function, while $\hat{g}(r)$ is calculated for slow changes in amplitude and phase, and is generated slowly by different functions $A'_\theta(r)$ and $|\Delta(r)|$. We substitute Eq. (20) into Eq. (17) and ignore the terms $\frac{d^2 g}{dr^2}$ compared to $k_\rho \frac{dg}{dr}$. These terms are of the order $1/(k_\rho \xi)$ or Δ/E_F, where ξ is the coherence distance, E_F is the Fermi level energy. Additionally, we ignore terms involving A_θ^2.

$$-\frac{i\hbar^2}{m}\tau_z \beta(r)\frac{d\hat{g}}{dr} + \Delta(r)\tau_x \hat{g} = \Big[(E + \frac{\mu e\hbar}{mcr})A_\theta(r)\Big]\hat{g}, \tag{21}$$

and where we have:

$$\beta(r) = \frac{k}{r}\left(r^2 - r_t^2\right)^{1/2}. \tag{22}$$

We write Eq. (21) in dimensionless form by changing the following variables:

$$x = \frac{2m\Delta_\infty}{\hbar^2 k_\rho}\left(r^2 - r_t^2\right)^{1/2}, \tag{23}$$

$$\lambda = \frac{E}{\Delta_\infty}, \tag{24}$$

$$F(x) = \frac{\mu e\hbar}{mcr\Delta_\infty}A_\theta(r), \tag{25}$$

$$\delta(r) = \frac{\Delta(r)}{\Delta_\infty}. \tag{26}$$

In Eq. (23), $r_t = \mu/k_\rho$ represents the radial distance of the turning point. The equation for $\hat{g}(x)$ becomes:

$$-2i\tau_z\beta(r)\frac{d\hat{g}}{dx} + \delta(x)\tau_x\hat{g} = \Big[\lambda + F(x)\Big]\hat{g}. \tag{27}$$

To solve this equation, we express $\hat{g}$ in the following form:

$$\hat{g} = A\begin{pmatrix} e^{i\eta/2} \\ e^{-i\eta/2} \end{pmatrix} e^{i\xi}, \tag{28}$$

where A is a normalization factor, and η and ξ are in general complex functions of r. We substitute Eq. (28) into Eq. (27) and obtain two coupled equations for the coefficients of the Hankel function coefficients.

$$\frac{d\eta}{dx} + \delta(x)\cos(\eta) = \lambda + F(x), \tag{29}$$

$$2\frac{d\zeta}{dx} = i\delta(x)\sin(\eta). \tag{30}$$

These equations are to be solved subject to appropriate boundary conditions, which determine the eigenvalues λ. As x (or r) $\to \infty$, the following conditions hold: $F(x) \to 0$, $\delta(x) \to 1$, and $\eta(x) \to \eta_\infty$, where $\cos\eta_\infty = \lambda = E/\Delta_\infty$. For the bound states in the core, $|E| < \Delta_\infty$, η is real, and ξ is purely imaginary [8].

In the presence of a vortex, the superconducting pairing potential $\Delta(r)$ vanishes locally at the vortex core and gradually increases away from the core. This spatial variation in the pairing potential creates localized quantum-bound states near the vortex core. These bound states, known as Bogoliubov-de Gennes states, are trapped within the vortex due to the confining potential caused by the spatial variations in the pairing potential.

These bound states are localized and have energies distinct from the continuum of quasiparticle states, forming discrete bound states within the superconducting gap. The energies of these bound states typically lie within the superconducting energy gap, meaning they exist at lower energies than the bulk superconducting gap predicted by BCS theory [9].

We use the asymptotic form of the Hankel function in the (WKB) approximation.

$$H_\mu^{(1).(2)} \sim \frac{\exp\left(\pm i\int_{r_t}^{r}\beta(r')\,dr'\right)}{(r^2 - r_t^2)^{1/4}}, \tag{31}$$

The quasi-classical approximation using the WKB method in superconductors, particularly in vortex lines, is an effective tool for calculating the energy and current of quantum-bound states in these systems. Bound states form near vortex lines due to spatial variations in the superconducting order parameter and pairing potential, and these states can be approximated using the WKB method. This method allows for approximating the wave functions and energies of these states and calculating the associated currents. In the next section, we will examine the current of a bound state in an isolated vortex line in a superconductor under the influence of temperature [10].

3 Current Density

To calculate the bound state density of the vortex line, we consider $A'_\theta(r) = 0$. From Eq. (1), with $A_\theta(r) = \hbar c/(2er)$, the equation for the current density is as follows:

$$j_\theta(r) = \frac{|e|\hbar}{m} \sum_n \Big\{ |v_n(r)|^2 \frac{\mu}{r} - \Big[(|u_n(r)|^2 + |v_n(r)|^2) \frac{\mu}{r} \Big] f(E_n) \Big\}, \tag{32}$$

where $|E| < \Delta_\infty$, and $n = (k_z, \mu)$.. We want to convert the summation into an integral.

$$j_\theta(r) = \frac{A^2|e|\hbar}{m} \int_{-kF}^{+k_F} \int_0^{rk_\rho} \Bigg\{ \frac{\mu r^{-1}}{\big[r^2 - (\mu/k_\rho)^2\big]^{1/2}} \mu r^{-1} - \frac{2\mu r^{-1}}{\big[r^2 - (\mu/k_\rho)^2\big]^{1/2}} \cdot \frac{1}{1 + e^{\frac{E}{K_B T}}} \Bigg\} d\mu \, dk_z. \tag{33}$$

The normalization constant A is given by:

$$A = \frac{1}{\sqrt{4\pi r_w}}, \tag{34}$$

where r_w represents the average extent of the wave function. In Eq. (32), we replace the bound state energy of the vortex line, which depends on μ.

The bound state energy in a superconductor's vortex core arises from quasiparticles trapped along the vortex line. These states, quantized around the vortex core, typically have energies below the superconducting energy gap and are calculated using the Bogoliubov-de Gennes equations.

These bound states play a critical role in the dynamics and magnetic behavior of superconductors dynamics and magnetic behavior with their energies being dependent on system properties such as the magnetic field and pairing interactions. In comparison, ultracold Fermi gases in superfluid states can also exhibit vortex lines and bound quasiparticle states. However, the intrinsic differences in particle types and interactions lead to variations in the energy of these states. In such systems, fermionic atoms pair up at extremely low temperatures, displaying behavior similar to superconductors. Yet, the energy of bound states is more sensitive to environmental parameters and experimental configurations [9–11]. Energy is calculated using Eq. (29) for a step pair potential in the vortex core, where $\delta(r) = 0$ for $r < r_c$ and $\delta(r) = 1$ for $r > r_c$, where, r_c is the vortex core radius, which is of the order of the coherence distance $\xi_0 = \hbar v_F/\pi\Delta_\infty$.

Finally, we obtain the following relations for energy:

$$E = -\Delta_\infty \sin\psi_0 = \Delta_\infty \Big[1 - \frac{1}{2}(x_\lambda^2/b^2)\Big], \tag{35}$$

$$\Psi_0 = -\frac{1}{2}\pi + \tan^{-1}(x_\lambda/b), \tag{36}$$

where λ is the penetration depth, and x_λ is the value of x corresponding to $r = \lambda$. Although this is a nonphysical approximation, it is qualitatively accurate when $r_c \ll \lambda$. b and b_c are defined as follows:

$$b = \frac{\mu}{\mu_c} b_c, \tag{37}$$

$$b_c = \frac{2m r_c \Delta_\infty r_c}{\hbar^2 k_\rho}, \tag{38}$$

$\mu_c = k_\rho r_c$ is the value of μ given at the turning point at $r = r_c$. The relationship between x and b_c is as follows:

$$x = b_c \sqrt{\left(\frac{r}{r_c}\right)^2 - \left(\frac{\mu}{\mu_c}\right)^2}, \tag{39}$$

where at $r = r_c$, we have:

$$x_c = x(r_c) = b_c \sqrt{1 - \left(\frac{\mu}{\mu_c}\right)^2}. \tag{40}$$

The vortex core radius in a superconductor corresponds to the distance from the vortex center where the superconducting order entirely collapses, allowing the magnetic flux to penetrate the material. This radius is approximately equal to the coherence length (ξ), which characterizes the spatial scale over which the superconducting order parameter decays. Within this core, the Cooper pairs are destroyed, and the supercurrent ceases, leading to the collapse of superconductivity and full magnetic penetration.

The core radius plays a significant role in determining the local current density around the vortex. The current density is significantly reduced near the vortex core because the superconducting order is weakened, and quasiparticles transition more easily to the normal state near the vortex core. However, as the distance from the core increases, the superconducting order is gradually restored, and the current density increases due to the reformation of Cooper pairs and the re-establishment of the supercurrent.

Suppose the core radius is smaller (associated with a shorter coherence length). In that case, the region of reduced current density is confined to a smaller area, resulting in a sharper increase in current density at a shorter distance from the core. Conversely, suppose the core radius is larger (associated with a longer coherence length). In that case, the reduced current density is spread over a larger region, causing the peak current density to occur farther from the core.

The vortex core radius directly influences superconductors' bound state current density distribution. Regions near the core exhibit reduced current density due to the collapse of the superconducting order. In contrast, farther from the core, the current density reaches higher values as superconductivity is restored [5–9].

By substituting Eqs. (34), (35), (37), and (38) into Eq. (33), we arrive at the bound state current equation for the isolated vortex line. The resulting integral is then solved numerically.

$$j_\theta(r) = \frac{1}{4\pi r_w}\frac{2e\hbar}{m}\int_{-k_F}^{k_F}\int_0^{rk_\rho}\left\{\frac{\mu}{r}\left[r^2-\left(\frac{\mu}{k_\rho}\right)^2\right]^{-\frac{1}{2}} - 2\left[r^2-\left(\frac{\mu}{k_\rho}\right)^2\right]^{-\frac{1}{2}}\left(\frac{\mu}{r}\right) \times \frac{1}{1+\exp\left(\frac{\Delta_\infty}{k_B T}\left[1-\frac{1}{2}\left(\frac{x_\lambda \mu_c \hbar^2 k_\rho}{2\mu m \Delta_\infty r_c}\right)^2\right]\right)}\right\} d\mu\, dk_z. \quad (41)$$

To plot the current density (in arbitrary units) as a function of temperature, we assume $\hbar = 1$, $k_B = 1$, and consider $b/x_\lambda < 1$ and $x_\lambda > x_c$. Fig. 1 shows the current density as a function of $\frac{T}{E_F}$, and Fig. 2 shows the current density as a function of $\frac{E_F}{T}$, where E_F represents the Fermi energy.

As the temperature increases, the weakening of the Cooper pairs decreases the bound state current density of the isolated vortex line.

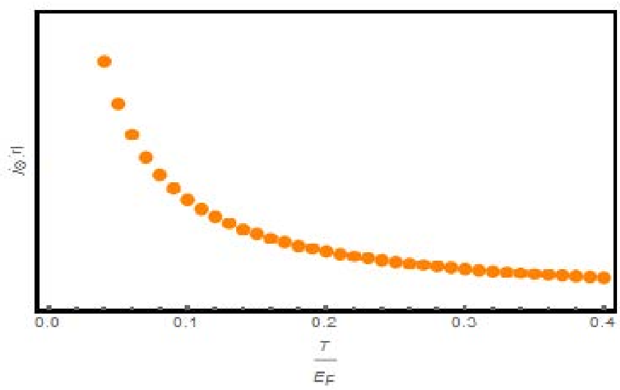

Fig. 1: The bound state current (in arbitrary units) of an isolated vortex line as a function of $\frac{T}{E_F}$.

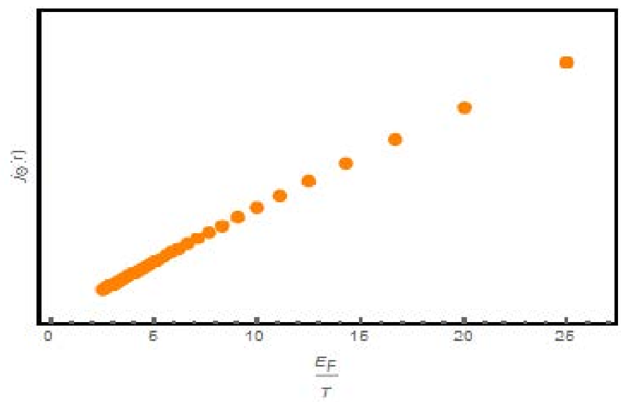

Fig. 2: The bound state current (in arbitrary units) of an isolated vortex line as a function of $\frac{E_F}{T}$.

4 Conclusion

We numerically investigated the isolated vortex line-bound state current with the WKB approximation and considered its temperature effects. According to the results, the bound state current density decreases as the temperature increases.

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TTMP vol 2 (1), 2025
doi: 10.30511/ttmp.2025.2053029.1046

Transactions in Theoretical
and Mathematical Physics

A class of integrable nonlinear evolution equations driven exclusively by extrinsic quadratic effects

A. Latifi[1,a]

[1] Department of Mechanics, Faculty of Sciences, Qom University of Technology, Qom, Iran

Received: 09 February 2025 / Accepted: 25 February 2025 / Published: 25 February 2025

Abstract In this work, we investigate a class of integrable nonlinear evolution equations where extrinsic quadratic contributions are the only source of nonlinearity. Within the Inverse Scattering Transform (IST) framework and using the ∂-bar method based on the Riemann-Hilbert approach, we extend the analysis to systems governed purely by singular dispersion relations, and we derive integrable equations that lack intrinsic nonlinearities and are driven only by the interaction between the evolution equations and their spectral problems. By suitable choices of singular dispersion relations, we derive the "truncated" Nonlinear Schrödinger (NLS) and Korteweg-de Vries (KdV) Equations that exhibit localized coherent structures and singular asymptotic behaviors that arise independently of initial conditions. The integrability of these systems is confirmed through their associated Lax pairs. To demonstrate the physical relevance of these systems, we derive them in the context of laser-plasma interactions, where they model the formation of propagating localized structures and energy transfer dynamics.

1 Introduction

The Inverse Scattering Transform (IST) [1, 2] has won physicists' consideration by offering a method to solve universal nonlinear evolution equations such as the nonlinear Schrödinger (NLS) and Korteweg-de Vries (KdV) equations. In this method, also called the Nonlinear Fourier Transform, a nonlinear evolution $q_t(x,t) = \mathcal{L}[q(x,t)]$, where $\mathcal{L}$ is a nonlinear operator, is associated to a convenient spectral operator. As is well known, the spectral operators associated with the NLS and the KdV equations are the Zakharov-Shabat [3] and the Sturm-Liouville operators [4], respectively. In this method, $q(x,0)$ plays the role of a "potential" for the spectral operator inducing spectral data at time $t=0$. A linear time evolution of these spectral data must be chosen obeying the so-called compatibility conditions [5]. Finally, $q(x,t)$ is reconstructed using the inverse scattering transform (IST).

An alternative approach, consists of writing the ∂-bar equation for the spectral operator [6]. In particular, the Zakharov-Shabat spectral operator (associated with the NLS equation) is written as follows:

$$\frac{\partial}{\partial \bar{k}}\psi(k) = \psi(k)R(k), \tag{1a}$$

$$\psi(k) = \mathbb{1} + O\left(\frac{1}{k}\right), \quad |k| \to \infty, \tag{1b}$$

where $\psi(k)$ and $R(k)$ are 2×2 matrices (for more details see Appendix A), $\partial/\partial\bar{k} = \partial/\partial k_R + \mathrm{i}\partial/\partial k_I$ is the $\overline{\partial}$-derivative with $k = k_R + \mathrm{i}\, k_I$, and $\mathbb{1}$ is the 2×2 unitary matrix. Note that a vanishing $\overline{\partial}$-derivative implies analyticity. $R(k)$ is the *jump operator* in the associated Riemann-Hilbert problem. The Sturm-Liouville spectral operator associated with the KdV equation is written as follows:

$$\frac{\partial}{\partial \bar{k}}\phi(k) = \phi(-k)r(k), \tag{2a}$$

$$\phi(k) = 1 + O\left(\frac{1}{k}\right), \quad |k| \to \infty, \tag{2b}$$

where $\phi(k)$ and $r(k)$ are given scalar distributions in $\mathbb{C}$ (for more details, see Appendix B), where $r(k)$ is the

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[a] e-mail: latifi@qut.ac.ir

jump function in the associated Riemann-Hilbert problem. R and r can be called "reflection coefficients" since they are related to spectral data. At this stage, one should choose a "simple" x and t dependency for R and r which will also imply, through (1a)-(1b) and (2a)-(2b), a dependency of ψ and ϕ on x and t.

To obtain the NLS equation with *source* (5a)-(5b)-(5c) from (1a)-(1b), one has to choose (Appendix A)

$$R_t = [R\ ,\ \Omega], \tag{3a}$$

$$R_x = [R\ ,\ \mathrm{i}k\sigma_3], \tag{3b}$$

where Ω is a 2×2 matrix and is the sum of a diagonal matrix of k-polynomials and a diagonal matrix of non-analytic (singular) functions of k. Likewise, to obtain the KdV equation with *source* (23a)-(23b) from (2a)-(2b), one must choose (Appendix B):

$$r_x(k) = 2\mathrm{i}kr(k), \tag{4a}$$

$$r_t(k) = [\beta(k) - \beta(-k)]r(k)k, \tag{4b}$$

where $\beta(k)$ is a sum of a k-polynomial and a non-analytic (singular) function of k. The time-derivatives of reflection coefficients $R_t(k,t)$ and $r_t(k,t)$ are called *dispersion relations*. If the dispersion relations do not contain any singular (non analytic) term, namely $\partial\Omega/\partial\bar{k} = 0$ and $\partial\beta/\partial\bar{k} = 0$, then one recovers the usual NLS and KdV equations without "source". Suppose the dispersion relation contains a singular term and its polynomial part. In that case, the compatibility condition yields the coupling of $q_t(x,t) = \mathcal{L}[q(x,t)]$ and its spectral problem [5].

These systems are integrable using IST. The Lax pair for the NLS and KdV coupled to their spectral problem are given in their most general form in [7] and [8], respectively.

In a more general context, the link between the Riemann Hilbert problem, the ∂-bar method and Lax pairs is presented in [9]. In the present work, we point out that if the dispersion relation contains no polynomials but only singular terms, then the evolution equation does not contain any intrinsic nonlinear or dissipative term; see Eqs. (9a) and (27a). The integrability of these equations is discussed in the Appendices, and their Lax pairs are given. These set of "truncated" coupled equations possess solutions evolving toward localized coherent structures with a singular asymptotic behaviour (blow up). The formation of this localized structure, *independent from the initial condition* $q(x,0)$ is due to the "extrinsic" nonlinearity and the "blow up" to the continuous energy transfer from the external media described by the spectral problem into the system described by the evolution equation. This work shows how the two major prototypes of universal nonlinear evolutions, namely the NLS and KdV equations, exhibit these properties. The relevant physical examples demonstrate the importance of this type of equations in physics.

In Sec. (2), starting from the NLS equation with source, we show how the choice of a proper singular dispersion relation leads to the "truncated" NLS equation coupled to its spectral operator. The lax pair is given, confirming its integrability. A related physical example is derived and solved. In Sec. (3), starting from the forced KdV equation, we show how the choice of a proper singular dispersion relation yields the "truncated" KdV equation coupled to its spectral operator. The Lax pair for this system is also given and confirms its integrability. In Sec. (4), using the multi-scale analysis, a related physical example, namely, a system of resonant coupled waves is derived in the context of the laser-plasma interaction. In Sec. (5), we present the method of solution to the system obtained is Sec. (3), based on the ∂-bar approach and we solve analytically the system. Finally, in Sec. (6), major results are discussed.

2 The truncated NLS equation coupled to its spectral problem and related physical model

We consider the following NLS equation coupled to its spectral problem

$$-\mathrm{i}q_t(x,t) + \frac{1}{2}q_{xx}(x,t) - q(x,t)|q(x,t)|^2 = \mathrm{i}\alpha\int_{-\infty}^{+\infty} d\lambda\ r_1(\lambda,x,t)\overline{r_2}(\lambda,x,t)\ , \tag{5a}$$

$$r_{1,x}(k,x,t) + \mathrm{i}kr_1(k,x,t) = qr_2(k,x,t)\ , \tag{5b}$$

$$r_{2,x}(k,x,t) - \mathrm{i}kr_2(k,x,t) = \bar{q}r_1(k,x,t), \tag{5c}$$

where α is a real constant and the over-head bar, the complex conjugation (throughout the paper).

The system (5a)-(5b)-(5c) associated to following conditions:

$$q(x,0) \in L^1(\mathbb{R})\ ;\ \lim_{x\to+\infty} r_1(k,x,t) = A(k,t)\mathrm{e}^{-\mathrm{i}kx}\ ;\ \lim_{x\to-\infty} r_2(k,x,t) = 0. \tag{6}$$

In the ∂-bar approach [10], (5a)-(5b)-(5c) associated to (6) has a singular dispersion relation, namely

$$\Omega(k,t) = \mathrm{i}k^2\sigma_3 - \frac{1}{\pi}\iint_{\mathbb{C}} \frac{d\lambda_R\, d\lambda_I}{\lambda - k} f(\lambda,t)\sigma_3, \tag{7}$$

where σ_3 is the Pauli matrix and $f(\lambda)$ any given distribution in the complex plane for $\lambda = \lambda_R + \mathrm{i}\lambda_I$. A physical application of the system (5)-(6) can be found in [11] for $f(\lambda,t) = (\mathrm{i}\pi/6)\delta(\lambda_I)\delta(\lambda_R - k_0)|A(k_R,t)|$ with $a_1(k_0,x,0) = 0$. The proof of integrability based on the ∂-bar approach is exhibited in Appendix A, while the Lax pair of the system can be found in [7].

Now, if we reduce the dispersion relation into its singular part

$$\Omega_s = -\frac{2\mathrm{i}}{\pi} P \int_{-\infty}^{+\infty} \frac{d\lambda}{\lambda - k} f(\lambda,t)\sigma_3, \tag{8}$$

where P denotes the Cauchy principal value of the integral. By setting $a_j(k,x,t) = r_j(k,x,t)e^{-\mathrm{i}kx}$, $j = 1,2$, system (5)-(6) is reduced to the "truncated" coupled system

$$q_t(x,t) = \int_{-\infty}^{+\infty} d\lambda\; f(\lambda,t)a_1\bar{a}_2, \tag{9a}$$

$$a_{1,x}(x,t) = q(x,t)a_2(x,t), \tag{9b}$$

$$a_{2,x}(x,t) - 2\mathrm{i}ka_2(x,t) = \bar{q}(x,t)a_1(x,t). \tag{9c}$$

Sys. (9), associated to initial/boundary conditions,

$$q(x,0) \in L^1(\mathbb{R})\ ;\ \lim_{x\to+\infty} a_1(k,x) = 1\ ;\qquad \lim_{x\to-\infty} a_2(k,x) = 0. \tag{10}$$

is also integrable. Its Lax pair is [7]

$$\mu_x + ik[\sigma_3,\mu] = Q\mu, \tag{11a}$$

$$\mu_t = (Cf)\mu\sigma_3 - \left((Cf)\,\mu\sigma_3\mu^{-1}\right)\mu, \tag{11b}$$

where $Cf(k)$ denotes the Cauchy integral of f:

$$Cf(k) = \frac{1}{2\mathrm{i}\pi}\int_{\mathbb{R}} \frac{f(l)}{l-k}dl, \qquad k \in \mathbb{C}, \tag{12}$$

and

$$Q = \begin{pmatrix} 0 & q \\ -\bar{q} & 0 \end{pmatrix}, \qquad \mu = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{13}$$

A physical application has been found [12] in the context of laser-plasma interaction, where a_1 and a_2 represent the small amplitude of laser beam components in the laboratory frame (T,z), namely

$$\begin{aligned}\mathcal{E} =& \varepsilon a_1(x,t)\exp\{[\mathrm{i}(\omega_1 T + k_1 z)]\} \\ &+ \varepsilon a_2(x,t)\exp\{[\mathrm{i}(\omega_2 T + k_2 z)]\} + O(\varepsilon^2),\end{aligned} \tag{14}$$

where $\varepsilon = (m_e/m_\mathrm{i})^2$ is the squared ratio of the electron and ion mass. q accounts for the fluctuations of electron density $n(z,T)$ around its average value n_0, through

$$\begin{aligned}& n(z,T)/(n_0) - 1 = \\ & \varepsilon(2k_1c^2)/(\mathrm{i}\omega^2)q(x,t)\exp\{[\mathrm{i}(\tilde{\omega}T + \tilde{k}z)]\} + O(\varepsilon^2),\end{aligned} \tag{15}$$

where s is the sound velocity, c is the speed of light, $\tilde{\omega}$ and $\tilde{k}$ are the frequency and the wave number mismatch between the two components a_1 and a_2 of the laser beam, respectively. $\tilde{\omega}$ and $\tilde{k}$ are expressed as follows:

$$\omega_1 = \omega_2 + \tilde{\omega}\ \ ;\ \ k_1 = k_2 + \tilde{k}. \tag{16}$$

In (9a), f is chosen to be

$$f = \frac{\omega_0^2}{\omega^2}\frac{Ze^2|A|^2}{2m_i m_e c_s^2 c}, \tag{17}$$

where $\omega_0 = 4\pi n_0 e^2/m_e$ is the plasma frequency, Ze is the plasma ion charge and A is the amplitude of the applied laser beam. We define the slow variables moving with the ion-acoustic waves as follows:

$$x = \epsilon(z + c_s T),\ \ t = \epsilon^2 T. \tag{18}$$

Within the framework defined by the variables (18), System (9) accounts for the Stimulated Brillouin Scattering effect and results from the low-frequency effect of the high-frequency electrostatic waves (ESW), $\mathcal{E}(x,t)$, by means of the ponderomotive force on the electrons, which acts as a source of ion-acoustic waves (ISW) $q(x,t)$. The time-asymptotic solution in the sharp line limit (i.e., as $A(k)$ tends to $\delta(k)$). Eqs. (9), completed by the initial-boundary conditions (10), read

$$a_1 = \frac{1+|\eta|^2}{1-|\eta|^2} \qquad a_2 = \frac{2\mathrm{i}\eta}{1-|\eta|^2}, \tag{19a}$$

$$q = -2\mathrm{i}\left(\frac{ft}{x}\right)^{1/2} \frac{\eta}{1-|\eta|^2}, \tag{19b}$$

where

$$\eta(x,t) = \frac{\mathrm{i}\rho\sqrt{\pi}}{2ft} u^{1/2}\left[1-\frac{27}{4u}+O(u^{-2})\right]e^{u/2}, \tag{20}$$

where

$$u(x,t) = 4\sqrt{fxt}, \qquad x>0, \tag{21}$$

and ρ is an arbitrary constant. One can see that $q(x,t)$ is a soliton-like solution for *any initial function* $q(x,0) \in L^1(\mathbb{R})$, representing the asymptotic behavior of the ion acoustic wave (IAW). Notice that $q(x,t)$ has singular points for $|\eta|^2 = 1$, that is, when $u(x,t)$ solves the equation

$$\left[\frac{2f}{\sqrt{\pi}\rho}\right]^2 t^2 e^{-u} = u\left[1-\frac{27}{4u}+O(u^{-2})\right]^{-2}. \tag{22}$$

3 The truncated KdV equation coupled to its spectral problem

We have proved [13] that the KdV equation coupled by its spectral problem

$$q_t + 6qq_x - q_{xxx} = \\ -2\frac{\partial}{\partial x}\int_{-\infty}^{+\infty} d\lambda\, \nu(\lambda)\,|u(\lambda,x,t)|^2, \tag{23a}$$

$$u_{xx} + k^2 u = qu, \tag{23b}$$

and associated with the following initial and boundary conditions

$$q(x,0)\in L^1(\mathbb{R}), \quad \lim_{x\to+\infty} u(k,x,t) = A(k,t)\mathrm{e}^{-\mathrm{i}kx}, \tag{24}$$

where $\nu(k)$ is a real function in $L^2(\mathbb{R})$, is integrable with the singular dispersion relation

$$\beta(k,t) = -4ik^3 - \mathrm{i}\int_{-\infty}^{+\infty} \frac{d\lambda}{\lambda-k}\nu(\lambda)\,|d(\lambda,t)|^2. \tag{25}$$

The proof of integrability of this System as well as of System (3) as a sub-case, based on the ∂-bar approach, is exhibited in Appendix B.

Now, if we reduce the dispersion relation to

$$\beta_r(k,t) = -\mathrm{i}\int_{-\infty}^{+\infty} \frac{d\lambda}{\lambda-k}\nu(\lambda)\,|d(\lambda,t)|^2, \tag{26}$$

System (23) is reduced to the "truncated" coupled system

$$q_t = -2\frac{\partial}{\partial x}\int_{-\infty}^{+\infty} d\lambda\, \nu(\lambda)\,|u(\lambda,x,t)|^2, \tag{27a}$$

$$u_{xx} + k^2 u = qu. \tag{27b}$$

The x-part of the Lax pair for System (27) is Eq. (27b) and its t-part is [8]:

$$u_t - \frac{1}{4k}\left[(H\nu)\left(u\bar{u} + \frac{1}{k^2}u_x\bar{u}_x + (u\hat{u})_x\right)\right]u_x \\ + \frac{1}{4\mathrm{i}}\left[(H\nu)\left(u\bar{u} - \frac{1}{k^2}u_x\bar{u}_x\right)\right]u = 0, \tag{28}$$

where

$$(H\nu)(k) = \frac{p}{\pi}\int_{\mathbb{R}} \frac{\nu(l)}{l-k}dl, \qquad k\in\mathbb{R}, \tag{29}$$

where H denotes the Hilbert transform and p denotes the principle value integral.

4 Derivation of a related physical example

In this section, we are going to use three set of variables: laboratory frame (ξ,τ), the dimensionless variables (X,T), defined in (32) and the slow variables (x,t), defined in (38). Let consider the propagation of a polarized electrostatic wave-packet (laser beam)

$$E(\xi,\tau) = \int_{-\infty}^{+\infty} d\omega \tilde{E}(\omega,\xi,T)\mathrm{e}^{-\mathrm{i}\omega\tau}, \tag{30}$$

in the laboratory frame (ξ,τ) with a frequency close to $\omega_0 = (4\pi e^2 n_0/m_e)^{1/2}$ in a uniform warm electron-cold ion plasma, where $\tilde{E}(\omega,\xi,T)$ is the slowly varying envelope of $E(\xi,\tau)$ and T the slow scaled time, defined below Eq. (32). The low frequency effect on the electrons due to the high frequency field E, results in the ponderomotive force

$$f_p = -\frac{e^2}{2m_e}\,\partial\xi\int_{-\infty}^{+\infty} d\omega\;\omega^{-2}\big|\tilde{E}(\omega,\xi,T)\big|^2, \tag{31}$$

which is the real part of $[\delta\xi(\partial E/\partial\xi)]$ around the average position $<\xi>$, where $\delta\xi$ is the solution of $m_e(\delta\xi)_{tt} = eE$. Using the small parameter of the system $\epsilon = (m_e m_i)^2$, where m_e is the mass of an electron and m_i the mass of an ion. The Debye wavelength is defined as $\lambda_D^2 = K_B T_e/4\pi n_0 e^2$, where T_e is the electron temperature and K_B is the Boltzmann constant. We set the following dimensionless set of variables

$$X = (\lambda_D)^{-1}\xi, \qquad T = \epsilon\omega_0\tau. \tag{32}$$

Using these variables and the dimensionless electrostatic field, the electrostatic potential and the velocity of ions, are defined as follows:

$$E' = E\frac{e}{\omega}\left(\frac{1}{2m_e K_B T_e}\right)^{-1/2} \quad ;$$

$$\phi' = \frac{e}{K_B T_e}\phi \;\; ; \;\; v_i' = v_i\left(\frac{m_e}{K_B T_e}\right)^{-1/2}, \tag{33}$$

where $-e$ is the charge of the electron. For $\omega^2 > {\omega_0}^2$, we can write the fluid-type equations [12,13]

$$\frac{\partial\phi'}{\partial X} - \frac{1}{(1+q_e)}\frac{\partial q_e}{\partial X} - \frac{\partial}{\partial X}\int_{-\infty}^{+\infty} d\omega\;\nu(\omega)\big|E'(\omega,X,T)\big|^2 = 0, \tag{34a}$$

$$\frac{\partial v_i'}{\partial T} + v_i'\,\frac{\partial v_i'}{\partial X} = -\frac{\partial\phi'}{\partial X}, \tag{34b}$$

$$\frac{\partial q_i}{\partial T} + \frac{\partial}{\partial X}\big[(1+q_i)\;v_i'\big] = 0, \tag{34c}$$

$$\frac{\partial^2\phi'}{\partial X^2} = q_e - q_i. \tag{34d}$$

Next, we expand ϕ, q_e q_i and v_i', in powers of ε as

$$q_i = \varepsilon q_i^{(1)} + \varepsilon^2 q_i^{(2)} + O(\varepsilon^3), \tag{35}$$

and, we define $\tilde{E}'$ and $\mathcal{E}$ by applying a multi-scale expansion to the amplitude of the electric field

$$E' = \varepsilon\mathcal{E} + O(\varepsilon^2), \qquad \tilde{E}' = \varepsilon^{3/4}\big(E' + O(\varepsilon)\big)\;. \tag{36}$$

In the the laboratory frame (ξ,t), the set of Eqs. (34) gives at first order

$$\frac{\partial^2 q}{\partial t^2} - \frac{\partial^2 q}{\partial\xi^2} = \varepsilon^{1/2}\frac{\partial^2}{\partial\xi^2}\int_{-\infty}^{+\infty} d\omega\;\nu(\omega)|\mathcal{E}(\omega,\xi,t)|^2, \tag{37}$$

where $q = q_e^{(1)} = q_i^{(1)}$. Introducing the slow variables

$$x = \varepsilon^{1/2}(X - T), \qquad t = \epsilon T, \tag{38}$$

co-moving with ISW at speed $C_s = (K_B T_e/m_i)^{1/2} = \varepsilon\omega_0\lambda_D$, Eq. (37) provides the evolution equation (27a).

On the other hand, the Maxwell equation for ESW,

$$\left[\frac{\partial^2}{\partial\tau^2} - 3(V_{Te})^2\frac{\partial^2}{\partial\xi^2}\right]E(\xi,\tau) = -\omega_0^2(1+q_e)E(\xi,\tau) \tag{39}$$

is obtained on the basis of the dispersion relation [14] $\omega^2 = \omega_p^2 + 3V_{Te}^2 k^2$, where $V_{Te} = \lambda_D\omega_0$ is the thermal electron velocity, k is the wave number and ω_p is the plasma frequency $\omega_p^2 = \omega_0^2(1+q_e)$, and, q_e is the fractional change in the electron density of average value n_0, namely $n_e = n_0[1+q_e(\xi,t)]$. Using the scalings and the expansions (32), (35), (36) and (38), the Maxwell equation gives at first order, the spectral problem (27b).

5 Method of solution

Based on the physical context described in the previous section, system (27a)-(27b) accounts for the following physical quantities: $u(\omega,x,t)$ represents the incoming electric field (laser beam), ω the frequency of the Pump closed to the plasma frequency and $q(x,t)$, the fractional change on electron density. By letting $\mu(k)$ be the profile of the source, in the (x,t)-frame moving with IAW, the physical system can be written as

$$q_t = \frac{\partial}{\partial x}\int_{-\infty}^{+\infty} d\lambda\ \mu(\lambda)\,|u(\lambda,x,t)|^2, \tag{40a}$$

$$u_{xx} + \lambda^2 u = qu. \tag{40b}$$

Notice that for convenience, we have slightly changed the system (27a)-(27b) by letting $\mu = -2\nu$. We associate a very general initial condition to system (40) by imposing $q(x,0)$ to obey the Faddeev condition, namely,

$$\int_{-\infty}^{+\infty} (1+|x|)\,|q(x,0)|dx < \infty. \tag{41}$$

In other words, any background noise in the fractional change of ion density in plasma will satisfy this condition. We also associate to system (40a)-(40b), the following boundary conditions

$$u(k,x,t) \to T(k,t)e^{\mathrm{i}kx}, \qquad x \to +\infty, \tag{42a}$$

$$\to e^{\mathrm{i}kx} + R(k,t)e^{-\mathrm{i}kx}, \qquad x \to -\infty, \tag{42b}$$

where, $R(k,t)$ and $T(k,t)$, stand for the reflection and transmission coefficients, respectively. The boundary conditions (42) are quite generic. Indeed, conditions (42) correspond to a normalized incoming laser beam from left to right, which is partially transmitted from to the right and seen at the vicinity of $+\infty$ as $T(k,t)e^{ikx}$ and partially reflected to the left and seen at the vicinity of $-\infty$, as $R(k,t)e^{-ikx}$.

In this section, we will obtain the time-asymptotic solution of system (40a)-(40b) complemented by the generic, initial/boundary conditions (41)-(42). The integral equation could be either the Marchenko equation obtained by the usual IST method [15] or the Cauchy-Green equation obtained by the ∂-bar approach. Here, we adopt the ∂-bar approach, which allows for obtaining directly the singular dispersion relation from the evolution equation and for writing the time evolution of the reflection coefficient in its simplest form [5].

By extending the parameter k to the complex plane, let us define the complex-valued function $f(k,x,t)$ as follows:

$$f(k,x,t) = u(k,x,t)e^{-ikx}, \tag{43}$$

and associate the following ∂-bar problem [16]

$$\frac{\partial}{\partial \bar{k}} f(k) = f(-k,t), \qquad k \in \mathbb{C}, \tag{44a}$$

$$f(k) = 1 + O(1/k), \qquad |k| \to \infty, \tag{44b}$$

to Eq.(40b). We consider the case where $q(x,0)$ has non-bound states, and we define $r(k,x,t)$ as follows:

$$r(k,x,t) = R(k,t)e^{2\mathrm{i}kx}\delta(k_I + 0), \tag{45}$$

where k_I stands for the imaginary part of k and δ for the Dirac distribution. We assume then, a simple time dependence for $r(k,x,t)$ in the form of

$$\frac{\partial}{\partial t} r(k) = [\beta(k) - \beta(-k)]r(k), \tag{46}$$

where β is a distribution obtained by the comparison of the evolution equation (40a) with

$$q_t = -\frac{1}{\pi}\iint_{\mathbb{C}} d\alpha \wedge d\bar{\alpha}\ u(\alpha)u(-\alpha)\frac{\partial}{\partial \bar{\alpha}}\beta(\alpha), \tag{47}$$

obtained from the compatibility condition $\partial_t(\partial_x^2 u) = \partial_x^2(\partial_t u)$. We remind that $d\alpha \wedge d\bar{\alpha} = -2\mathrm{i}\ d\alpha_R\ d\alpha_I$ where α_R and α_I are the real and imaginary parts of α, respectively. Notice that in Eq. (46), we have chosen odd coefficients because any even part can be scaled off through a gauge transformation of $f(\lambda)$. In our case, we choose:

$$\beta(k) = -\frac{1}{2}\mathrm{i}\int_{-\infty}^{+\infty} \frac{dl}{l-k}\,\frac{\mu(l)}{a\bar{a}(l)}, \tag{48}$$

where $a(k)$ is a scattering coefficient of the potential u in (40b). For clarity, let us recall briefly the definition of the scattering coefficients $a(k)$ and $b(k)$ and their relation to the reflection and transmission coefficients $R(k)$ and $T(k)$. Let us consider the Schrödinger equation

$$\psi_{xx}(k,x,t) + (k^2 - q)\psi(k,x,t) = 0, \quad k \in \mathbb{C}, \tag{49}$$

where ψ has the following asymptotic behavior

$$\psi(k,x,t) \to e^{\mathrm{i}kx}, \qquad x \to +\infty, \tag{50a}$$

$$\to a(k,t)e^{\mathrm{i}kx} + b(k,t)e^{-\mathrm{i}kx}, \quad x \to -\infty. \tag{50b}$$

For $k = \alpha > 0$, we have [16, 17],

$$u(\alpha,x,t) = \frac{1}{a(k,t)}\psi(k,x,t)\Big|_{k=\alpha}, \tag{51}$$

and

$$T(\alpha,t) = \frac{1}{a(k,t)}\Big|_{k=\alpha}, \qquad R(\alpha,t) = \frac{b(k,t)}{a(k,t)}\Big|_{k=\alpha}. \tag{52a}$$

Now, inserting Eq. (48) into Eq. (46), yields

$$\frac{\partial}{\partial t}R(k,t) = R(k,t)\int_{-\infty}^{+\infty}\frac{-\mathrm{i}k}{l^2-(k+\mathrm{i}\epsilon)^2}\frac{\mu(l)}{a\bar{a}(l)}dl. \tag{53}$$

The potential $q(x,t)$ is obtained by solving the basic integral equation

$$f(k,x,t) = 1 + \frac{1}{2\mathrm{i}\pi}\iint_{\mathbb{C}}\frac{dl\wedge d\bar{l}}{l-k}\, f(-l,x,t)R(l,t)\, e^{2\mathrm{i}kx}\delta(k_I+\mathrm{i}\epsilon), \tag{54}$$

which solves the ∂-bar problem (44). The solution $q(x,t)$ of (40b) can be obtained from f as follows [5]:

$$q = -2\mathrm{i}\frac{\partial}{\partial x}f^{(1)}(x,t)\ , \tag{55}$$

where $f^{(1)}$ is the coefficient of $1/k$ in the Laurent series expansion of $f(k,x,t)$. We now have to solve the evolution given by Eq. (53), which can be written as

$$\frac{\partial}{\partial t}R(k,t) = R(k,t)\left(\frac{\pi\mu(k)}{a\bar{a}(k)} - \mathrm{i}k\ P\int_{-\infty}^{+\infty}\frac{dl}{l^2-k^2}\frac{\mu(l)}{a\bar{a}(l)}\right), \tag{56a}$$

$$\frac{\partial}{\partial t}\overline{R}(k,t) = \overline{R}(k,t)\left(\frac{\pi\mu(-k)}{a\bar{a}(k)} + \mathrm{i}k\ P\int_{-\infty}^{+\infty}\frac{dl}{l^2-k^2}\frac{\mu(l)}{a\bar{a}(l)}\right), \tag{56b}$$

where P is the principal value of the integral and $\overline{R}(k) = R(-k)$. Notice that the potential $q(x,t)$ is real. Therefore, $\overline{R}(k)$ must be equal to the complex conjugated of $R(k)$, and $\bar{a}(k) = a(-k)$. The assumption of a symmetric profile $\mu(k)$, namely $\mu(-k) = \mu(k)$ implies

$$\frac{\partial}{\partial t}(R\overline{R}) = \frac{2\pi\mu}{a\bar{a}}R\overline{R}. \tag{57}$$

Now using the basic relation $a\bar{a} - b\bar{b} = 1$ [15] and Eqs. (51)-(52), one can solve Eq. (57) by eliminating $a\bar{a}$, and obtain

$$|R(k,t)|^2 = \frac{|R(k,0)|^2}{|R(k,0)|^2 + [1-|R(k,0)|^2]\exp[-2\mu(k)\pi t]}. \tag{58}$$

Equations (56a)-(56b) can now be solved and yield

$$R(k,t) = \frac{R(k,0)e^{\mathrm{i}\theta(k,t)}}{\{|R(k,0)|^2 + (1-|R(k,0)|^2)e^{-2\mu(k)\pi t}\}^{1/2}}, \tag{59a}$$

$$\theta(k,t) = \frac{k}{2\pi}P\int\frac{dl}{l^2-k^2}\times \ln\{|R(l,0)|^2 + (1-|R(l,0)|^2)e^{-2\mu(l)\pi t}\}. \tag{59b}$$

Note that for $\mu > 0$, when $t \to +\infty$, $|R(k,t)| \to 1$. Here it worths to notice that we have the remarkable property that the phase of $R(k,t)$ approaches a constant as t yields to infinity. Indeed, using the fact that the function $|R(k,t)|^2 = R(k)R(-k)$ is even in k, we have

$$\theta(k,t) \to \theta_\infty(k) = -\frac{k}{\pi}P\int\frac{dl}{l-k}\ln|R(l,0)|. \tag{60}$$

The solution f of the integral equation (54) approaches, asymptotically, the solution f_∞ of the integral equation

$$f_\infty(k,x) = 1 - \frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{dl e^{2\mathrm{i}lx}}{l-(k+\mathrm{i}0)}\times f_\infty(-l,x)\exp[\mathrm{i}\theta_\infty(l) + \mathrm{i}\arg R(l,0)]. \tag{61}$$

To evaluate the asymptotic behavior of IAW from Eqs. (61) and (55), we must choose a model for $R(k,0)$ [15]:

$$R(k,0) = -\exp\left(\frac{-wk}{k-\mathrm{i}\kappa}\right), \tag{62}$$

where the constant w measures the width in k space of the continuous spectrum, and κ determines how fast the spectum drops off in k space. For large k, $R(k,0) \to -e^{-w}$. Therefore, w must be chosen quite large. With this choice,

$$f(k,x,t) = 1 + \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{dl f(-l,x,t)}{l-(k+\mathrm{i}\epsilon)})\times \\ \exp\left[2\mathrm{i}lx + \frac{\kappa l}{2(l+\mathrm{i}\kappa)} - \frac{\kappa l}{l-\mathrm{i}\kappa}\right]. \tag{63}$$

The behavior of Eq. (63) is evaluated by the steepest descent method. Let

$$g(l) = 2\mathrm{i}lx + \frac{wl}{2(l+\mathrm{i}\kappa)}, \qquad \tilde{g}(l) = 2\mathrm{i}lx - \frac{wl}{2(l-\mathrm{i}\kappa)}. \tag{64}$$

The stationary points are given by

$$x<0, \qquad \frac{\partial}{\partial l}g(l) = 0 \Rightarrow l^{\pm} = -\mathrm{i}\kappa \pm \sqrt{\frac{\kappa w}{4|x|}}, \tag{65a}$$

$$x>0, \qquad \frac{\partial}{\partial l}\tilde{g}(l) = 0 \Rightarrow \tilde{l}^{\pm} = \mathrm{i}\kappa \mp \mathrm{i}\sqrt{\frac{\kappa w}{4x}}. \tag{65b}$$

For $x<0$, the path of integration can follow the path of the steepest descent, namely $\Im[g(l)] = \Im[g(l^{\pm})]$. In this case $(x<0)$,

$$f(k) \to 1 - \frac{1}{\pi}\sum_{\pm}\frac{R_0(l^{\pm})}{l^{\pm}-k} f(-l^{\pm})\times \\ \sqrt{\frac{-2\pi}{g''(l^{\pm})}}\exp\left[g(l^{\pm})\right]. \tag{66}$$

Using (55), we obtain

$$q(x) = \frac{\partial}{\partial x}\sum_{\pm}\frac{-4\mathrm{i}l^{\pm}C^{\pm}\exp[g(l^{\pm})]}{l+C^{\pm}\exp[g(l^{\pm})]}, \tag{67}$$

where

$$C^{\pm} = \frac{R_0(l^{\pm})}{2\pi l^{\pm}}\sqrt{\frac{-2\pi}{g''(l^{\pm})}}. \tag{68}$$

The approximation in Eq. (66) is valid as long as the second term of the expansion is smaller than the first term, which occurs taken $\kappa w|x| \gg 1$.

For $x>0$, one can evaluate f by the method of stationary phase, following the path of integration $\Re[\tilde{g}(l)] = \Re[\tilde{g}(l^{-})]$, which yields

$$f(k) \to 1 + \frac{\mathrm{i}}{\pi}\frac{f(-\tilde{l}^{-})}{\tilde{l}^{-}-k}\exp\left\{\left[\frac{w\tilde{l}^{-}}{2(\tilde{l}^{-}+\mathrm{i}\kappa)}\right]\right\}\times \\ \sqrt{\frac{\pi e}{|\tilde{g}''(\tilde{l}^{-})|}}\exp\{[\tilde{g}(l^{-})]\}. \tag{69}$$

Again, using Eq. (55), we obtain

$$q(x) = \frac{\partial}{\partial x}\frac{-4\mathrm{i}\tilde{l}^{-}\tilde{C}\exp\left[\tilde{g}(\tilde{l}^{-})\right]}{l+\tilde{C}\exp\{[\tilde{g}(\tilde{l}^{-})]\}}, \tag{70}$$

where

$$\tilde{C} = \frac{\sqrt{\kappa w e\pi}(4x)^{3/4}}{2\pi(w-\sqrt{\kappa w/4x})}\exp\left\{\left[-\left(\frac{w}{2}\right)\frac{2\kappa x-\sqrt{\kappa wx}}{4\kappa x-\sqrt{\kappa wx}}\right]\right\}. \tag{71}$$

In this case, the approximation is valid only if $\kappa wx \gg \kappa^2$. Otherwise, $\tilde{l}^{-}$ moves into the lower half plane and it would no more be possible to ignore the contribution of other terms in the Laurent expansion of f.

6 Results and conclusion

In this work, we aim to highlight two integrable systems of coupled waves that we call the "truncated NLS with source" (9) and the " forced truncated KdV equation" (27). The physical context we have chosen to discuss these systems is the laser-plasma interaction. However, these are systems of equations which can appear in many other physical contexts. Indeed, we deal with a low frequency initial "noise" as a generic initial condition $q(x,0)$ evolving in time toward a coherent structure due to a nonlinear interaction with high-frequency external fields. This allows us to refer to this phenomenon as a "universal behavior".

It is worth to note that the only reason we call Eq. (9a), "truncated NLS" even though (9a) does not contain the key terms $q|q|^2$ and q_{xx} of the NLS equation, is that in the context of the IST method, both (9a) and NLS equations are associated to the same spectral operator, namely to the Zakharov-Shabat operator. It is the same for the so-called "forced truncated KdV" equation (27a) which does not contain the key terms qq_{xx} and q_{xxx} of the KdV equation but again in the IST approach, both (27a) and the KdV equations have the Sturm-Liouville operator as the associated spectral problem in common.

In both cases described by Eqs. (9) and, (27), i.e. the "truncated NLS with source" and " forced truncated KdV equation" respectively, the only nonlinearities arise from the nonlinear quadratic contribution of an external field. In our physical context, this contribution results from the coupling between the IAW and the electrostatic waves. Using the integrability of these systems through the ∂-bar formulation of the Inverse Scattering Transform method (see Appendices A and B), we have demonstrated the formation of localized coherent structures due to this extrinsic nonlinearity, which is valid *for any generic initial condition.*

Appendix A

Here we prove the integrability of the system

$$Q_t - \frac{\mathrm{i}}{2}\sigma_3 Q_{xx} + \mathrm{i}\sigma_3 Q^3 = \mathrm{i}\left[\sigma_3, \frac{1}{2\pi\mathrm{i}}\iint_{\mathbb{C}} d\lambda\wedge d\bar{\lambda} g(\lambda)\mu(\lambda)\sigma_3\mu^{-1}(\lambda)\right], \tag{A.1a}$$

$$\psi_x(k) + \mathrm{i}[\sigma_3, \psi(k)] = Q\psi(k). \tag{A.1b}$$

A reduction of the above system (A.1) yields

$$\begin{cases} -\mathrm{i}q_t + \frac{1}{2}q_{xx} - q|q|^2 = \mathrm{i}\alpha\int_{-\infty}^{+\infty} d\lambda\ a_1\overline{a_2}\ , \\ a_{1,x} + \mathrm{i}ka_1 = qr_2\ , \\ a_{2,x} - \mathrm{i}ka_2 = \bar{q}r_1, \end{cases} \tag{A.2}$$

where α is a real constant, the over-head bar represents the complex conjugation, q is a function of (x,t) and a is a function of (k,x,t). We have

$$q(x,0)\in L^1(\mathbb{R}),\quad \lim_{x\to+\infty} a_1(k,x,t) = A(k,t)\mathrm{e}^{-\mathrm{i}kx}, \quad \lim_{x\to-\infty} a_2(k,x,t) = 0. \tag{A.3}$$

The integrability of the "truncated" system (9) can be seen as a sub-case of (A.2)-(A.3). The second and the third equations of System (A.2) are the vectorial form of the Zakharov-Shabat spectral problem [3] that can be written in the general matrix form [18]

$$\psi_x = U\ \psi, \qquad U = -k\ \sigma_3 + Q, \qquad Q = \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix}. \tag{A.4}$$

(A.2) is recovered by the reduction $r = \bar{q}$. Here, ψ is a 2×2 matrix, built with two independent column-vector solutions $(\psi_1\ ,\psi_2)$ and is completely determined by its asymptotic behavior. Then, the set of differential equations $\psi_x = U\ \psi$ can be equivalently written as a set of Volterra integral equations. For convenience, let us write these equations for the matrix $\mu(k,x)$, defined by

$$\mu = \psi\exp(\mathrm{i}k\sigma_3 x)\ \Rightarrow\ \mu_x = \mathrm{i}k[\mu\ ,\ \sigma_3] + Q\mu\ . \tag{A.5}$$

Two dependent solutions μ^+ and μ^- can be defined through

$$\begin{cases} \mu_{11}^+ = 1 - \int_x^{+\infty} dx'\, q\,\mu_{21}^+ \\ \mu_{21}^+ = \int_{-\infty}^{x} dx'\,\bar{q}\,\mu_{11}^+\,\mathrm{e}^{2\mathrm{i}k(x-x')} \\ \mu_{12}^+ = -\int_x^{+\infty} dx'\, q\,\mu_{22}^+\,\mathrm{e}^{-2\mathrm{i}k(x-x')} \\ \mu_{22}^+ = 1 - \int_x^{+\infty} dx'\,\bar{q}\,\mu_{12}^+ \end{cases} \tag{A.6}$$

$$\begin{cases} \mu_{11}^- = 1 - \int_x^{+\infty} dx'\, q\,\mu_{21}^- \\ \mu_{21}^- = \int_x^{+\infty} dx'\,\bar{q}\,\mu_{11}^-\,\mathrm{e}^{2\mathrm{i}k(x-x')} \\ \mu_{12}^- = -\int_{-\infty}^{x} dx'\, q\,\mu_{22}^-\,\mathrm{e}^{-2\mathrm{i}k(x-x')} \\ \mu_{22}^- = 1 - \int_x^{+\infty} dx'\,\bar{q}\,\mu_{12}^- \end{cases} \tag{A.7}$$

One can easily verify, by using the Leibniz formula, that μ satisfies Eq. (A.5). In the reduction $r = \bar{q}$, one has

$$Q = \sigma_1\ \bar{Q}\ \sigma_1, \qquad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{A.8}$$

and it is easy to prove that

$$\mu^+(k,x) = \sigma_1\ \overline{\mu^-}(\bar{k},x)\ \sigma_1\ . \tag{A.9}$$

The set of Volterra equations (A.7) can be mapped into a Riemann-Hilbert problem as follows: μ^+, (respectively μ^-) is holomorphic in the complex upper half plane $\Im(k) >$

0 (respectively in the complex lower half plane $\Im(k) < 0$). Let us write

$$\mu = \begin{cases} \mu^+ & \text{in} \quad \Im(k) > 0 \\ \\ \mu^- & \text{in} \quad \Im(k) < 0 \end{cases} . \tag{A.10}$$

The matrix μ is holomorphic every where in the complex plane, except on the real axis, where it is discontinuous. Writing the Riemann-Hilbert problem consists of expressing this discontinuity in terms of μ^+ (or μ^-). To do so, let us compute

$$D(k,x) = \left[\mu_1^+(k+\mathrm{i}0,x) - \mu_1^-(k-\mathrm{i}0,x)\right]\mathrm{e}^{2\mathrm{i}kx}, \quad x \in \mathbb{R}. \tag{A.11}$$

from the integral equations (A.7) and obtain the following Volterra equations for D_1 and D_2, what constitutes the components of D,

$$\begin{cases} D_1 = -\int_x^{+\infty} dx' \; q \; D_2 \; \mathrm{e}^{2\mathrm{i}k(x'-x)} \\ \\ D_2 = \alpha^+(k) - \int_x^{+\infty} dx' \; \bar{q} \; D_1 \end{cases} . \tag{A.12}$$

Here,

$$\alpha^+(k) = \int_{-\infty}^{+\infty} dx' \; \bar{q}(x') \; \mu_{11}^+(k,x') \; \mathrm{e}^{-2\mathrm{i}kx'}, \;\; \Im(k) = 0^+ \; . \tag{A.13}$$

An integral equation having the same Green function as Eq. (A.12) can be obtained readily from Eq. (A.7). Indeed, the vector

$$L = \mu_2^+(k+\mathrm{i}0,x)\alpha^+(k), \qquad k \in \mathbb{R}. \tag{A.14}$$

is also a solution of Eq. (A.12). General theorems about integral equations allow to prove that Eq. (A.12) possesses only the trivial solution $\alpha = 0$. Then, we have $D = L$, which gives the following Riemann-Hilbert problem

$$\mu_1^+(k+\mathrm{i}0\;,\;x) - \mu_1^-(k-\mathrm{i}0\;,\;x) = \alpha^+(k)\;\mathrm{e}^{2\mathrm{i}kx}\;\mu_2^+(k+\mathrm{i}0\;,\;x)\;. \tag{A.15}$$

Hence,

$$D'(k,x) = \left[\mu_2^+(k+\mathrm{i}0,x) - \mu_2^-(k-\mathrm{i}0,x)\right]\mathrm{e}^{-2\mathrm{i}kx}, \quad x \in \mathbb{R}. \tag{A.16}$$

This leads to the second Riemann-Hilbert problem:

$$\mu_2^+(k+\mathrm{i}0,x) - \mu_2^-(k-\mathrm{i}0,x) = \alpha^-(k)\mathrm{e}^{-2\mathrm{i}kx}\mu_2^-(k-\mathrm{i}0,x). \tag{A.17}$$

Using Eq. (A.7), we obtain:

$$\alpha^-(k) = -\int_{-\infty}^{+\infty} q(x')\mu_{22}^-(k,x')\mathrm{e}^{2\mathrm{i}kx'}dx' \equiv -\overline{\alpha^+}(k), \quad \Im(k) = 0^-. \tag{A.18}$$

Eqs. (A.15) and (A.17) can be written as:

$$\mu^+ - \mu^- = (\mu_1^-, \mu_2^+)S, \tag{A.19}$$

where

$$S(k,x) = \mathrm{e}^{-\mathrm{i}k\sigma_3 s}\begin{pmatrix} 0 & -\overline{\alpha^+}(k) \\ \alpha^+(k) & 0 \end{pmatrix}\mathrm{e}^{\mathrm{i}k\sigma_3 x}, \quad k \in \mathbb{R}. \tag{A.20}$$

The matrix S satisfies the reduction $S(k,x) = -\sigma_1\bar{S}(k,x)\sigma_1$. The function $\alpha^+(k)$ is called the reflection coefficient. $\alpha^+(k)$ and $q(x)$ are equivalent in the sense that, given $q(x)$, one solves the Volterra equations (referencing Eq. (A.7)) and computes $\alpha^+(k)$ through:

$$\alpha^+(k) = \lim_{x \to +\infty} \mu_{21}^+(k,x)\mathrm{e}^{-2\mathrm{i}kx}. \tag{A.21}$$

This solves the direct problem.

The inverse problem consists of constructing $Q(x)$ from a given function $S(k,x)$ and solving the Riemann-Hilbert problem (A.19) to obtain $\mu(k,x)$. Once μ is obtained, $q(x)$ can be computed as follows: Write the Laurent expansion,

$$\begin{cases} \mu_{11}^- = 1 + \dfrac{1}{k}\mu_{11}^{-\,(1)} + \dfrac{1}{k^2}\mu_{11}^{-\,(2)} + \ldots \\ \\ \mu_{21}^- = \dfrac{1}{k}\mu_{21}^{-\,(1)} + \dfrac{1}{k^2}\mu_{21}^{-\,(2)} + \ldots \end{cases} , \tag{A.22}$$

obtained from Eq. (A.7). Then, insert Eq. (A.22) in Eq.(A.5) and use the Liouville theorem to obtain

$$q = 2\mathrm{i}\, \overline{\mu_{21}^{-}}^{(1)}, \tag{A.23}$$

which gives $q(x)$ from $\mu(k,x)$. To solve the Riemann-Hilbert problem, it is convenient to use the "∂-bar problem" formulation:

$$\frac{\partial\mu}{\partial\bar{k}} \doteq \frac{1}{2}\left[\mu(k)\ \delta^{+}(k_I) - \mu(k)\ \delta^{-}(k_I)\right], \tag{A.24}$$

where

$$\frac{\partial}{\partial\bar{k}} = \frac{\mathrm{i}}{2}\left(\frac{\partial}{\partial k_R} + \mathrm{i}\frac{\partial}{\partial k_I}\right), \tag{A.25}$$

with $k = k_R + \mathrm{i}\, k_I$, $\delta^{+} = \delta(k_I - \mathrm{i}0)$, $\delta^{-} = \delta(k_I + \mathrm{i}0)$, where δ is the Dirac distribution. Eq. (A.19) becomes

$$\frac{\partial\mu}{\partial\bar{k}} = \mu\ R, \qquad R = \frac{1}{2}\ S\begin{pmatrix}\delta^{+} & 0\\ 0 & \delta^{-}\end{pmatrix}, \tag{A.26}$$

and the generalized Cauchy formula reads

$$\mu(k,x) = \frac{1}{2\mathrm{i}\pi}\int_{\partial\mathcal{D}}\frac{d\lambda}{\lambda-k}\mu(\lambda,x) + \frac{1}{2\mathrm{i}\pi}\iint_{\mathcal{D}}\frac{d\lambda\wedge d\bar{\lambda}}{\lambda-k}\mu(\lambda,k)R(\lambda,k) \tag{A.27}$$

where $\mathcal{D}$ denotes the complex plane.

The Laurent expansion (A.22), allows for the computation of the first integral in Eq. (A.27) and, taking into account the analytic properties of μ, the second integral of (A.27) is reduced to an integration over the real line. Consequently, for the first column vector μ_1 with the reduction of (A.9), we have

$$\mu_1^{-}(k,x) = \begin{pmatrix}1\\0\end{pmatrix} + \frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{+\infty}\frac{d\lambda}{\lambda-k}\times \sigma_1\ \overline{\mu_1^{-}}(\bar{\lambda},x)\ \alpha^{+}(\lambda)\ \mathrm{e}^{2\mathrm{i}\lambda x}\ . \tag{A.28}$$

A similar equation holds for the second column vector.

Hence, the inverse problem (construction of $q(x)$ from $\alpha^{+}(k)$) is resolved by solving the Cauchy-Green integral equation and by calculating $q(x)$ from Eq. (A.23). Notice that the preceding formalism remains valid if $q(x)$ is assumed to depend also on a real external parameter t (time). Then, the eigenfunction $\mu(k,x)$ and the spectral transform $R(k,x)$ will also depend on t.

Next, we must construct integral evolutions from a simple choice of a given time dependence of the spectral transform $R(k,x,t)$. We start with the following generic ∂-bar problem

$$\begin{cases}\dfrac{\partial}{\partial\bar{k}}\ \mu(k) = \mu(k)\ R(k)\\[2ex] \mu(k) = 1 + O(\frac{1}{k})\ \ ,\ \ |k|\to\infty\end{cases}, \tag{A.29}$$

and ask for an (x,t)-dependence of R through the set of equations

$$\begin{cases}R_t = [R\ ,\ \Omega]\\[1ex] R_x = [R\ ,\ \Lambda]\end{cases}, \tag{A.30}$$

where Λ and Ω are given distributions of $k\in\mathbb{C}$, and are functions of x and t. It is a simple task to check the following relations

$$\begin{cases}\dfrac{\partial}{\partial\bar{k}}(\mu_x\mu^{-1} - \mu\Lambda\mu^{-1}) = -\mu\dfrac{\partial\Lambda}{\partial\bar{k}}\mu^{-1}\\[2ex] \dfrac{\partial}{\partial\bar{k}}(\mu_t\mu^{-1} - \mu\Omega\mu^{-1}) = -\mu\dfrac{\partial\Omega}{\partial\bar{k}}\mu^{-1}\end{cases}. \tag{A.31}$$

By integrating the above equations, one obtains:

$$\mu_x(k,x,t) = \left[U - \frac{1}{2\pi\mathrm{i}}\iint_{\mathbb{C}}\frac{d\lambda\wedge d\bar{\lambda}}{\lambda-k}\mu(\lambda,x,t) \times\frac{\partial\Lambda(\lambda,x,t)}{\partial\bar{\lambda}}\mu^{-1}(\lambda,x,t)\right]\mu(k,x,t) + \mu(k,x,t)\Lambda(k,x,t), \tag{A.32}$$

$$\mu_t(k,x,t) = \left[V - \frac{1}{2\pi\mathrm{i}}\iint_{\mathbb{C}}\frac{d\lambda\wedge d\bar{\lambda}}{\lambda-k}\mu(\lambda,x,t) \times\frac{\partial\Omega(\lambda,x,t)}{\partial\bar{\lambda}}\mu^{-1}(\lambda,x,t)\right]\mu(k,x,t) + \mu(k,x,t)\Omega(k,x,t). \tag{A.33}$$

The matrices U and V, referred to as the "constants of integration of the $\bar\partial$ operator," are given by:

$$\begin{cases} U(k,x,t) = \\ \qquad -\mathrm{Pol}_k\left[\mu(k,x,t)\Lambda(k,x,t)\mu^{-1}(k,x,t)\right], \\ V(k,x,t) = \\ \qquad -\mathrm{Pol}_k\left[\mu(k,x,t)\Omega(k,x,t)\mu^{-1}(k,x,t)\right]. \end{cases} \tag{A.34}$$

Here, $\mathrm{Pol}_k[\cdots]$ denotes the polynomial part in k of the given expression.

To obtain Eq. (A.5), we adopt the following choice:

$$\Lambda = \mathrm{i}k\sigma_3. \tag{A.35}$$

The compatibility condition $\mu_{xt} = \mu_{tx}$, when $\partial\Lambda/\partial\bar k = 0$, leads to:

$$U_t(k) - V_x(k) + [U(k),V(k)] = -\frac{1}{2\pi \mathrm{i}}\iint_{\mathbb{C}} \frac{d\lambda\wedge d\bar\lambda}{\lambda-k}\times \left[U(\lambda)-U(k),\mu(\lambda)\frac{\partial\Omega(\lambda)}{\partial\bar\lambda}\mu^{-1}(\lambda)\right]. \tag{A.36}$$

With the choice:

$$\Omega = \mathrm{i}k^2\sigma_3 + \frac{1}{2\mathrm{i}\pi}\iint_{\mathbb{C}}\frac{d\lambda\wedge d\bar\lambda}{\lambda-k}g(\lambda)\sigma_3, \tag{A.37}$$

the time evolution (A.36) becomes:

$$Q_t - \frac{\mathrm{i}}{2}\sigma_3 Q_{xx} + \mathrm{i}\sigma_3 Q^3 = \mathrm{i}\left[\sigma_3, \frac{1}{2\pi\mathrm{i}}\iint_{\mathbb{C}} d\lambda\wedge d\bar\lambda g(\lambda)\mu(\lambda)\sigma_3\mu^{-1}(\lambda)\right]. \tag{A.38}$$

This equation describes the evolution in (A.33). The explicit time evolution of the reflection coefficient $\alpha = \alpha^+(k)$ is then:

$$\alpha_t(k_R,t) = 2\mathrm{i}\left[k_R^2 - \frac{\mathrm{i}}{\pi}\iint_{\mathbb{C}}\frac{d\lambda_R d\lambda_I}{\lambda-(k_R-\mathrm{i}0)}g(\lambda)\right]\alpha(k_R,t). \tag{A.39}$$

In the specific case:

$$g(\lambda) = \frac{\mathrm{i}\pi}{6}\delta(\lambda_I)\delta(\lambda_R-k_0)|A(k_R)|, \tag{A.40}$$

we find:

$$\alpha(k,t) = \alpha(k,0)\exp\left[2\mathrm{i}\left(k^2+\frac{1}{6}\int_{-\infty}^{+\infty}\frac{dkA(k)}{k_0-k}\right)t\right]. \tag{A.41}$$

Clearly, the function $\alpha(k)$ has an essential singularity at $k = k_0$. Hence, $\alpha(k)$ is not defined for $k = k_0$ unless $a(k_0,0) \equiv 0$, which is required by the system (A.2). For $q(x,0)$ vanishing at both ends of the x-axis, consistency requires that $a_1\bar a_2$ also vanishes. Consequently, using Eqs. (A.17) and (A.23), $\alpha(k)$ must vanish for $k = k_0$. This condition determines the parameter k_0 from $q(x,0)$, as the solution of:

$$\alpha(k_0,0) = \int_{-\infty}^{+\infty} dx'\,\bar q(x')\overline{\mu_{22}^-(k_0,x',0)}\mathrm{e}^{-\mathrm{i}k_0x'} = 0. \tag{A.42}$$

This demonstrates that the initial value $q(x,0)$ determines the small correction to the wave number of the scattered ESW.

Appendix B

In this section, we prove the integrability of the system:

$$u_{xx} + \lambda^2 u = qu, \quad q = q(x,t), \quad u = u(\lambda,x,t), \tag{B1a}$$

$$q_t + 6qq_x - q_{xxx} = -\frac{\partial}{\partial x}\iint_{\mathbb{C}} d\lambda\wedge d\bar\lambda\, u(\lambda,x,t)u(-\lambda,x,t)\mu(\lambda,t), \tag{B1b}$$

where $\mu(\lambda,t)$ is an arbitrary distribution in $\mathbb{C}$, and $d\lambda\wedge d\bar\lambda = -2\mathrm{i}\,d\lambda_R$ for $\lambda = \lambda_R + \mathrm{i}\lambda_I$. These equations are associated with the initial and boundary conditions:

$$q(x,0) = q_0(x), \tag{B2a}$$

$$\lim_{x\to+\infty} u(\lambda,x,t) = d(\lambda,t)e^{-\mathrm{i}\lambda(x-\lambda^2 t)}. \tag{B2b}$$

The integrability of the "truncated system" (27) will be shown as a sub-case of (B1b)-(B2a)-(B2b).

We start by writing the $\bar{\partial}$-problem for the scalar ϕ:

$$\frac{\partial}{\partial \bar{\lambda}}\phi(\lambda) = \phi(-\lambda, t), \quad \lambda \in \mathbb{C}, \tag{B3a}$$

$$\phi(\lambda) = 1 + O(1/\lambda), \quad |\lambda| \to \infty, \tag{B3b}$$

where $r(\lambda)$ is a given distribution in $\mathbb{C}$. We restrict our study to the case where $\phi(\lambda)$ has only simple poles or discontinuities along lines in the λ-plane.

The solution of (B3a) satisfying (B3b) is given by the following integral equation:

$$\phi(\lambda) = 1 + \frac{1}{2\mathrm{i}\pi}\iint_{\mathbb{C}} \frac{dl \wedge d\bar{l}}{l-\lambda}\phi(-l)r(l). \tag{B4}$$

This leads to the asymptotic series:

$$\phi(\lambda) = \sum_{j=0}^{n-1} \lambda^{-j}\phi^{(j)} + O(\lambda^{-n}), \quad \phi^{(0)} = 1. \tag{B5}$$

The (x,t)-dependence for ϕ is obtained by requiring the "simplest integrable" (x,t)-dependence for $r(\lambda)$:

$$\frac{\partial}{\partial x}r(\lambda) = [\alpha(\lambda) - \alpha(-\lambda)]r(\lambda), \tag{B6a}$$

$$\frac{\partial}{\partial t}r(\lambda) = [\beta(\lambda) - \beta(-\lambda)]r(\lambda). \tag{B6b}$$

The choice of $\alpha(\lambda) = \mathrm{i}\lambda$ fixes the principal spectral problem. For this choice, the function:

$$\psi(\lambda, x, t) = \phi(\lambda, x, t)e^{-\mathrm{i}\lambda x} \tag{B7}$$

solves the Schrödinger spectral problem for

$$q = -2\mathrm{i}\frac{\partial}{\partial x}\phi^{(1)}(x,t). \tag{B8}$$

The proof of the above statement is given in [19] and relies on a comparison of the asymptotic expansions of $\psi_{xx} + \lambda^2\psi$ and ψ. In our treatment, we assume $r(\lambda)$ is such that the integral equation (B4) has a unique solution.

ψ and u solve the same scalar second-order differential equation. The function u is defined by its asymptotic behavior, while ψ is determined by its behavior in the complex λ-plane. Therefore, to relate ψ and u, one must find the behavior of ψ as $x \to \infty$, which is feasible at least when $q(x,t)$ is piecewise continuous, bounded, and vanishes rapidly as $|x| \to \infty$.

The integral equation (B4) provides a solution (u,q) of the system (B1b) through (B8) and:

$$u(\lambda, x, t) = \psi(\lambda, x, t)d(\lambda, t)e^{\mathrm{i}\lambda^3 t}. \tag{B9}$$

We now relate the dispersion relation $\beta(\lambda)$ to integrable evolution equations in the (x,t)-space. The auxiliary spectral problem is constructed by analyzing the spectral Wronskian:

$$\widehat{W}[\psi(\lambda), F(\lambda)] = \frac{1}{2\mathrm{i}\lambda}[\psi(\lambda)F(-\lambda) - \psi(-\lambda)F(\lambda)] \stackrel{\mathrm{def}}{=} b(\lambda). \tag{B10}$$

When $\beta(\lambda)$ is chosen as:

$$\beta(\lambda) = \mathrm{i}\lambda \sum_{j=0}^{n} \beta_{2j}\lambda^{2j} + \beta_s(\lambda), \tag{B11a}$$

$$\beta_s(\lambda) = O(1/\lambda), \quad |\lambda| \to \infty, \tag{B11b}$$

the well-known Korteweg-de Vries hierarchy of nonlinear evolution equations is recovered in the absence of $\beta_s(\lambda)$.

The nonlinear evolution equations are obtained from the compatibility condition:

$$\left[\frac{\partial^2}{\partial x^2} + \lambda^2 - q, \frac{\partial}{\partial t} - b\frac{\partial}{\partial x} + \frac{b_x}{2}\right]\psi = 0. \tag{B12}$$

This system is integrable by means of the spectral transform.

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TTMP vol 2 (1), 2025
doi: 10.30511/ttmp.2025.2053217.1047

Transactions in Theoretical and Mathematical Physics

(2+1)-dimensional Chern-Simons bi-gravity with AdS Lie bialgebra as an interacting theory of two massless spin-2 fields

S. Hoseinzadeh[1], **A. Rezaei-Aghdam**[1a]

[1] Department of Physics, Faculty of Science, Azarbaijan Shahid Madani University, 53714-161, Tabriz, Iran

Received: 11 February 2025 / Accepted: 26 February 2025 / Published: 26 February 2025

Abstract We introduce a new Lie bialgebra structure for the anti de Sitter (AdS) Lie algebra in (2+1)-dimensional spacetime. By gauging the resulting *AdS Lie bialgebra*, we write a Chern-Simons gauge theory of bi-gravity involving two dreibeins rather than two metrics, which describes two interacting massless spin-2 fields. Our ghost-free bi-gravity model which has no any local degrees of freedom, has also a suitable free field limit. By solving its equations of motion, we obtain a *new black hole* solution which has two curvature singularities and two horizons. We also study cosmological implications of this massless bi-gravity model.

1 Introduction

There are different theories of gravity in three dimensional spacetime and each of them has own advantages and has been widely studied. General relativity is a classical theory which describes interactions of a single massless spin-2 particle (graviton) [1–5]. Three-dimensional general relativity, without cosmological constant, is equivalent to a Chern-Simons gauge theory with the Poincaré gauge group ISO(2,1) [1]. But, the Chern-Simons gauge theories with gauge groups SO(2,2) or SO(3,1) are equivalent to adding negative or positive cosmological constants to three-dimensional general relativity, respectively [1].

It has been shown that does not exist any consistent theory (with at most two derivatives of the fields) involving interactions of many massless spin-2 fields in spacetime dimensions $d > 3$, because of the appearance of an unphysical scalar mode of negative energy (Boulware-Deser ghost) in such theories, or their discontinuity in the number of local degrees of freedom at their free field limits [6]. Although, in (2+1)-dimensional spacetime an exotic consistent interacting theory of many massless spin-2 fields has been constructed in [7], but the physical consequences of such interacting model has not been studied in detail. Theories which describe massless spin-2 fields in (2+1)-dimensional spacetime, have no local degrees of freedom, hence the Chern-Simons theory (with any gauge group) which is a topological model and has no local degrees of freedom [21], is a suitable candidate to construct a (2+1)-dimensional interacting theory of massless spin-2 fields.

On the other hand, in past years, "massive gravity" theories which have local degrees of freedom and describe the interactions of the massive spin-2 fields (gravitons), have been developed. Massive gravity theories have been greatly studied after resolution of their theoretical difficulties (see for a review [8, 9]). Topologically massive gravity [10–13], new massive gravity (NMG) [14–17] and general massive gravity [17] are three higher derivative theories of massive gravity involving auxiliary fields. dRGT massive gravity [18–21] is a bi-metric theory of massive gravity, and describes a massive together with a massless spin-2 particles. The non-dynamical reference metric of the dRGT model is promoted to a dynamical metric by introducing a kinetic term for it, resulting in the zwei-dreibein gravity (ZDG) [22, 23] (see also [24, 25]). The ZDG model has been generalized to obtain a parity-violating model which is called General Zwei-Dreibein Gravity (GZDG) [21, 26]. These massive gravity models have not Chern-Simons formulations, but they are Chern-Simons-like theories of gravity (see for a review [21]). In ref. [27], the GZDG$^+$ model has been introduced by adding a constraint term to the GZDG model for fixing torsion. Moreover, during the past few years two different extensions of the Poincaré algebra, i.e. the Maxwell algebra [28–35] and the semi-simple extension of the Poincaré algebra (AdS-Lorentz algebra) [35–38] have been applied to construct some group-theoretical gravity theories in four and three

[a]e-mail: rezaei-a@azaruniv.ac.ir

spacetime dimensions. Recently, we have studied a (2+1)-dimensional interacting model of two massless spin-2 fields by gauging a new Lie algebra [39]. Now, in this paper, we are interested in the study of an interacting theory of two massless spin-2 fields which obtains by gauging a new Lie bialgebra. The resulting bi-gravity model, just like the ZDG model, has been formulated in terms of two dreibeins rather than two metrics. But, unlike the ZDG model, it is a massless zwei-dreibein gravity theory.

Formulating new theories of the gravitational interaction is useful to understand the recent observational data in cosmology, which indicate that the expansion of the universe is accelerating. One of the possibilities for constructing new theories of gravity is extending known classical field theories to include additional spin-2 fields and interactions, which can modify the general relativity at large distances. Bi-metric theory of gravity, which describes the interactions of two different spin-2 fields, is therefore an interesting candidate to explain the accelerated expansion of the universe. One of the motivations of massive and bi-metric theories of gravity is that the interactions could change some of the dynamics of the gravitational theory, and therefore by changing the long-distance behavior of the gravitational fields, make them candidate theories of dark matter and energy.

The outline of the paper is as follows: In section two, we construct a new *Lie bialgebra* using the AdS Lie algebra so(2,2) in (2+1)-dimensional spacetime. In Section three, using the obtained Lie bialgebra (and the corresponding Manin triple), we propose a Chern-Simons gauge invariant bi-metric theory of gravity involving two different dreibein fields which describes *two interacting massless spin-2 fields*. We compare our bi-gravity model with the “massive gravity” theories such as NMG and ZDG. We also solve the equations of motion using the BTZ black hole metric for one of the metrics, and obtain a new black hole in the other metric solution. In section four, we study cosmological implications of the model, and show that it admits a homogeneous and isotropic Friedmann-Robertson-Walker solution. Some concluding remarks and discussions are given at the end.

2 Ads Lie bialgebra

In this section, we introduce a new bialgebra [40, 41] which is obtained by use of the AdS Lie algebra in (2+1)-dimensional spacetime. In (2+1)-dimensional spacetime, the commutation relations of the six-dimensional AdS Lie algebra are as follows [1]:

$$[J_a,J_b]=\epsilon_{abc}J^c,\quad [J_a,P_b]=\epsilon_{abc}P^c,\quad [P_a,P_b]=\frac{1}{\ell^2}\epsilon_{abc}J^c,\quad (1)$$

where $\ell^{-2}=-\Lambda$ is a constant, $\epsilon_{012}=-1$, and J_a and P_a $(a=0,1,2)$ are the Lorentz and translation generators, respectively [1]. The algebra indices a,b,c can be raised and lowered by (2+1)-dimensional Minkowski metric $\eta_{ab}=diag(-1,1,1)$. By letting the basis of the AdS algebra as $\{X_1,...,X_6\}=\{J_0,J_1,J_2,P_0,P_1,P_2\}$, and using the structure constants ${f_{ab}}^c$ of the AdS Lie algebra (1) and the following Jacobi and mixed-Jacobi identities [41]:

$$\tilde{f}^{ab}{}_n\tilde{f}^{nc}{}_m+\tilde{f}^{ca}{}_n\tilde{f}^{nb}{}_m+\tilde{f}^{bc}{}_n\tilde{f}^{na}{}_m=0,$$

$$f_{mc}{}^a\tilde{f}^{bm}{}_n-f_{mn}{}^a\tilde{f}^{bm}{}_c-f_{mc}{}^b\tilde{f}^{am}{}_n+f_{mn}{}^b\tilde{f}^{am}{}_c=f_{cn}{}^m\tilde{f}^{ab}{}_m,$$

respectively, we obtain the following structure constants $\tilde{f}^{ab}{}_c$ of the dual Lie algebra:

$$\begin{array}{llll}\tilde{f}^{54}{}_6=-a, & \tilde{f}^{15}{}_3=a, & \tilde{f}^{26}{}_1=a, & \tilde{f}^{21}{}_6=-\Lambda a,\\ \tilde{f}^{56}{}_4=-a, & \tilde{f}^{35}{}_1=a, & \tilde{f}^{24}{}_3=a, & \tilde{f}^{23}{}_4=-\Lambda a,\end{array}\qquad (2)$$

where a is an arbitrary constant. By letting the basis of the dual Lie algebra as

$$\{\tilde{X}^1,...,\tilde{X}^6\}=\{\tilde{P}_0,\tilde{P}_1,\tilde{P}_2,\tilde{J}_0,\tilde{J}_1,\tilde{J}_2\},$$

the commutation relations of the dual Lie algebra can be written in the following form:

$$\begin{array}{ll}[\tilde{J}_1,\tilde{J}_b]=-a\epsilon_{1bc}\tilde{J}^c, & [\tilde{J}_b,\tilde{P}_1]=-a\epsilon_{1bc}\tilde{P}^c,\\ [\tilde{J}_1,\tilde{P}_b]=-a\epsilon_{1bc}\tilde{P}^c, & [\tilde{P}_1,\tilde{P}_b]=a\ell^{-2}\epsilon_{1bc}\tilde{J}^c,\end{array}\qquad (3)$$

where $\tilde{J}_a$ and $\tilde{P}_a$ $(a=0,1,2)$ are the generators of spacetime rotation and translation related to the dual geometric structures such as metric and spin connection (see below). The dual Lie algebra (3) is very similar to the AdS Lie algebra (1), but in (3) we have the commutation relations between generators with indice "1" (J_1,P_1) and generators with indice "$j=0,2$" (J_j,P_j), only. In other words, generators with indice "$j=0,2$" (J_j,P_j) commute with each other. The commutation relations (1) together with (3) describe *AdS Lie bialgebra*. Now, using $[X_a,\tilde{X}^b]=\tilde{f}^{bc}{}_aX_c+f_{ca}{}^b\tilde{X}^c$ [40, 41], one can obtain the commutation relations between the generators of the AdS Lie algebra J_a,P_a and the generators of the dual Lie algebra $\tilde{J}_a,\tilde{P}_a$ as follows:

[1]Here, we use $J^a=\frac{1}{2}\epsilon^{abc}J_{bc}$ for the Lorentz generators J_{ab}.

$$[J_b,\tilde{P}_1]=\epsilon_{1bc}(aP^c-\tilde{P}_c),\quad [P_b,\tilde{J}_1]=-\epsilon_{1bc}(aP^c+\tilde{P}_c),$$
$$[J_b,\tilde{J}_1]=-\epsilon_{1bc}(aJ^c+\tilde{J}_c),\quad [P_b,\tilde{P}_1]=\ell^{-2}\epsilon_{1bc}(aJ^c-\tilde{J}_c),$$
$$[P_i,\tilde{P}_j]=\ell^{-2}[J_i,\tilde{J}_j]=\ell^{-2}\epsilon_{1i}{}^{j}(-aJ_1+\tilde{J}_1),$$
$$[J_i,\tilde{P}_j]=[P_i,\tilde{J}_j]=\epsilon_{1i}{}^{j}(aP_1+\tilde{P}_1),$$
$$[P_1,\tilde{P}_b]=\ell^{-2}[J_1,\tilde{J}_b]=-\ell^{-2}\epsilon_{1bc}\tilde{J}^c,$$
$$[J_1,\tilde{P}_b]=[P_1,\tilde{J}_b]=-\epsilon_{1bc}\tilde{P}^c, \tag{4}$$

where the indices i and j $(i,j=0,2)$ are the algebra indices. The commutation relations (4) together with (1) and (3), describe *AdS Manin triple* which is a 12-dimensional Lie algebra.

3 (2+1)-dimensional Chern-Simons gravity with Ads Lie bialgebra

In this section, we use the AdS Lie bialgebra, which is discussed in the previous section, to construct a new (2+1)-dimensional Chern-Simons bi-gravity. Using the relation $f_{AB}{}^{C}\ \Omega_{CD}+f_{AD}{}^{C}\ \Omega_{CB}=0$ [42], [2] an ad-invariant metric $\Omega_{AB}=\langle X_A,X_B\rangle$ for the AdS Manin triple is obtained as follows:

$$\langle J_a,P_b\rangle=-\alpha\,\eta_{ab},\qquad \langle J_a,\tilde{P}_b\rangle=\beta\delta_{ab}+a\alpha\,\delta_{a1}\delta_{b1},$$
$$\langle \tilde{J}_a,\tilde{P}_b\rangle=a^2\alpha\,\delta_{a1}\delta_{b1},\qquad \langle P_a,\tilde{J}_b\rangle=\beta\delta_{ab}-a\alpha\,\delta_{a1}\delta_{b1},$$
$$\langle J_a,J_b\rangle=\langle P_a,P_b\rangle=\langle \tilde{J}_a,\tilde{J}_b\rangle=0,$$
$$\langle \tilde{P}_a,\tilde{P}_b\rangle=\langle J_a,\tilde{J}_b\rangle=\langle P_a,\tilde{P}_b\rangle=0, \tag{5}$$

where $\delta_{ab}=diag(1,1,1)$ is the Kronecker delta function, and α and β are arbitrary constants. The ad-invariant metric should be non-degenerate, and then, we have $\beta\neq 0$. Now, we use the AdS *Manin triple* to construct a gauge symmetric Chern-Simons action, $I_{cs}=\frac{1}{4\pi}\int_M\Big(\langle h\wedge dh\rangle+\frac{1}{3}\ \langle h\wedge[h\wedge h]\rangle\Big)$, where $h=h_\mu\ dx^\mu$ is an AdS Manin triple valued Murer-Cartan one-form gauge field as follows:

$$h_\mu=h_\mu{}^{B}X_B=e_\mu{}^{a}P_a+\omega_\mu{}^{a}J_a+\tilde{e}_\mu{}^{a}\tilde{P}_a+\tilde{\omega}_\mu{}^{a}\tilde{J}_a, \tag{6}$$

where the Greek indices $\mu=0,1,2$ are the spacetime indices, $e_\mu{}^a$ and $\omega_\mu{}^a$ are the ordinary dreibein and spin connection, and $\tilde{e}_\mu{}^a,\tilde{\omega}_\mu{}^a$ are dreibein and spin connection corresponding to the generators of the dual Lie algebra, respectively. In this point of view, we obtain a gauge theory which has two metric tensors $g_{\mu\nu}=e_\mu{}^a e_\nu{}^b\eta_{ab}$ and $f_{\mu\nu}=\tilde{e}_\mu{}^a\tilde{e}_\nu{}^b\eta_{ab}$. We use the infinitesimal gauge parameter $u=\rho^aP_a+\tau^aJ_a+\tilde{\rho}^a\tilde{P}_a+\tilde{\tau}^a\tilde{J}_a$ together with the commutation relations (1),(3),(4) and the gauge transformations $h_\mu\rightarrow h'_\mu=U^{-1}h_\mu U+U^{-1}\partial_\mu U$, with $U=e^{-u}\simeq 1-u$ and $U^{-1}=e^{u}\simeq 1+u$, to obtain the following transformations for the gauge fields:

$$\delta e_\mu{}^c=-\partial_\mu\rho^c+\epsilon^{abc}(\rho_a\ \omega_{\mu b}+\tau_a\ e_{\mu b})$$
$$+a\epsilon^{ibc}\Big(\rho_i\ \tilde{\omega}_\mu{}^b-\tilde{\tau}^b e_{\mu i}-(-1)^c(\tau_i\ \tilde{e}_\mu{}^b-\tilde{\rho}^b\omega_{\mu i})\Big),$$
$$\delta\omega_\mu{}^c=-\partial_\mu\tau^c+\epsilon^{abc}(\ell^{-2}\rho_a\ e_{\mu b}+\tau_a\ \omega_{\mu b})$$
$$-a\epsilon^{ibc}\Big(\ell^{-2}(\rho_i\,\tilde{e}_\mu{}^b-\tilde{\rho}^b e_{\mu i})-(-1)^c(\tau_i\,\tilde{\omega}_\mu{}^b-\tilde{\tau}^b\omega_{\mu i})\Big),$$
$$\delta\tilde{e}_\mu{}^c=-\partial_\mu\tilde{\rho}^c+\epsilon^{abc}\Big(\rho^b\tilde{\omega}_{\mu a}-\tilde{\tau}_a e_\mu{}^b+\tau^b\tilde{e}_{\mu a}-\tilde{\rho}_a\omega_\mu{}^b\Big)$$
$$-a\epsilon^{1bc}\Big(\tilde{\tau}_b\ \tilde{e}_\mu{}^1-\tilde{\rho}^1\tilde{\omega}_{\mu b}+\tilde{\tau}^1\tilde{e}_{\mu b}-\tilde{\rho}_b\ \tilde{\omega}_\mu{}^1\Big),$$
$$\delta\tilde{\omega}_\mu{}^c=-\partial_\mu\tilde{\tau}^c+\epsilon^{abc}\Big(\ell^{-2}(\rho^b\tilde{e}_{\mu a}-\tilde{\rho}_a e_\mu{}^b)+\tau^b\tilde{\omega}_{\mu a}-\tilde{\tau}_a\omega_\mu{}^b\Big)$$
$$+a\epsilon^{1bc}\Big(\tilde{\tau}_b\ \tilde{\omega}_\mu{}^1-\tilde{\tau}^1\tilde{\omega}_{\mu b}+\ell^{-2}(\tilde{\rho}^1\tilde{e}_{\mu b}-\tilde{\rho}_b\tilde{e}_\mu{}^1)\Big). \tag{7}$$

The Ricci curvature two-form $\mathcal{R}=\mathcal{R}_{\mu\nu}dx^\mu\wedge dx^\nu$ can be written as:

$$\mathcal{R}_{\mu\nu}=\partial_{[\mu}h_{\nu]}+[h_\mu,h_\nu]=\mathcal{R}_{\mu\nu}^{\ A}X_A$$
$$=T_{\mu\nu}{}^{a}\ P_a+R_{\mu\nu}{}^{a}\ J_a+\tilde{T}_{\mu\nu}{}^{a}\ \tilde{P}_a+\tilde{R}_{\mu\nu}{}^{a}\ \tilde{J}_a, \tag{8}$$

such that the torsion $T_{\mu\nu}{}^a$ and the standard Riemannian curvature $R_{\mu\nu}{}^a$ are as follows:

$$T_{\mu\nu}{}^{j}=\partial_{[\mu}e_{\nu]}^{\ j}+\epsilon_{ab}{}^{j}\omega_{[\mu}^{\ a}e_{\nu]}^{\ b}+a\epsilon_{1b}{}^{j}\Big(\omega_{[\mu}^{\ b}\tilde{e}_{\nu]}^{\ 1}-e_{[\mu}^{\ b}\tilde{\omega}_{\nu]}^{\ 1}\Big),$$
$$T_{\mu\nu}{}^{1}=\partial_{[\mu}e_{\nu]}^{\ 1}+\epsilon_{ab}{}^{1}\omega_{[\mu}^{\ a}e_{\nu]}^{\ b}+a\epsilon_{1a}{}^{b}\Big(\omega_{[\mu}^{\ a}\tilde{e}_{\nu]}^{\ b}+e_{[\mu}^{\ a}\tilde{\omega}_{\nu]}^{\ b}\Big),$$
$$R_{\mu\nu}{}^{j}=\partial_{[\mu}\omega_{\nu]}^{\ j}+\frac{1}{2}\epsilon_{ab}{}^{j}\Big(\omega_{[\mu}^{\ a}\omega_{\nu]}^{\ b}+\ell^{-2}e_{[\mu}^{\ a}e_{\nu]}^{\ b}\Big)$$
$$+a\epsilon_{1b}{}^{j}\Big(\ell^{-2}e_{[\mu}^{\ b}\tilde{e}_{\nu]}^{\ 1}-\omega_{[\mu}^{\ b}\tilde{\omega}_{\nu]}^{\ 1}\Big),$$
$$R_{\mu\nu}{}^{1}=\partial_{[\mu}\omega_{\nu]}^{\ 1}+\frac{1}{2}\epsilon_{1ab}\Big(\omega_{[\mu}^{\ a}\omega_{\nu]}^{\ b}+\ell^{-2}e_{[\mu}^{\ a}e_{\nu]}^{\ b}\Big)$$
$$-a\epsilon_{1a}{}^{b}\Big(\ell^{-2}e_{[\mu}^{\ a}\tilde{e}_{\nu]}^{\ b}+\omega_{[\mu}^{\ a}\tilde{\omega}_{\nu]}^{\ b}\Big), \tag{9}$$

and in the same way, the field strengths $\tilde{T}_{\mu\nu}{}^a$ and $\tilde{R}_{\mu\nu}{}^a$ which can be interpreted as dual torsion and dual Riemannian curvature respectively, have the following forms:

[2] $f_{AB}{}^{C}$ is the structure constant of the AdS Manin triple.

$$\tilde{T}_{\mu\nu}{}^{j} = \partial_{[\mu}\tilde{e}_{\nu]}{}^{j} - \epsilon_{1bj}\left(e_{[\mu}{}^{b}\tilde{\omega}_{\nu]}{}^{1} + \omega_{[\mu}{}^{b}\tilde{e}_{\nu]}{}^{1}\right)$$
$$+\epsilon_{1b}{}^{j}\left(a\tilde{e}_{[\mu}{}^{1}\tilde{\omega}_{\nu]}{}^{b} - a\tilde{\omega}_{[\mu}{}^{1}\tilde{e}_{\nu]}{}^{b} - e_{[\mu}{}^{1}\tilde{\omega}_{\nu]}{}^{b} - \omega_{[\mu}{}^{1}\tilde{e}_{\nu]}{}^{b}\right),$$
$$\tilde{T}_{\mu\nu}{}^{1} = \partial_{[\mu}\tilde{e}_{\nu]}{}^{1} + \epsilon_{1a}{}^{b}\omega_{[\mu}{}^{a}\tilde{e}_{\nu]}{}^{b} + \epsilon_{1a}{}^{b}e_{[\mu}{}^{a}\tilde{\omega}_{\nu]}{}^{b},$$
$$\tilde{R}_{\mu\nu}{}^{j} = \partial_{[\mu}\tilde{\omega}_{\nu]}{}^{j} - \epsilon_{1bj}\left(\ell^{-2}e_{[\mu}{}^{b}\tilde{e}_{\nu]}{}^{1} + \omega_{[\mu}{}^{b}\tilde{\omega}_{\nu]}{}^{1}\right)$$
$$-\epsilon_{1b}{}^{j}\left(\ell^{-2}e_{[\mu}{}^{1}\tilde{e}_{\nu]}{}^{b} + a\tilde{\omega}_{[\mu}{}^{1}\tilde{\omega}_{\nu]}{}^{b} - \ell^{-2}a\tilde{e}_{[\mu}{}^{1}\tilde{e}_{\nu]}{}^{b} + \omega_{[\mu}{}^{1}\tilde{\omega}_{\nu]}{}^{b}\right),$$
$$\tilde{R}_{\mu\nu}{}^{1} = \partial_{[\mu}\tilde{\omega}_{\nu]}{}^{1} + \epsilon_{1a}{}^{b}\omega_{[\mu}{}^{a}\tilde{\omega}_{\nu]}{}^{b} + \ell^{-2}\epsilon_{1a}{}^{b}e_{[\mu}{}^{a}\tilde{e}_{\nu]}{}^{b}. \tag{10}$$

Using (1) as well as (3)-(6), one obtains the following Chern-Simons bi-gravity model with the AdS Manin triple as a gauge symmetry:

$$I = I^{e,\omega}(e,\omega) + I^{\tilde{e},\tilde{\omega}}(\tilde{e},\tilde{\omega}) + I^{int}(e,\omega,\tilde{e},\tilde{\omega}), \tag{11}$$

where the first term is

$$I^{e,\omega} = -4\hat{\alpha}G\ I_{EC}(\omega,e,\Lambda),$$

and the Eistein-Cartan action I_{EC} is

$$I_{EC}(\omega,e,\Lambda) = -\frac{1}{16\pi G}\int_M d^3x\,\epsilon^{\mu\nu\rho}e_{\mu}{}^{c}\Big(D_\nu\omega_{\rho c} - \frac{\Lambda}{3}\epsilon_{abc}e_{\nu}{}^{a}e_{\rho}{}^{b}\Big).$$

The second term in (11) is the Einstein-Cartan action with $\tilde{\omega}_\mu{}^j = \tilde{e}_\mu{}^j = 0,\ \tilde{\omega}_\mu{}^1 \neq 0,\ \tilde{e}_\mu{}^1 \neq 0$, as follows:

$$I^{\tilde{e},\tilde{\omega}} = -\frac{\hat{\alpha}a^2}{4\pi}\int_M d^3x\ \epsilon^{\mu\nu\rho}\tilde{e}_{\mu}{}^{1}\partial_{[\nu}\tilde{\omega}_{\rho]}{}^{1}.$$

The third term in (11) includes some interaction terms between the fields $\{e_\mu{}^a, \omega_\mu{}^a\}$ and the fields $\{\tilde{e}_\mu{}^a, \tilde{\omega}_\mu{}^a\}$ as follows:

$$I^{int} = \int_M \frac{d^3x}{4\pi}\epsilon^{\mu\nu\rho}\Big\{a\hat{\alpha}\Big(\tilde{\omega}_\mu{}^1 D_\nu e_\rho{}^1 - \tilde{e}_\mu{}^1(D_\nu\omega_\rho{}^1 + \ell^{-2}\epsilon_{1bc}e_\nu{}^b e_\rho{}^c)\Big)$$
$$+\hat{\beta}\Big(\tilde{\omega}_\mu{}^c D_\nu e_\rho{}^c + \tilde{e}_\mu{}^c(D_\nu\omega_\rho{}^c + \ell^{-2}\epsilon_{ab}{}^c e_\nu{}^a e_\rho{}^b)\Big)$$
$$+2a\hat{\beta}\epsilon_{1b}{}^c\Big(\tilde{\omega}_\mu{}^1(\omega_\nu{}^b\tilde{e}_\rho{}^c + e_\nu{}^b\tilde{\omega}_\rho{}^c) - \tilde{e}_\mu{}^1(\omega_\nu{}^b\tilde{\omega}_\rho{}^c - \ell^{-2}\tilde{e}_\nu{}^b e_\rho{}^c)\Big)\Big\},$$

where $D_\nu\omega_\rho{}^c$ and $D_\nu e_\rho{}^c$ are the covariant derivatives with respect to the spin connection $\omega_\mu{}^c$ as follows:

$$D_\nu\omega_{\rho c} = \partial_{[\nu}\ \omega_{\rho]c} + \epsilon_{abc}\ \omega_\nu{}^a\omega_\rho{}^b,$$
$$D_\nu e_{\rho c} = \partial_{[\nu}\ e_{\rho]c} + \epsilon_{abc}\ \omega_{[\nu}{}^a e_{\rho]}{}^b. \tag{12}$$

The Chern-Simons bi-gravity model (11) which is invariant under the gauge transformations (7), has no any local degrees of freedom, and is a ghost-free model which describes two interacting massless spin-2 fields in (2+1)-dimensional spacetime. Two dreibeins in the action (11) are related to their corresponding metric tensors as follows:

$$g_{\mu\nu} = e_\mu{}^a e_\nu{}^b\eta_{ab}, \qquad f_{\mu\nu} = \tilde{e}_\mu{}^a\tilde{e}_\nu{}^b\eta_{ab}. \tag{13}$$

In the absence of the interaction terms I^{int}, the free field limit of (11),

$$I^{free} = I^{e,\omega}(e,\omega) + I^{\tilde{e},\tilde{\omega}}(\tilde{e},\tilde{\omega}), \tag{14}$$

similar to (11) has no any local degrees of freedom, and is invariant under the following gauge transformations:

$$\delta e_\mu{}^c = -\partial_\mu\rho^c + \epsilon^{abc}(\rho_a\omega_{\mu b} + \tau_a e_{\mu b}), \qquad \delta\tilde{e}_\mu{}^c = -\partial_\mu\tilde{\rho}^c,$$
$$\delta\omega_\mu{}^c = -\partial_\mu\tau^c + \epsilon^{abc}(\ell^{-2}\rho_a e_{\mu b} + \tau_a\omega_{\mu b}), \quad \delta\tilde{\omega}_\mu{}^c = -\partial_\mu\tilde{\tau}^c. \tag{15}$$

Now, by assuming the following relations among the fields and constants:

$$\beta\,\tilde{e}_\mu{}^a = -\frac{1}{m^2}f_{\mu a}, \qquad \beta\,\tilde{\omega}_\mu{}^a = h_{\mu a},$$
$$\alpha = -\sigma, \quad \alpha\ell^{-2} = \Lambda_0, \quad a\ell^{-2} = 2m^2\beta, \tag{16}$$

the Chern-Simons action (11) can be rewritten in the following form:

$$I = \frac{1}{2\pi}I_{NMG} + \frac{1}{4\pi}\int d^3x\,\epsilon^{\mu\nu\rho}\Big\{-a^2\alpha\tilde{e}_\mu{}^1\partial_{[\nu}\tilde{\omega}_{\rho]}{}^1$$
$$+a\alpha\Big(\tilde{\omega}_\mu{}^1 D_\nu e_\rho{}^1 - \tilde{e}_\mu{}^1(D_\nu\omega_\rho{}^1 + \ell^{-2}\epsilon_{1bc}e_\nu{}^b e_\rho{}^c)\Big)$$
$$-a\beta\ell^{-2}\epsilon_{1bc}e_\mu{}^1\tilde{e}_\nu{}^b\tilde{e}_\rho{}^c + \beta\ell^{-2}\epsilon_{ab}{}^c\tilde{e}_\mu{}^c e_\nu{}^a e_\rho{}^b$$
$$+2a\beta\ \epsilon_{1b}{}^c\Big(\tilde{\omega}_\mu{}^1(\omega_\nu{}^b\tilde{e}_\rho{}^c + e_\nu{}^b\tilde{\omega}_\rho{}^c) - \tilde{e}_\mu{}^1\omega_\nu{}^b\tilde{\omega}_\rho{}^c\Big)\Big\}, \tag{17}$$

where I_{NMG} is the NMG model involving a pair of the auxiliary fields $f_\mu{}^c, h_\mu{}^c$, as follows: [14]

$$I_{NMG} = \frac{1}{2}\int d^3x\,\epsilon^{\mu\nu\rho}\Big\{-\sigma e_\mu{}^c D_\nu\omega_{\rho c} + \frac{\Lambda_0}{3}\epsilon_{abc}e_\mu{}^a e_\nu{}^b e_\rho{}^c$$
$$+h_\mu{}^c D_\nu e_{\rho c} - \frac{1}{m^2}f_\mu{}^c\Big(D_\nu\omega_{\rho c} + \epsilon_{abc}(e_\nu{}^a f_\rho{}^b + f_\nu{}^a e_\rho{}^b)\Big)\Big\}. \tag{18}$$

Again, using the following redefinition of the fields and constants,

$$e_1^a \equiv e^a, \qquad \omega_1^a \equiv \omega^a, \qquad e_2^a \equiv \tilde{e}^a, \qquad \omega_2^a \equiv \tilde{\omega}^a,$$
$$a^2=1, \quad \alpha=M_P, \quad \ell^{-2}=-\alpha_1 m^2, \quad \beta=M_P(a-\frac{\beta_1}{\alpha_1}),$$

the Chern-Simons model (11) can be rewritten in another form as follows:

$$\begin{aligned} I =& \frac{1}{2\pi} I_{ZDG}(\sigma=-1,\tilde{e}_\mu{}^j=\tilde{\omega}_\mu{}^j=0) \\ &+\frac{1}{4\pi}\int_M d^3x\epsilon^{\mu\nu\rho}\Big\{a\alpha\Big(\tilde{\omega}_\mu{}^1 D_\nu e_\rho{}^1-\tilde{e}_\mu{}^1 D_\nu\omega_\rho{}^1\Big) \\ &+\beta\Big(\tilde{\omega}_\mu{}^c D_\nu e_\rho{}^c+\tilde{e}_\mu{}^c D_\nu\omega_\rho{}^c+\ell^{-2}\epsilon_{ab}{}^j e_\mu{}^a e_\nu{}^b\tilde{e}_\rho{}^j\Big) \\ &+2a\beta\epsilon_{1b}{}^c\Big(\tilde{\omega}_\mu{}^1(\omega_\nu{}^b\tilde{e}_\rho{}^c+e_\nu{}^b\tilde{\omega}_\rho{}^c)-\tilde{e}_\mu{}^1(\omega_\nu{}^b\tilde{\omega}_\rho{}^c-\ell^{-2}\tilde{e}_\nu{}^b e_\rho{}^c)\Big)\Big\}, \end{aligned}$$

where $I_{ZDG}(\sigma=-1,\tilde{e}_\mu{}^j=\tilde{\omega}_\mu{}^j=0)$ is the ZDG action [22]:

$$\begin{aligned} I_{ZDG}=&-\frac{1}{2}M_p\int d^3x\epsilon^{\mu\nu\rho}\Big\{\sigma e_{1\mu}{}^c D_\nu\omega_{1\rho c}+e_{2\mu}{}^c D_\nu\omega_{2\rho c} \\ &+\frac{1}{3}\alpha_1 m^2\epsilon_{abc}e_{1\mu}{}^a e_{1\nu}{}^b e_{1\rho}{}^c+\frac{1}{3}\alpha_2 m^2\epsilon_{abc}e_{2\mu}{}^a e_{2\nu}{}^b e_{2\rho}{}^c \\ &-\beta_1 m^2\epsilon_{abc}e_{1\mu}{}^a e_{1\nu}{}^b e_{2\rho}{}^c-\beta_2 m^2\epsilon_{abc}e_{1\mu}{}^a e_{2\nu}{}^b e_{2\rho}{}^c\Big\}, \end{aligned} \quad (19)$$

with the sign parameter $\sigma=-1$ and the fields $\tilde{e}_\mu{}^j=\tilde{\omega}_\mu{}^j=0$, $(j=0,2)$, where M_p is the Planck mass, α_1 and α_2 are cosmological parameters, β_1 and β_2 are coupling constants, and $e_{I\mu}{}^a$ and $\omega_{I\mu}{}^a$ $(I=1,2)$ are pairs of the dreibein and spin connection one-forms, respectively. Note that the zero values of the fields $\tilde{e}_\mu{}^j=\tilde{\omega}_\mu{}^j=0$, in ZDG action is imposed by the dual Lie algebra (3). Variations of the action (11) with respect to the fields $e_{\mu a},\tilde{e}_{\mu a},\omega_{\mu a}$ and $\tilde{\omega}_{\mu a}$ give the corresponding equations of motion, respectively:

$$T_{\nu\rho}{}^a=\tilde{T}_{\nu\rho}{}^a=R_{\nu\rho}{}^a=\tilde{R}_{\nu\rho}{}^a=0, \quad (20)$$

where $T_{\nu\rho}{}^a,\tilde{T}_{\nu\rho}{}^a,R_{\nu\rho}{}^a$ and $\tilde{R}_{\nu\rho}{}^a$ are defined in (9)-(10).

3.1 Black hole solution

Now, we use the BTZ black hole metric $g_{\mu\nu}$ [43] to obtain the following solution for the equations of motion (20):

$$ds^2=-N^2(r)dt^2+\frac{dr^2}{N^2(r)}+r^2(N^\varphi(r)dt+d\varphi)^2, \quad (21)$$

$$df^2=-\frac{4N_f^4}{a^2N^2}dt^2+\frac{4(\Lambda_f N^2-\Lambda N_f^2)^2}{a^2\Lambda^2N^6}dr^2+r^2(N_f^\varphi dt+d\varphi)^2, \quad (22)$$

where $ds^2=g_{\mu\nu}dx^\mu dx^\nu$, $df^2=f_{\mu\nu}dx^\mu dx^\nu$, and

$$N^2(r)=-M+\frac{r^2}{\ell^2}+\frac{J^2}{4r^2}, \qquad N^\varphi(r)=-\frac{J}{2r^2},$$
$$N_f^2(r)=-M_f+\frac{r^2}{\ell_f^2}+\frac{J^2}{4r^2}, \qquad N_f^\varphi(r)=-\frac{J+2Dr^2}{ar^2},$$

$\{x^0,x^1,x^2\}=\{t,r,\varphi\}$ are the coordinates of the spacetime, M,J,D,M_f and ℓ_f are arbitrary constants, and the spin connections $\omega_\mu{}^a(r)$ and $\tilde{\omega}_\mu{}^a(r)$ are obtained as follows:

$$\begin{aligned} \omega^0=&2DNdt+(1-a)Nd\varphi, \\ \omega^1=&\Big(\frac{2JD}{r}(2-\frac{\ell^2}{\ell_f^2})+(\ell^{-2}+2\ell_f^{-2})r\Big)dt \\ &+\frac{J}{2r}\Big(2(1-a)(2-\frac{\ell^2}{\ell_f^2})-1\Big)d\varphi, \\ \omega^2=&-\frac{J}{2r^2N}dr, \\ \tilde{\omega}^0=&\frac{2D}{aN}\Big(2N_f^2+N^2(1-2\frac{\ell^2}{\ell_f^2})\Big)dt \\ &+N\Big\{\frac{2(1-a)}{a}\Big(\frac{N_f^2}{N^2}-\frac{\ell^2}{\ell_f^2}\Big)-1\Big\}d\varphi, \\ \tilde{\omega}^1=&\Big(\frac{JD}{ar}(\frac{2\ell^2}{\ell_f^2}-3)-\frac{2r}{a\ell_f^2}\Big)dt+\frac{J}{2r}\Big(\frac{2(a-1)}{a}(1-\frac{\ell^2}{\ell_f^2})+1\Big)d\varphi, \\ \tilde{\omega}^2=&\frac{J}{ar^2N^3}\Big(N^2-N_f^2\Big)dr. \end{aligned} \quad (23)$$

The Kretschmann scalar $K=R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ for the metric $f_{\mu\nu}$ is proportional to N_f^{-8}, and then $f_{\mu\nu}$ has two curvature singularities at

$$r_{s\pm}=\sqrt{\frac{\ell_f}{2}\Big(\ell_f M_f\pm\sqrt{\ell_f^2M_f^2-J^2}\Big)}, \qquad |\ell_f M_f|>|J|, \quad (24)$$

where $N_f(r)$ vanishes. $f_{\mu\nu}$ has also two horizons at

$$r_\pm=\sqrt{\frac{\ell}{2}\Big(\ell M\pm\sqrt{\ell^2M^2-J^2}\Big)}, \qquad |\ell M|>|J|, \quad (25)$$

where $N(r)$ vanishes. We use suitable values of the arbitrary constants M,M_f,ℓ and ℓ_f to have $r_+>r_{s+}$, such that r_+ is the event horizon of the black hole. Then, depending on the values of these constants, we have three different situations: $r_->r_{s+}$, $r_{s+}>r_->r_{s-}$ and $r_-<r_{s-}$.

To investigate the asymptotic behavior of this solution, we keep only the dominant terms. For very large values of r, ds^2 has the following form:

$$ds^2 \;\sim\; -\frac{r^2}{\ell^2}dt^2 + \frac{\ell^2}{r^2}dr^2 + r^2 d\varphi^2,$$

which is the AdS spacetime. But The metric df^2, for large values of r, approaches to the following one:

$$df^2 \sim -\frac{4\ell^2 r^2}{a^2\ell_f^4}dt^2 + \frac{4\ell^{10}\Big(M/\ell_f^2 - M_f/\ell^2\Big)^2}{a^2 r^6}dr^2 + \Big(\frac{2Dr}{a}dt - r d\varphi\Big)^2,$$

which is clearly different from the AdS spacetime. This *new black hole* is different from the black hole solutions of the three dimensional *f*–*g* theory, which are asymptotically AdS and have coordinate singularities [44, 45].

4 Cosmological implications

We study the homogeneous and isotropic cosmology of our massless bi-gravity model (11) using the following Friedmann-Robertson-Walker (FRW) Ansatz for both metrics:

$$ds^2 = -N^2(t)\; dt^2 + A^2(t)\Big(\frac{dr^2}{1-kr^2} + r^2 d\varphi^2\Big), \tag{26}$$

and

$$df^2 = -X^2(t)\; dt^2 + Y^2(t)\Big(\frac{dr^2}{1-kr^2} + r^2 d\varphi^2\Big), \tag{27}$$

where $A(t)$ and $Y(t)$ are the spatial scale factors of the FRW metrics $g_{\mu\nu}$ and $f_{\mu\nu}$, respectively, and $N(t)$ and $X(t)$ are their lapse functions. The constant k in both metrics (26) and (27) is the spatial curvature, whose positive, vanishing and negative values ($k = 1, 0, -1$) correspond to the closed, flat and open universes, respectively.

We solve the equations of motion (20), and obtain the following equations:

$$(ab-1)\dot{A}(t) - \xi(t)\, N(t) = 0, \tag{28}$$

and

$$\xi(t)\dot{A}(t) - a\,\xi^2(t)X(t) + \Big(\frac{ab-1}{\ell^2}A^2(t) - k\Big)N(t) = 0, \tag{29}$$

together with a relation between two scale factors as:

$$Y(t) = b\; A(t), \tag{30}$$

and the following relations for the spin connection fields:

$$\begin{aligned}
\omega^0(t) &= \frac{\sqrt{1-kr^2}}{ab-1}\; d\varphi,\\
\omega^1(t) &= \frac{r\Big(kab - (ab-1)^2\;\xi^2(t)\Big)}{(ab-1)^2\;\xi(t)}\; d\varphi,\\
\omega^2(t) &= \frac{\xi(t)}{\sqrt{1-kr^2}}\; dr,\\
\tilde{\omega}^0(t) &= \frac{-a\, b^2\sqrt{1-kr^2}}{(ab-1)^2}\; d\varphi,\\
\tilde{\omega}^1(t) &= -\frac{br\Big(kab + \ell^{-2}(ab-1)^3\; A^2(t)\Big)}{(ab-1)^2\;\xi(t)}\; d\varphi,\\
\tilde{\omega}^2(t) &= \frac{b\,\ell^{-2}(ab-1)\; A^2(t)}{\xi(t)\sqrt{1-kr^2}}\; dr,
\end{aligned} \tag{31}$$

where b is an arbitrary constant, dot denotes the time derivative ($\dot{A} \equiv \frac{dA}{dt}$), and

$$\xi(t) = \sqrt{-k - \ell^{-2}(ab-1)^2 A^2(t)}, \qquad |A(t)| < \frac{\ell\sqrt{-k}}{|ab-1|}, \tag{32}$$

which implies that we have an open universe with negative spatial curvature ($k = -1$), where the radial coordinate r is defined on $0 \leq r < +\infty$. Solving the equations (28) and (29) give the following relations for $N(t)$ and $X(t)$ in terms of the scale factor $A(t)$:

$$N(t) = \frac{ab-1}{\xi(t)}\dot{A}(t), \tag{33}$$

and

$$X(t) = -\frac{bk}{\xi^3(t)}\dot{A}(t). \tag{34}$$

The equations (28) and (29) do not restrict the scale factor $A(t)$ of the FRW metric (26), and then $A(t)$ is an arbitrary function of the timelike coordinate t. Using the following coordinate transformation

$$\hat{t} \equiv \ell\; arcsin\Big(\frac{ab-1}{\ell}A(t)\Big), \tag{35}$$

the FRW metric (26) can be rewritten as:

$$ds^2 = -d\hat{t}^2 + \hat{a}^2(\hat{t})\Big(\frac{dr^2}{1+r^2} + r^2 d\varphi^2\Big), \tag{36}$$

where the scale factor is

$$\hat{a}(\hat{t}) = \frac{\ell\, sin(\hat{t}/\ell)}{ab-1}, \tag{37}$$

which is obviously an oscillating solution. The Hubble parameter for this solution is obtained as follows:

$$H(\hat{t}) \equiv \frac{\dot{\hat{a}}}{\hat{a}} = \frac{1}{\ell} cot(\hat{t}/\ell). \tag{38}$$

Its deceleration parameter is

$$q(\hat{t}) \equiv -\frac{a\ddot{\hat{a}}}{\dot{\hat{a}}^2} = tan^2(\hat{t}/\ell), \tag{39}$$

which is obviously positive and implies that the expansion of the universe is decelerating. Using another coordinate transformation as follows:

$$t' \equiv \frac{bkA(t)}{\sqrt{1-\ell^{-2}(ab-1)^2 A^2(t)}}, \tag{40}$$

the second FRW metric (27) can be rewritten in the following form:

$$df^2 = -dt'^2 + \bar{a}^2(t')\Big(\frac{dr^2}{1+r^2} + r^2 d\varphi^2\Big), \tag{41}$$

where the scale factor is

$$\bar{a}(t') = \frac{b\, t'}{\sqrt{b^2 + \ell^{-2}(ab-1)^2\, t'^2}}, \tag{42}$$

The Hubble and deceleration parameters for this solution are

$$H(t') = \frac{b^2}{t\Big(b^2 + \ell^{-2}(ab-1)^2\, t'^2\Big)}, \tag{43}$$

and

$$q(t') = \frac{3(ab-1)^2\, t'^2}{\ell^2 b^2}, \tag{44}$$

respectively. The positive deceleration parameter (44) implies that the universe which is described by the FRW metric (41) has a decelerating expansion.

5 Conclusions

We have obtained a Lie bialgebra for the AdS Lie algebra in (2+1)-dimensional spacetime. Applying a Manin triple corresponding to the AdS Lie bialgebra as a gauge symmetry algebra of the Chern-Simons theory, we have introduced a new (2+1)-dimensional bi-metric gravity model. Our ghost-free Chern-Simons bi-gravity action is an exactly soluble model without any local degrees of freedom which describes two interacting massless spin-2 fields. Its free field limit has also no local degrees of freedom, and is a ghost-free action. Our model is different from known three dimensional bi-metric massive gravity theories such as dRGT and ZDG models. The black hole solution (21)-(23) of our model is different from the previously obtained three dimensional bi-gravity black hole solutions [44, 45]. In our solution, one of the metrics is the BTZ black hole metric, and the other metric is a new black hole metric with two curvature singularities and two horizons unlike two coordinate singularities of the previously obtained solutions of the three dimensional $f-g$ theory. Our solution is also different from the other solutions in its asymptotic behaviour, and unlike other solutions, it has not asymptotically AdS form. Our bi-gravity model admits a homogeneous and isotropic FRW cosmological solution with two different scale factors in two metrics $g_{\mu\nu}$ and $f_{\mu\nu}$, which describe two universes with the decelerating expansions. It is also interesting to study details of the new black hole metric (22) as well as gravity/CFT correspondence at the boundary of the bi-gravity model (11), which we leave them to later. Chern-Simons formulation of our interacting model simplifies its quantization, which may be interesting in the context of quantum gravity. Study of (3+1)-dimensional version of the AdS Lie bialgebra, and resultant (3+1)-dimensional gauge invariant interacting model is also a useful task which may have interesting features.

Acknowledgments: We would like to express our heartfelt gratitude to M.M. Sheikh-Jabbari, F. Darabi, M.R. Setare and F. Loran for their useful comments and discussions. This research was supported by a research fund No. 217D4310 from Azarbaijan Shahid Madani university.

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TTMP vol 2 (1), 2025
doi: 10.30511/ttmp.2025.2053874.1049

Transactions in Theoretical
and Mathematical Physics

Hamiltonian Formulation of Scalar Density Field Theory

Alireza Shariati Joni[a,1]

[1]Department of Physics, Isfahan University of Technology, Isfahan 84156-83111, Iran

Received: 11 February 2025 / Accepted: 26 February 2025 / Published: 26 February 2025

Abstract In the ADM formalism, spacetime is foliated by spacelike hypersurfaces, dividing spacetime into space and time. In this formalism, the evolution of fields can be separated into evolution on these hypersurfaces and evolution from one hypersurface to the next. Teitelboim showed that by solving the algebra of the generators under certain assumptions, the Hamiltonian of scalar field, Maxwell's field, and general relativity can be obtained [1]. In this paper, we solve this algebra for scalar density field theory in arbitrary curved spacetime and demonstrate that scalar density fields with weights zero and one are permissible within the Hamiltonian formalism.

1 Introduction

The scalar density field is defined by its properties under a general coordinate transformation:

$$\phi(x') = \phi(x)\left|\frac{\partial x}{\partial x'}\right|^{\lambda}. \tag{1}$$

In this expression,$|\frac{\partial x}{\partial x'}|$ denotes the Jacobi determinant of the transformation, and x represents a point in "space". We regard the parameter λ as the weight of the scalar density field. By selecting $\lambda = 0$, the scalar field theory will be obtained. The effect of a weighted scalar field is not exclusively significant in curved spacetime. Experiments conducted in particle physics laboratories are all performed in an approximately flat spacetime. For instance, we know that the pseudoscalar pion field acquires a negative sign under a parity transformation, for which the Jacobi determinant is -1 [2]. If we assume that this field is described by equation (1), we find that the pion is a scalar density field with an odd weight. To determine the weight of pseudoscalar fields, one must resort to field theory in curved spacetime.

Spacetime is decomposed into space and time. This decomposition leads to the emergence of the Dirac algebra among the Hamiltonian generators. The diffeomorphism generators can be determined by identifying which class of tensors the fields of the theory belong to. Using the Dirac algebra, the time evolution generators can then be derived.

The only scalar density fields that can satisfy the entire algebra are those with weight one or zero. In Minkowski spacetime, their solutions are the same.

The Lagrangian formulation can be investigated by extending the three-dimensional diffeomorphism framework to the four-dimensional case. By expressing the corresponding terms in the Hamiltonian of scalar density field, it is shown that these terms do not yield the correct Hamiltonian form through a Legendre transformation, except in the case of a weightless scalar field.

2 ADM Decomposition

To describe this approach, we first decompose the four dimensional spacetime with metric $g_{\mu\nu}$ into space plus time. The spacetime is hyperbolic, and on every hyperbolic manifold, there exists a temporal function, such as τ, such that each surface of constant time is a Cauchy surface with metric h_{ij} where $i, j = \{1,2,3\}$ [3]. Therefore, the manifold can be foliated by these Cauchy surfaces. An important characteristic of a hyperbolic manifold is that every null or timelike curve intersects each Cauchy surface exactly once. Hence, τ can be considered as a representation of time, and the Cauchy surfaces can be regarded as representations of space [4]. The spatial part of the metric, h_{ij} can be obtained from the ADM metric [5, 6]:

[a]e-mail: ar.shariati.j@gmail.com

$$g_{\mu\nu} = \begin{pmatrix} -N^2 + N^i N_i & N_j \\ N_k & h_{jk} \end{pmatrix}. \tag{2}$$

N is called the lapse function, and N_i is known as the shift function. The upper index is defined using the relation $N^i = h^{ij}N_j$, where h^{ij} is the inverse of the metric with the condition $h_{ik}h^{kj} = \delta_i^j$.

The result of this decomposition is the emergence of Dirac algebra among the Hamiltonian generators of the theory. This algebra also arises from other ideas such as parametrized systems or the principle of path independence [7–9].

The generators are decomposed into one generator normal to the constant-time hypersurface $\mathscr{H}_\perp$ and three generators parallel to the hypersurface $\mathscr{H}_i$, which are also known as generators of infinitesimal diffeomorphisms [6].

We define the weight under spatial coordinate transformations like equation (1). For example, the determinant of the metric h is a scalar density with weight 2.

The Hamiltonian is obtained as a linear combination of these generators using the lapse and shift functions:

$$\mathscr{H} = N\mathscr{H}_\perp + N^i\mathscr{H}_i. \tag{3}$$

The algebra of these generators is as follows [9]:

$$\{\mathscr{H}_\perp(x), \mathscr{H}_\perp(x')\} = \delta_{,i}(x,x')\left(h^{ij}(x)\mathscr{H}_j(x) + h^{ij}(x')\mathscr{H}_j(x')\right). \tag{4}$$

$$\{\mathscr{H}_i(x), \mathscr{H}_\perp(x')\} = \delta_{,i}(x,x')\mathscr{H}_\perp(x). \tag{5}$$

$$\{\mathscr{H}_i(x), \mathscr{H}_j(x')\} = \delta_{,i}(x,x')\mathscr{H}_j(x) + \delta_{,j}(x,x')\mathscr{H}_i(x'). \tag{6}$$

The symbol δ denotes the Dirac delta distribution defined as [9]:

$$\int d^3x\, h^{\frac{1}{2}}\ \delta(x,x')\ F(x) = F(x'). \tag{7}$$

$\delta_{,j} := \frac{\partial \delta}{\partial x^j}$, and $\{,\}$ denotes the Poisson bracket given by

$$\{\phi(x), \pi(x')\} = \delta(x,x'), \tag{8}$$

implying that the weight of π, the conjugate momentum of the scalar density field, equals $1-\lambda$.

Teitelboim, in his doctoral dissertation [1], has shown that if we decompose $\mathscr{H}_\perp$ into gravitational and matter parts, i.e., $\mathscr{H}_\perp = \mathscr{H}_\perp^{\text{matter}} + \mathscr{H}_\perp^{\text{gravitational}}$, the matter part is ultralocal with respect to the metric, meaning that it does not depend on the derivative of the metric field. This result follows from his first assumption that the matter part does not include the conjugate momentum of the metric. Another result of this assumption is that Dirac algebra (4) is closed for matter part (same as (6)):

$$\{\mathscr{H}_\perp^{\text{matter}}(x), \mathscr{H}_\perp^{\text{matter}}(x')\} = \delta_{,i}(x,x')\left(h^{ij}(x)\mathscr{H}_j^{\text{matter}}(x) + h^{ij}(x')\mathscr{H}_j^{\text{matter}}(x')\right), \tag{9}$$

$$\{\mathscr{H}_i^{\text{matter}}(x), \mathscr{H}_j^{\text{matter}}(x')\} = \delta_{,i}(x,x')\mathscr{H}_j^{\text{matter}}(x) + \delta_{,j}(x,x')\mathscr{H}_i^{\text{matter}}(x'). \tag{10}$$

His second and third assumptions are that $\mathscr{H}_\perp$ is ultralocal with respect to the conjugate momentum of the matter field and is of second order.

Teitelboim demonstrated that the algebra (4)-(6) can be solved as follows. Initially, by using the following rule to calculate the variation of a function of canonical variables such as F:

$$\delta F = \int d^3x\, h^{\frac{1}{2}}\{F,\ \mathscr{H}_\mu\}N^\mu. \tag{11}$$

If F belongs to a class of (weighted) tensor fields, the generators of infinitesimal diffeomorphism can be derived by comparing the result of (11) with its Lie derivative. Then from equations (5) and (6), we find that the weights of $\mathscr{H}_\perp$ and $\mathscr{H}_i$ are equal to 1. Then, from equation (4), $\mathscr{H}_\perp$ will be obtained.

In the next section, we will derive the properties of the Hamiltonian generators of matter for the scalar density field theory.

3 Hamiltonian Generators of Scalar Density Field

3.1 Diffeomorphism Generators

For an infinitesimal spatial diffeomorphism $x^i \to x'^i = x^i + N^i$ we have:

$$\delta\phi(y) = \phi'(y) - \phi(y) = \int d^3x\ \{\phi(y),\ \mathscr{H}_i(x)\}N^i(x). \tag{12}$$

From equation (1) we can find:

$$\delta\phi = \lambda\phi N^j_{,j} + N^j\phi_{,j}. \tag{13}$$

This implies that the generators of diffeomorphism can be expressed in the following form:

$$\mathscr{H}_i^{\text{matter}} = -\lambda\phi\pi_{,i} + (1-\lambda)\phi_{,i}\pi. \tag{14}$$

The transformation of the momentum conjugate to the scalar density field under a diffeomorphism can be investigated:

$$\delta\pi = \int d^3x\, N^i(x)\{\pi,\, \mathscr{H}_i(x)\} = (\pi N^i)_{,i} - \lambda\pi N^i{}_{,i}\,. \tag{15}$$

which is in a form that we expect from a scalar density with weight $1-\lambda$. Diffeomorphism generators satisfy Dirac algebra (6). The detailed calculation can be found in Appendix B.

3.2 Generator of Time Evolution

For the remainder of this paper, the superscript "matter" for generators will be dropped unless stated otherwise. For a field with zero weight, the generator must reduce to that of the Klein-Gordon field theory. Therefore, in accordance with Teitelboim's third assumption, we express the form of $\mathscr{H}_\perp$ as a second-order function of the matter momentum:

$$\mathscr{H}_\perp = \frac{1}{2}h^{\lambda-\frac{1}{2}}\pi^2 + V. \tag{16}$$

V includes a mass term, interaction terms, and local terms involving derivatives of the field. We assume that V is in the following form:

$$V = A[\phi] + B^i[\phi]\phi_{,i} + C^{ij}[\phi]\phi_{,ij} + D^{ij}[\phi]\phi_{,i}\phi_{,j}. \tag{17}$$

By substituting (16) into the Jacobi identity:

$$\{\{\phi(y),\, \mathscr{H}_\perp(x)\},\, \mathscr{H}_\perp(x')\} - \{\{\phi(y),\, \mathscr{H}_\perp(x')\},\, \mathscr{H}_\perp(x)\} = \{\phi(y),\, \{\mathscr{H}_\perp(x),\, \mathscr{H}_\perp(x')\}\}, \tag{18}$$

we find that our assumptions about the form of $\mathscr{H}_\perp$ could satisfy Jacobi identity. By a similar calculation to Appendix A one could prove that V includes only the first derivative, the second derivative, and the product of two first derivatives of the field. Terms involving other combinations of field derivatives must have zero coefficients.

3.2.1 Covariant Derivative of Scalar Density Field

Dirac algebra (5) forces the $\mathscr{H}_\perp$ to be a scalar density with weight one. Note that (5) is not closed for matter part and the gravitational part must be considered in diffeomorphism generators [1]:

$$\{\mathscr{H}_i^{\text{matter}}(x) + \mathscr{H}_i^{\text{gravitational}}(x), \mathscr{H}_\perp^{\text{matter}}(x')\} = \delta_{,i}(x,x')\mathscr{H}_\perp^{\text{matter}}(x), \tag{19}$$

The gravitational part of diffeomorphism generator is:

$$\mathscr{H}_i^{\text{gravitational}} = h_{jk,i}p^{jk} - 2(p^{jk}h_{ij})_{,k}. \tag{20}$$

p^{ij} is the momentum corresponding to the metric field with definition:

$$\{h_{ij}(x),\ p^{kl}(x')\} = \delta_i^k\delta_j^l\delta(x,\ x'). \tag{21}$$

The partial derivative of a scalar density field does not transform as a weighted tensor under coordinate transformations. Therefore, by defining connection γ_i, we must express the covariant derivative of the field in the following form, which transforms as a vector density with weight λ under coordinate transformations [10]:

$$\nabla_i\phi := \phi_{,i} + \lambda\gamma_i\phi. \tag{22}$$

Vector density is also defined by its properties under a general coordinate transformation:

$$V_i'(x') = V_l(x)\left|\frac{\partial x}{\partial x'}\right|^\lambda \frac{\partial x^l}{\partial x'^i}. \tag{23}$$

Although the connection is not unique, if we assume it depends only on the metric, the only form of the connection, using (5), would be as follows:

$$\gamma_i = -\frac{1}{2}h^{jk}h_{jk,i}. \tag{24}$$

This connection can be descriptively written as a canonical transformation of the scalar field ψ with zero weight:

$$g^{\lambda/2}\psi_{,i} = (\psi g^{\lambda/2})_{,i} + \lambda\left[-\frac{1}{2}g^{kj}g_{kj,i}\right](\psi g^{\lambda/2}). \quad (25)$$

In this case, the canonical transformation is defined as $\psi = \phi h^{\frac{-\lambda}{2}}$ and $\pi_\psi = \pi h^{\frac{\lambda}{2}}$. This choice of degrees of freedom preserves the Poisson bracket algebra. It is evident that applying this covariant derivative violates Teitelboim's first assumption, as $\mathscr{H}_\perp$ is not ultralocal with respect to the metric. Any other scalar density with weight λ such as ω (excluding ϕ), could be used to define a vector density of the form:

$$\nabla_i^\omega \phi = \phi_{,i} - (ln(\omega))_{,i}. \quad (26)$$

For example we could use determinant of momentum conjugate to metric p:

$$\nabla_i^p \phi = \phi_{,i} - p^{\frac{\lambda}{2}}(p^{\frac{-\lambda}{2}})_{,i}. \quad (27)$$

Which is also a canonical transformation from scalar field in the form $\psi = \phi p^{\frac{\lambda}{2}}$ and $\pi_\psi = \pi p^{-\frac{\lambda}{2}}$. But this connection is not consistent with first assumption of Teitelboim. we could also find an appropriate connection using momentum conjugate to scalar density field as:

$$\nabla_i^\pi \phi = \phi_{,i} - \frac{\lambda}{1-\lambda}(ln(\pi))_{,i}, \quad (28)$$

which is a canonical transformation in the form $\psi = \phi\pi^{\frac{-\lambda}{1-\lambda}}$ and $\pi_\psi = (1-\lambda)\pi^{\frac{1}{1-\lambda}}$. This connection violates Teitelboim's second assumption.

3.2.2 Calculating $\mathscr{H}_\perp$

If we write $\mathscr{H}_\perp$ using a connection of the form equation (24), by applying the Jacobi identity from equation (18) for a second-order Hamiltonian in terms of momenta similar to (16), we have:

$$\mathscr{H}_\perp = \frac{1}{2}h^{\lambda-\frac{1}{2}}\pi^2 + \sum_n m_n h^{\frac{1}{2}-\lambda n}\phi^{2n} + B^i[\phi]\nabla_i\phi + C^{ij}[\phi]\nabla_i\nabla_j\phi + D^{ij}[\phi]\nabla_i\phi\nabla_j\phi. \quad (29)$$

The parameters m_n are numerical constants. Since odd powers of the field would lead to the absence of a lower energy bound, we have discarded them. The coefficients B^i, C^{ij}, D^{ij} are ultralocal with respect to the matter field and are given by the following relations (detailed calculation can be found in Appendix A):

$$D^{ij}[\phi] = (\frac{1}{2}-\lambda)h^{ij}h^{\frac{1}{2}-\lambda}. \quad (30)$$

$$C^{ij}[\phi] = -\lambda\phi h^{ij}h^{\frac{1}{2}-\lambda}. \quad (31)$$

$$B^i[\phi] = (1-\lambda)\phi h^{\frac{1}{2}}h^{ij}(h^{-\lambda})_{,j}. \quad (32)$$

The solution (29), when combined with the properties (30), (31), and (32), as well as the Jacobi identity (18), also satisfies (4). It is necessary to check for weight of this solution again. Terms (30) and (31) satisfy equation (5) but (32) contains the partial derivative of the metric's determinant which is not a tensor. Therefore, we are forced to choose only $\lambda = \{0,1\}$. It is evident that for a scalar field with $\lambda = 0$, the generator reduces to that of the Klein-Gordon field:

$$\mathscr{H}_\perp = \frac{1}{2}h^{-\frac{1}{2}}\pi^2 + \frac{1}{2}h^{ij}h^{\frac{1}{2}}\phi_{,i}\phi_{,j} + \sum_n m_n h^{\frac{1}{2}}\phi^{2n}. \quad (33)$$

And for $\lambda = 1$ we have:

$$\mathscr{H}_\perp = \frac{1}{2}h^{\frac{1}{2}}\pi^2 - \frac{1}{2}h^{ij}h^{-\frac{1}{2}}\nabla_i\phi\nabla_j\phi + \sum_n m_n h^{\frac{1}{2}-n}\phi^{2n} - \phi h^{ij}h^{-\frac{1}{2}}\nabla_i\nabla_j\phi. \quad (34)$$

4 Lagrangian Formulation

In the previous section, within the Hamiltonian formulation, the coordinate transformations and the definition of scalar density were established using equation (1) and in three-dimensional space. However, in studying the Lagrangian formulation, we must consider them within the framework of spacetime transformations [11]:

$$\phi(x') = \phi(x)\left|\frac{\partial x}{\partial x'}\right|^w. \quad (35)$$

Here, x denotes a spacetime coordinate, and w is the weight of scalar density field. In the Lagrangian formulation, the kinetic term is expressed as follows:

$$\mathscr{L} = \frac{1}{2} g^{\frac{1}{2}-w} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi. \quad (36)$$

Where the covariant derivative similar to (24) is:

$$\nabla_\mu \phi := \phi_{,\mu} - \frac{1}{2} w (ln(g))_{,\mu} \phi, \quad (37)$$

which is a vector density with weight w. g^w in equation (36) is added to ensure that the action is scalar. By performing a Legendre transformation, the Hamiltonian is obtained:

$$\mathscr{H} = -\frac{1}{2} N^{2w+1} h^{w-\frac{1}{2}} \pi^2 + \frac{w}{2} \phi \pi (ln(hN^2))_{,0} + \pi \nabla_\mu \phi N^i - \frac{1}{2} N^{1-2w} h^{\frac{1}{2}-w} h^{ij} \nabla_i \phi \nabla_j \phi. \quad (38)$$

Here we have used the metric (2), and the canonical momentum is expressed as:

$$\pi := \frac{\partial \mathscr{L}}{\partial \phi_{,0}} = g^{\mu 0} \nabla_\mu \phi g^{\frac{1}{2}-w}. \quad (39)$$

This Hamiltonian differs from (3) unless for $w = 0$, in which case it reduces exactly to the Klein-Gordon field.

5 Conclusions and Discussion

A weighted scalar field exhibits behavior distinct from the Klein-Gordon field, which can be used to describe the differences between scalar and pseudo-scalar fields under parity transformation. In this paper, by analyzing the solution of the Dirac algebra for the scalar density field within the Hamiltonian formulation, we investigated its properties, including its dependence on metric derivatives, which is inconsistent with Teitelboim's first assumption. We demonstrated that the Hamiltonian is constructed from combinations of field derivatives that differ from those of the Klein-Gordon theory. In the Hamiltonian formalism, scalar density fields with weight one and zero are permissible. Furthermore, this Hamiltonian cannot be derived via a Legendre transformation from the corresponding Lagrangian. However, for a scalar field with zero weight, all such inconsistencies vanish, and the generators of the Klein-Gordon field theory are recovered.

The approach presented in this paper is not the only method for studying the theory of scalar density fields. For example, appropriate connections were derived from the combination of the conjugate momentum of the metric and the conjugate momentum of the scalar density field. Using these connections would introduce new properties to the theory. However, they violate Teitelboim's first and second assumptions, respectively. Moreover, the use of h to adjust the weight of terms in the Hamiltonian and Lagrangian is not a critical choice. Employing other weighted tensors could potentially resolve some of the challenges in this theory.

Acknowledgements This paper is based on the author's master's thesis, supervised by Prof. Farhang Loran at Isfahan University of Technology.

Appendix A: Calculation of Time Evolution Generators

In this section, we provide detailed calculation of equations (30), (31), (32). If we put equation (29) in Dirac algebra (9) the LHS is:

$$\begin{aligned}
&\{\mathscr{H}_\perp(x), \mathscr{H}_\perp(x')\} = \\
&\int d^3y (\pi(x') h^{\lambda-\frac{1}{2}}(x') \delta(x', y) \left[B^i(x) \delta_{,i}(x,y) \right. \\
&+ 2D^{ij}(x) \nabla_i \phi(x) \delta_{,i}(x,y) \\
&\left. + C^{ij}(x)(\delta_{,ij} + 2\gamma_j(x) \delta_{,i}(x,y) - \Gamma^k_{ij}(x) \delta_{,k}(x,y)) \right] - x \leftrightarrow x' \\
&= \pi(x) h^{\lambda-\frac{1}{2}(x)} \left[B^i(x') \delta_{,i}(x,x') + 2D^{ij}(x') \nabla_j \phi(x') \delta_{,i}(x,x') \right. \\
&- C^{ij}(x') \delta_{,ij}(x,x') + 2C^{ij}(x') \gamma_j(x') \delta_{,i}(x,x') \\
&\left. - \Gamma^k_{ij}(x') \delta_{,k}(x,x') C^{ij}(x') \right] - x \leftrightarrow x'. \quad (A.1)
\end{aligned}$$

Where Γ^i_{jk} is the Christoffel symbol corresponding to h^{ij}. Note that $\nabla_i \nabla_j \phi$ is calculated as:

$$\nabla_i \nabla_j \phi = (\nabla_j \phi)_{,i} + \lambda \gamma_i \nabla_j \phi - \Gamma^k_{ij} \nabla_k \phi. \quad (A.2)$$

Using arbitrary test functions $N(x)$, $N'(x')$ we can integrate over x, x':

$$\begin{aligned}
&\int d^3x d^3x' \, N(x) N'(x') \{\mathscr{H}_\perp(x), \mathscr{H}_\perp(x')\} \\
&= \int d^3x \left[-(N\pi h^{\lambda-\frac{1}{2}})_{,i} \left[N' B^i + 2N' D^{ij} \nabla_j \phi + 2C^{ij} \gamma_j N' \right. \right. \\
&\left. \left. - N' \Gamma^i_{jk} C^{jk} \right] - (N\pi h^{\lambda-\frac{1}{2}})_{,ij} N' C^{ij} - N \leftrightarrow N' \right]. \quad (A.3)
\end{aligned}$$

After simplifying symmetric terms under interchange of x and x', we have:

$$\int d^3x d^3x'\ N(x)N'(x')\{\mathscr{H}_\perp(x),\mathscr{H}_\perp(x')\}$$
$$=\int d^3x(NN'_{,i}-N'N_{,i})\left[h^{\lambda-\frac{1}{2}}\pi B^i+2\pi h^{\lambda-\frac{1}{2}}D^{ij}\nabla_j\phi\right.$$
$$+2C^{ij}\gamma_j\pi h^{\lambda-\frac{1}{2}}-\Gamma^i_{jk}\pi h^{\lambda-\frac{1}{2}}C^{jk}-2\pi h^{\lambda-\frac{1}{2}}C^{ij}_{,j}$$
$$\left.+\left(C^{ij}\pi h^{\lambda-\frac{1}{2}}\right)_{,j}\right]. \quad (A.4)$$

Momentum and its derivatives are independent in phase space; therefore, we separate terms depending on momentum P^i and parts depending on derivative of momentum Q^{ij}.

$$\int d^3x d^3x'\ N(x)N'(x')\{\mathscr{H}_\perp(x),\mathscr{H}_\perp(x')\}$$
$$=\int d^3x(NN'_{,i}-N'N_{,i})[\pi P^i+\pi_{,i}Q^{ij}] \quad (A.5)$$

P^i and Q^{ij} are:

$$Q^{ij}=C^{ij}h^{\lambda-\frac{1}{2}}. \quad (A.6)$$

$$P^i=h^{\lambda-\frac{1}{2}}\pi B^i+2\pi h^{\lambda-\frac{1}{2}}D^{ij}\nabla_j\phi+2C^{ij}\gamma_j\pi h^{\lambda-\frac{1}{2}}$$
$$-\Gamma^i_{jk}\pi h^{\lambda-\frac{1}{2}}C^{jk}-2\pi h^{\lambda-\frac{1}{2}}C^{ij}_{,j}+(C^{ij}h^{\lambda-\frac{1}{2}})_{,j}. \quad (A.7)$$

Similar calculation could be done on the RHS of (9):

$$\int d^3x d^3x'\ N(x)N'(x')\delta_{,i}\left(\mathscr{H}_i(x)h^{ij}(x)+\mathscr{H}_i(x')h^{ij}(x')\right)$$
$$=\int d^3x(NN'_{,i}-N'N_{,i})h^{ij}\left[-\lambda\phi\pi_{,i}+(1-\lambda)\pi\phi_{,i}\right]. \quad (A.8)$$

By equating (A.4) and (A.8) terms with derivative of momentum π must cancel each other so we have:

$$C^{ij}=-\lambda\phi h^{ij}h^{\frac{1}{2}-\lambda}. \quad (A.9)$$

From the terms with derivative of field ϕ in (A.7) one can find:

$$D^{ij}=(\frac{1}{2}-\lambda)h^{ij}h^{\frac{1}{2}-\lambda}, \quad (A.10)$$

and from the remaining terms:

$$0=B^ih^{\frac{1}{2}-\lambda}+2(\frac{1}{2}-\lambda)h^{ij}\lambda\gamma_j\phi-2\lambda h^ij\phi\gamma_j+\Gamma^i_{jk}\lambda\phi h^{kj}$$
$$+h^{-\frac{1}{2}+\lambda}\lambda\phi(h^{ij}h^{\frac{1}{2}-\lambda})_{,j}-\lambda\phi h^{ij}h^{\frac{1}{2}-\lambda}(h^{-\frac{1}{2}+\lambda})_{,j}. \quad (A.11)$$

After some simplification we obtain:

$$B^i[\phi]=(1-\lambda)\phi h^{\frac{1}{2}}h^{ij}(h^{-\lambda})_{,j}. \quad (A.12)$$

Appendix B: Calculation of Diffeomorphism Generators

Diffeomorphism generators must satisfy Dirac algebra (5). If we put (14) in this algebra, we obtain:

$$\{\mathscr{H}_i(x),\ \mathscr{H}_j(x')\}=$$
$$\int d^3y\left[((1-\lambda)\pi(x)\delta_{,i}(x,y)-\lambda\delta(x,y)\pi(x)_{,i})\right.$$
$$((1-\lambda)\delta(x',y)\phi_{,j}(x')$$
$$-\lambda\phi(x')\frac{\partial\delta(x',y)}{\partial x'^j})-((1-\lambda)\delta(x,y)\phi_{,i}(x)-\lambda\phi(x)\delta_{,i})$$
$$\left.((1-\lambda)\pi(x')\frac{\partial\delta(x',y)}{\partial x'^j}-\lambda\delta(x',y)\pi_{,j}(x'))\right]. \quad (B.13)$$

By integrating using arbitrary test functions $N(x)$ and $N'(x')$, we have:

$$\int dxdx'N(x)N'(x')\ \{\mathscr{H}_i(x),\ \mathscr{H}_j(x')\}=$$
$$\int d^3x\big(-NN'(1-\lambda)^2\phi_{,j}\pi_{,i}-N'N_{,i}(1-\lambda)^2\pi\phi_{,j}$$
$$-\lambda(1-\lambda)N'N\phi_{,j}\phi_{,i}+\lambda^2N'\phi[N_{,j}\pi_{,i}+N\pi_{,ij}]$$
$$+\lambda(1-\lambda)N'\phi[\pi N]_{,ij}+NN'\lambda(1-\lambda)\phi_{,i}\pi_{,j}$$
$$+\lambda^2N'\pi_{,j}(N\phi)_{,i}-(1-\lambda)^2N'\pi(N\phi_{,i})_{,j}$$
$$-\lambda(1-\lambda)\pi N'[N\phi]_{,ij}\big). \quad (B.14)$$

After simplifying, we obtain:

$$\int dxdx'N(x)N'(x')\ \{\mathscr{H}_i(x),\ \mathscr{H}_j(x')\}$$
$$=\int dxdx'N(x)N'(x')\ \left(\mathscr{H}_i(x')\delta_{,j}(x,x')+\mathscr{H}_j(x)\delta_{,i}(x,x')\right). \quad (B.15)$$

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TTMP vol 2 (1), 2025
doi: 10.30511/ttmp.2025.2053798.1048

Transactions in Theoretical and Mathematical Physics

Crypto-Fiat Exchange Rates Network as indicator of macroeconomic dynamics

Y. Shahmari[1a], **H. Zare**[2b], **H. Hatami**[1c], **S. Rouhani**[3d]

[1] Physics Department, Sharif University of Technology, Tehran PO Box 11155-9161, Iran
[2] Department of Mathematical Sciences, Sharif University of Technology,P.O. Box 11155-9415, Tehran, Iran
[3]Physics Department and Research Center for High Energy Physics, Sharif University of Technology, Tehran PO Box 11155-9161, Iran

Received: 18 February 2025 / Accepted: 27 February 2025 / Published: 27 February 2025

Abstract The relationship between cryptocurrencies and fiat currencies reflects the dynamic connection between them, shaped by investors, macroeconomic forces, regulatory frameworks etc. In this article, we have shown how markets have evolved from a Bitcoin-centric ecosystem to a more Interconnected network, reflecting broader changes in the maturity of the financial system and geopolitical strategies

1 Introduction

A fundamental question in the crypto markets is: which cryptocurrency is the driver of the market movements? The immediate response may be that Bitcoin is the leader and other cryptos follow. A similar question in fiat currencies would be whether the Euro follows the Dollar or vice versa. We have attempted to answer these questions by looking at the Crypto/fiat exchange rates across diverse geographic markets and cryptocurrencies. There exist many crypto-fiat exchanges, where one may buy or sell cryptocurrencies. They facilitate conversion and transfer between traditional currencies (fiat) and cryptocurrencies, in such exchanges, these conversion rates are quoted and the rates change with time rapidly. This is certainly not exclusive to a single platform and may take other forms. However, such exchanges occur on a large scale. So we may think of a network of crypto/fiat pairs affecting the exchange rate of other such pairs. There is a question about the main drivers of pair exchange rates. Some good guesses exist in the market that surely bitcoin is the driver since it enjoys the largest circulating capital, high liquidity, and market dominance. However, this is not always the case. If you think of such an exchange rate as a stochastic time series, we face a deeper question of what we mean by calling a stochastic process the main driver node of the network of all crypt-fiat pairs. There are various ways of finding the driver and the derived between two-time series[1][2][3] but each of them has its flaws. We may examine the cross-correlation between the two rates at the simplest level. This leaves us with a correlation matrix. The returns can be expressed as a linear combination of uncorrelated (neither necessarily independent) random variables, using the eigenvectors of this matrix. The eigenvalues correspond to the squared volatilities of these components. However, In the case of Gaussian returns, the principal components are independent.
Estimating of full correlation matrix with empirical data is tough. This is depicted by the empirical spectrum of eigenvalues in the correlation matrix, against a completely random correlation matrix for which the analytic form is derived for large dimensions. The rates as a whole are in simple terms reflected by the largest eigenvalues while most eigenvalues can be attributed to a purely random correlation matrix. Volatility correlations cannot be described in terms of factor models either. There are many methods for the construction of arbitrarily correlated non-Gaussian random variables, and each of them is associated with different multivariate distribution.[4–6]
By constructing the maximal spanning tree, we can talk about the dominance of the main cluster. The central node of that main cluster can be regarded as the main driver of the network. Then, we can talk about the stability of the network using the largest eigenvalue and the spectral gap of the system. These results can be used to find the noise-dominated vs. signal-dominated parts of the network over time.

Our analysis yields three main contributions: (1) The existence of community leadership roles in cryptocurren-

[a]e-mail: Shahmari.acer@physics.sharif.edu
[b]e-mail: Hooman.zare99@sharif.edu
[c]e-mail: Hanie.hatami@physic.sharif.edu
[d]e-mail: (Corresponding author) srouhani@sharif.edu

cies that conflict the Bitcoin supremacy; (2) the signs of geo-distributed fiat currency activities signaling regional market integration; (3) Exposure of the critical thresholds where the network is resilient/impermeable through spectral/MST stability metrics, representing phase-transition behavior in market topology.

2 Methods

2.1 Data and Sources

The data used in this study consists of time-series data on prices, returns, market capitalizations, and volumes of various crypto-fiat currency pairs. [Appendix Appendix A] Two distinct datasets were utilized:

- Historical data from November 9, 2017, to August 21, 2022.
- Recent data from May 23, 2023, to August 19, 2024.

The data was gathered using the CoinGecko API and data center. The time-series data includes daily exchange rates and returns for each currency pair. The historical data required minimal preprocessing, with only a few instances of missing or NaN values corrected by carrying forward the previous values. The newer data contained no missing or NaN values.

2.2 Correlation Analysis

The Pearson correlation coefficient ρ between two time series X and Y is calculated as:

$$\rho_{XY} = \frac{\text{cov}(X,Y)}{\sigma_X \sigma_Y} = \frac{\sum_{i=1}^{n}(X_i - \bar{X})(Y_i - \bar{Y})}{\sqrt{\sum_{i=1}^{n}(X_i - \bar{X})^2}\sqrt{\sum_{i=1}^{n}(Y_i - \bar{Y})^2}}, \quad (1)$$

where $\text{cov}(X,Y)$ is the covariance of X and Y, and σ_X and σ_Y are the standard deviations of X and Y, respectively. The Pearson correlation coefficient was calculated to form a cross-correlation matrix ρ for each pair of crypto and fiat currencies. This matrix highlights the linear relationships between the daily returns of different currency pairs. The correlations were sorted and visualized to identify significant relationships. Although the cross-correlation matrix provides a snapshot of the relationships between currency pairs, it does not capture the temporal dynamics of these relationships.[7]

We are not interested in the global correlations of the market, because they miss the dynamics of observables of the market. A sliding window approach was employed to capture the temporal dependencies of correlations [8, 9]. For a given window size w, the moving correlation $\rho(t)$ at time t is computed using the data from $t - w$ to t. This results in a time series of correlations, illustrating how the relationships between currency pairs evolve. The decay could be modeled as a stochastic process affecting the entries of the correlation matrix.[10]

Using the sliding window method, we expect the determination of covariances to be noisy, and therefore the empirical correlation matrix to be to a large extent random, i.e. the structure of the matrix to be dominated by measurement noise. This is because for M time-series, the correlation matrix contains $M(M-1)/2$ entries, which must be determined from M time-series of length N. For the rolling window, N is considerably close to M.

2.3 Eigenvalue Analysis

Eigenvalues capture the key elements of a matrix. In random matrix theory (RMT), the Marchenko-Pastur distribution describes the eigenvalues of a random matrix [11]. Meaning that if the time series are continuous and independently, identically distributed (i.i.d.), the eigenvalues of the correlation matrix follow this distribution. But, in the case of crypto-fiat pairs, time-series consisted of a fixed crypto and varying fiats are far from independent and it is well known in the literature that the returns of economic time series are jumpy [12], hence discontinuous [13]. This means that we need to construct a new theorem for the crypto-fiat time-series.

The largest eigenvalue (λ_{max}) usually corresponds to the market mode, explaining the maximum variance [14]. In crypto-fiat markets, the largest eigenvalues are indicators of a dominant currency or overall market sentiment. The smallest eigenvalues (λ_{min}) however, relate to specific, maybe anti-correlated pairs and finds the fastest modes and fastest responses of the system. In traditional RMT, the bulk eigenvalues are semicircle (Wigner) or Marchenko-Pastur [15], but we will argue that the power law distribution of eigenvalues are more from heavy-tailed data or scale-free networks.

2.4 Non-Gaussian returns

If the returns are non-Gaussian we need to construct non-Gaussian correlated variables. Bouchaud et. al. [16] showed that correlated Gaussian variables can be transformed as a linear superposition of independent random Gaussian variables.

Let independent non-Gaussian factors F_a with distributions $P_a(f_a)$ be linearly combined as:

$$X_i = \sum_{a=1}^{M} b_{a,i} f_a, \tag{2}$$

where $b_{a,i}$ is a basis matrix diagonalizing the correlation matrix of X_i. The inverse transform is:

$$f_i = \sum_{i=1}^{M} b_{a,i} X_a. \tag{3}$$

$P_a(f_a)$ can be any arbitrary probability density function. An interesting case is when $P_a(f_a)$ is scale free:

$$P_a(f_a) \approx \lim_{|f_a| \to \pm\infty} \frac{\mu A_{a,\mu}}{|f_a|^{1+\mu}}. \tag{4}$$

Here, the variables X_i inherit tails with exponent μ and amplitude:

$$A_i^\mu = \sum_{a=1}^{M} |b_{ai}|^\mu A_a^\mu. \tag{5}$$

For Lévy stable factors ($\mu < 2$), X_i become multivariate Lévy variables.

We can involve non-linear transformations of correlated Gaussian variables. Let's define:

$$\tilde{X}_i = \sum_{a=1}^{M} b_{ai} f_a, \quad \text{with} \quad f_a \sim \mathcal{N}(0,1), \tag{6}$$

and map these to non-Gaussian variables via $X_i = G_i(\tilde{X}_i)$. The marginals $P_i(X_i)$ are enforced using:

$$P_i(X_i) = P_G(\tilde{X}_i) \left| \frac{d\tilde{X}_i}{dX_i} \right|, \tag{7}$$

where P_G is the Gaussian density. This corresponds to a Gaussian copula framework. Generalizing, let $u_i = P_{i<\text{arbitrary number}}(X_i)$, where $P_{i<}$ is the cumulative distribution of X_i. The copula $C(u_1, \ldots, u_N)$, defined as the joint distribution of u_i, isolates dependency structures. The inverse reconstruction is:

$$X_i = P^{-1}_{i<\text{same arbitrary number}}(u_i). \tag{8}$$

The next step is to incorporate stochastic volatility. Consider:

$$X_i = g \sum_{a=1}^{M} b_{ai} \tilde{f}_a, \tag{9}$$

where $\tilde{f}_a$ are Gaussian factors and g which is a common volatility factor follows an inverse gamma distribution. This generates multivariate Student's t-distributed returns.

To find out whether the linear model is better or not, we need to find a comparison metric. Define generalized kurtosis metrics; For the linear model, transform factors f_a to Gaussians $\tilde{f}_a = G_a^{-1}(f_a)$ and we compute:

$$K_{ab} = \langle \tilde{f}_a^2 \tilde{f}_b^2 \rangle - \langle \tilde{f}_a^2 \rangle \langle \tilde{f}_b^2 \rangle - 2 \langle \tilde{f}_a \tilde{f}_b \rangle^2, \tag{10}$$

averaging over factor pairs:

$$\overline{K}_l = \frac{1}{M(M-1)} \sum_{a \neq b} K_{ab}. \tag{11}$$

For the non-linear model, map X_i to Gaussians $\tilde{X}_i = G_i^{-1}(X_i)$, we need to extract factors f_a, and compute:

$$\overline{K}_{nl} = \frac{1}{M(M-1)} \sum_{a \neq b} \left(\langle f_a^2 f_b^2 \rangle - \langle f_a^2 \rangle \langle f_b^2 \rangle \right). \tag{12}$$

Empirical tests yield $\overline{K}_{nl} = 0.221$ and $\overline{K}_l = 0.296$, favoring the Gaussian copula. Residual $\overline{K}_{nl}$ highlights unmodeled correlations from volatility fluctuations.

Dividing returns by the volatility proxy g (the "variety") reduces residuals to $\overline{K}_{nl} = 0.087$ and $\overline{K}_l = 0.066$. The model:

$$X_i = g \sum_{a=1}^{M} b_{ai} \tilde{f}_a, \tag{13}$$

with g diagonalizing volatility correlations, suppresses cross-kurtosis effectively.

While Gaussian copulas outperform linear non-Gaussian models[17], explicit volatility factors are critical for accurate dependency structures[18]. This aligns with the heavy-tailed, volatility-clustered nature of financial returns, favoring for models that integrate stochastic volatility.

2.5 Probability Distribution of Eigenvalues

In our case, correlation matrices are all symmetric. Hence, an ensemble of symmetric matrices with independent entries are required to find the noise component. Note that if there is any dominant mode for the system with meaningful information, the independence hypothesis will not hold, thus some eigenvalues will be outliers of the spectral density function we are going to derive. For any arbitrary random matrix A with finite variance and zero mean entries, we are interested in the family of $\{C_{M\times M} - A_{M\times M}\}_{A\in \mathbf{Sym}_{M\times M}(\mathcal{N}(0,.))}$ matrices where there exists at least one eigenvalue which is invariant upto finite fluctuations, a.s.

Let A be a symmetric random matrix of size $M \times M$ and its off-diagonal entries are i.i.d. random variables with zero mean and variance σ^2/M. The set of eigenvalues $\{\lambda_a\}_{a=1}^{M}$ of A satisfy eigenvalue equation:

$$A\boldsymbol{\nu}_a = \lambda_a \boldsymbol{\nu}_a. \tag{14}$$

We define the empirical spectral density as follows:

$$\rho_{empirical}(\lambda) := \frac{1}{M}\sum_{a=1}^{M} \delta(\mu - \mu_a), \tag{15}$$

where δ is a 1-point Dirac delta distribution function. Define resolvent $\mathcal{G}(\lambda)$ of A as:

$$\mathcal{G}_{ij}(\lambda) := ((\lambda\mathbf{I} - A)^{-1})_{ij}. \tag{16}$$

Where $\mathbf{I}$ is the identity matrix. Trace of the resolvent is connected to the eigenvalues with:

$$\mathbf{Tr}(\mathcal{G}(\lambda)) = \sum_{a=1}^{M} \frac{1}{\lambda - \lambda_a}. \tag{17}$$

By using Sokhotski–Plemelj theorem, we want to decompose the singular function $\frac{1}{x-i\epsilon}$ into its principal value and a Dirac delta function, meaning we can make a new representation for 1-point Dirac distribution functions that helps us to calculate $\rho_e(\lambda)$ at the limit $M \to \infty$.

For the limit $\epsilon \to 0$ for $\epsilon \in \mathcal{R}^+$, the Sokhotski–Plemelj theorem states:

$$\frac{1}{x - i\epsilon} = \mathcal{P}\frac{1}{x} + i\pi\delta(x), \tag{18}$$

where $\mathcal{P}\frac{1}{x}$ denotes the Cauchy principal value of $\frac{1}{x}$. This formula is useful in the study of spectral properties of operators, as it allows us to extract the imaginary part of the resolvent, which is directly related to the spectral density.

By adding a small imaginary part $i\epsilon$ to λ in equation (17) and applying the Sokhotski–Plemelj formula, we can isolate the imaginary part of the resolvent:

$$\frac{1}{\lambda - \lambda_a - i\epsilon} = \mathcal{P}\frac{1}{\lambda - \lambda_a} + i\pi\delta(\lambda - \lambda_a). \tag{19}$$

Taking the imaginary part of the trace of the resolvent, we obtain:

$$\Im\left[\mathrm{Tr}\,\mathcal{G}(\lambda - i\epsilon)\right] = \pi\sum_{a=1}^{M} \delta(\lambda - \lambda_a). \tag{20}$$

Thus, the spectral density can be written as:

$$\rho(\lambda) = \lim_{\epsilon\to 0^+} \frac{1}{M\pi}\Im\left[\mathrm{Tr}\,\mathcal{G}(\lambda - i\epsilon)\right]. \tag{21}$$

This expression connects the spectral density to the resolvent, providing a practical way to compute $\rho(\lambda)$ for large matrices.

Next step is constructing the recursive relation for resolvents, such that the empirical distribution $\rho_e(\lambda)$ approaches the spectral density of the random operator, a.s. Consider extending the matrix $A^{(M)}$ of size $M \times M$ to a larger matrix $A^{(M+1)}$ of size $(M+1) \times (M+1)$. The new matrix $A^{(M+1)}$ includes additional entries A_{0i} for $i = 1, \ldots, M$, and A_{00}. Note that the matrix A^{M+1} is symmetric, meaning that the elements A_{i0} have their respective couples in the A_{0i}.

The resolvent $\mathcal{G}^{(M+1)}(\lambda)$ of the extended matrix satisfies the following relation:

$$\frac{1}{\mathcal{G}_{00}^{(M+1)}(\lambda)} = \lambda - A_{00} - \sum_{i,j=1}^{M} A_{0i}A_{0j}\mathcal{G}_{ij}^{(M)}(\lambda). \tag{22}$$

This recursive formula allows us to compute the resolvent of the larger matrix $A^{(M+1)}$ in terms of the resolvent of the smaller matrix $A^{(M)}$. The key idea is to express the diagonal element $\mathcal{G}_{00}^{(M+1)}(\lambda)$ in terms of the elements of $\mathcal{G}^{(M)}(\lambda)$.

In the limit $M \to \infty$, the off-diagonal terms $\mathcal{G}_{ij}^{(M)}(\lambda)$ for $i \neq j$ become negligible, scaling as $\mathcal{O}(1/\sqrt{M})$. Additionally, the diagonal term A_{00} scales as $\mathcal{O}(1/\sqrt{M})$, so it can be neglected compared to λ. This leads to the simplified recursion relation:

$$\frac{1}{\mathcal{G}_\infty(\lambda)} \approx \lambda - \sigma^2 \mathcal{G}_\infty(\lambda), \tag{23}$$

where σ^2 is the variance of the matrix elements. Solving this quadratic equation yields the limiting form of the resolvent $\mathcal{G}_\infty(\lambda)$, which is used to derive the spectral density $\rho(\lambda)$ in the large M limit. This yields:

$$\mathcal{G}_\infty(\lambda) = \frac{1}{2}\left(\frac{\lambda}{\sigma^2} - \sqrt{\frac{\lambda^2}{\sigma^4} - \frac{4}{\sigma^2}}\right). \tag{24}$$

This solution is chosen to ensure the correct behavior in the limit $\sigma \to 0$, where the resolvent reduces to $\mathcal{G}_\infty(\lambda) = \frac{1}{\lambda}$. The spectral density $\rho(\lambda)$ for $M \to \infty$ is given by:

$$\rho(\lambda) = \frac{1}{2\pi}\sqrt{4/\sigma^2 - \lambda^2/\sigma^4} \quad \text{for } (\lambda/\sigma)^2 \leq 4, \tag{25}$$

and zero otherwise. This is the well-known Wigner semi-circle law, which describes the distribution of eigenvalues of large symmetric random matrices.

Resolvent $\mathcal{G}_C(\nu)$ of the matrix C, can be written as:

$$\mathcal{G}_C(\nu) = \frac{\partial}{\partial \nu} \log\det(\nu \mathbf{I} - C). \tag{26}$$

Using the identity for the determinant of a symmetric matrix $\mathbf{X}$:

$$\det(\mathbf{X})^{-1/2} = \int \exp\left(-\frac{1}{2}\boldsymbol{\phi}^\top \mathbf{X} \boldsymbol{\phi}\right) \prod_{i=1}^{M} \frac{d\phi_i}{\sqrt{2\pi}}, \tag{27}$$

we express $\mathcal{G}_C(\nu)$ in terms of an integral over auxiliary variables $\boldsymbol{\phi}$. By averaging over the random matrix A performing a saddle-point approximation and using a replica trick to handle the quenched disorder arising from the randomness in A, we can find the spectrum density.

Let $C = AA^\top$, where A is an $M \times N$ matrix with independent and identically distributed (iid) entries of variance σ^2/N. The eigenvalues ν_a of C are related to those of A via $\nu_a = \lambda_a^2$.

For $M, N \to \infty$ with $c = N/M \geq 1$, the spectral density of C is given by:

$$\rho(\nu) = \frac{c}{2\pi\sigma^2}\sqrt{\frac{(\nu_{\max} - \nu)(\nu - \nu_{\min})}{\nu}}, \tag{28}$$

where the spectral edges are:

$$\nu_{\min}^{\max} = \sigma^2\left(1 + \frac{1}{c} \pm 2\sqrt{\frac{1}{c}}\right), \tag{29}$$

and $\nu \in [\nu_{\min}, \nu_{\max}]$. This is the Marčenko-Pastur distribution, which describes the eigenvalue distribution of large sample covariance matrices.

For finite N, the sharp edges of the spectral density become blurred. Let $\lambda_{\max/\min}^{(N)}$ denote the empirical extremal eigenvalues of C for finite N. Then:

$$\lim_{N\to\infty} \mathbb{P}\left(\lambda_{\max}^{(N)} > \nu_{\max}\right) = 0, \tag{30}$$

with a similar result holding for $\lambda_{\min}^{(N)}$. This means that, as N increases, the probability of finding eigenvalues outside the theoretical bounds $[\nu_{\min}, \nu_{\max}]$ vanishes.

2.6 Maximal Spanning Tree Algorithm

The maximal spanning tree (MST) graph is constructed based on the correlation matrix of crypto-fiat pairs [19]. Consider $M(M-1)/2$ entries of correlation matrix C from the daily returns of crypto-fiat pairs. First, rank the C_{ij} in descending order. Nodes of this tree are crypto-fiat pairs, and edges must be added iteratively from the decreasing list of C_{ij}. This process should be repeated for all the correlation entries, unless adding a link between the pair under consideration creates a loop in the graph, in which case skip that edge.

Once this process in finished, one finds a tree constructed based on the largest correlation values. This tree is named maximal spanning tree. Figure (2) finds the maximal spanning tree for two datasets, based on a fixed fiat. There are several methods to cluster MST and find the center node, but the figure is informative on its own.

2.7 Spectral Gap Analysis

Eigenvalues of correlation matrix C which is positive semi-definite, can be written in the following way:

$$\lambda_1 > \lambda_2 > \cdots > \lambda_M \geq 0. \tag{31}$$

From that, we can define spectral gap $\Delta\lambda$ as the difference between two largest eigenvalues.

$$\Delta\lambda := \lambda_1 - \lambda_2. \tag{32}$$

Mathematically, we can fix some bounds on $\Delta\lambda$. Since diagonal entries are all 1, $\mathbf{Tr}(C) = M$, meaning that $\sum_{i=1}^{M} \lambda_i = M$. For λ_1 we have:

$$\lambda_1 = M - \sum_{i=2}^{M} \lambda_i \tag{33}$$

For spectral gap, we get to the following relation:

$$\Delta\lambda = M - \lambda_2 - \sum_{i=2}^{M} \lambda_i. \tag{34}$$

We can find two bounds on $\Delta\lambda$. An upper bound can be found set λ_1, due to λ_2 being positive. A lower bound can be found by the using equation (31) on equation (34), getting to:

$$\Delta\lambda \geq M - \lambda_2 - (M-1)\lambda_2 = M(1 - \lambda_2). \tag{35}$$

This lower bound is not necessarily negative, but since the eigenvalues are all positively sorted, the spectral gap is always positive.

For a large spectral gap, we can deduce that λ_2 is small, meaning that the largest cluster of data is due to one dominant eigenvalue. Using principle component analysis (PCA), λ_1 is variance explained by first principal component (PC1), and $\Delta\lambda$ is excess variance in PC1 over PC2. PCA can be regarded as intrinsic dimensionality of the whole data. By using PCA stability theorem, we can show that how much this system is stable.

First, consider $\tilde{C} = C + E$ to be a perturbed matrix with $\|E\|_2 < \frac{\Delta\lambda}{2}$. Let v_1 be the dominant eigenvector of C and $\tilde{v}_1$ of $\tilde{C}$. By the Davis-Kahan $\sin\theta$ theorem, the angle θ between v_1 and $\tilde{v}_1$ satisfies:

$$\sin\theta \leq \frac{\|E\|_2}{\Delta\lambda - \|E\|_2}. \tag{36}$$

Given $\|E\|_2 < \frac{\Delta\lambda}{2}$, the denominator $\Delta\lambda - \|E\|_2 > \frac{\Delta\lambda}{2}$. Thus:

$$\sin\theta \leq \frac{\|E\|_2}{\Delta\lambda/2} \implies \theta \leq \arcsin\left(\frac{2\|E\|_2}{\Delta\lambda}\right). \tag{37}$$

For small θ, $\sin\theta \approx \theta$, so $\|\tilde{v}_1 - v_1\|_2 \approx 2\sin(\theta/2) \leq \frac{2\|E\|_2}{\Delta\lambda}$. So the v_1 and $\tilde{v}_1$ are related by:

$$\|\tilde{v}_1 - v_1\|_2 \leq \frac{2\|E\|_2}{\Delta\lambda}. \tag{38}$$

Thus, a large $\Delta\lambda$ stabilizes v_1.
This result can be shown using the von neumann entropy. The von Neumann entropy is defined:

$$S(C) = -\sum_{i=1}^{n} \frac{\lambda_i}{n} \log\frac{\lambda_i}{n}. \tag{39}$$

Let $\lambda_1 = \frac{M}{M} + \frac{\Delta\lambda}{M} = 1 + \frac{\Delta\lambda}{M}$, and $\lambda_i = \frac{M-\Delta\lambda}{M(M-1)}$ for $i \geq 2$. Expanding $S(C)$:

$$S(C) = -\frac{1+\frac{\Delta\lambda}{M}}{M}\log\left(\frac{1+\frac{\Delta\lambda}{M}}{M}\right) - \sum_{i=2}^{M} \frac{\frac{M-\Delta\lambda}{M(M-1)}}{M}\log\left(\frac{\frac{M-\Delta\lambda}{M(M-1)}}{M}\right). \tag{40}$$

Using $\log(1+x) \approx x$ for small $\Delta\lambda/M$:

$$S(C) \approx \log M - \frac{\Delta\lambda}{M} + \mathcal{O}\left(\frac{\Delta\lambda^2}{M^2}\right). \tag{41}$$

Now, for a large $\Delta\lambda \gg \lambda_2$, we want to show the graph G with adjacency matrix C contains a dominant cluster.

By using the eigenvectors of C, we can find spectral clustering partitions G. By the Perron-Frobenius theorem, v_1 has non-negative entries and a large $\Delta\lambda$ implies v_1 dominates the spectrum, so its entries $v_1(i)$ reflect node's "importance." we can partition nodes via:

$$\begin{aligned} \text{Dominant cluster} &= \{i \mid v_1(i) \geq \tau\}, \\ \text{Outliers} &= \{i \mid v_1(i) < \tau\}, \end{aligned} \tag{42}$$

For a threshold τ, since $\lambda_2 \ll \Delta\lambda$, the residual eigenvalues $\lambda_2, \ldots, \lambda_n$ correspond to weak connections, justifying the partition.

For a random matrix C_{rand} with i.i.d. entries (mean 0, variance σ^2), Wigner's semicircle law states eigenvalues lie in $[-2\sigma\sqrt{M}, 2\sigma\sqrt{M}]$. Normalizing C as a correlation matrix (unit diagonal), $\sigma^2 = \frac{1}{M}$. The largest eigenvalue λ_1 of C_{rand} converges to $2\sqrt{M}$. If $\Delta\lambda = \lambda_1 - \lambda_2 > 2\sqrt{M}$, then $\lambda_1 > 2\sqrt{M}$, violating the random matrix threshold, implying structured correlations.

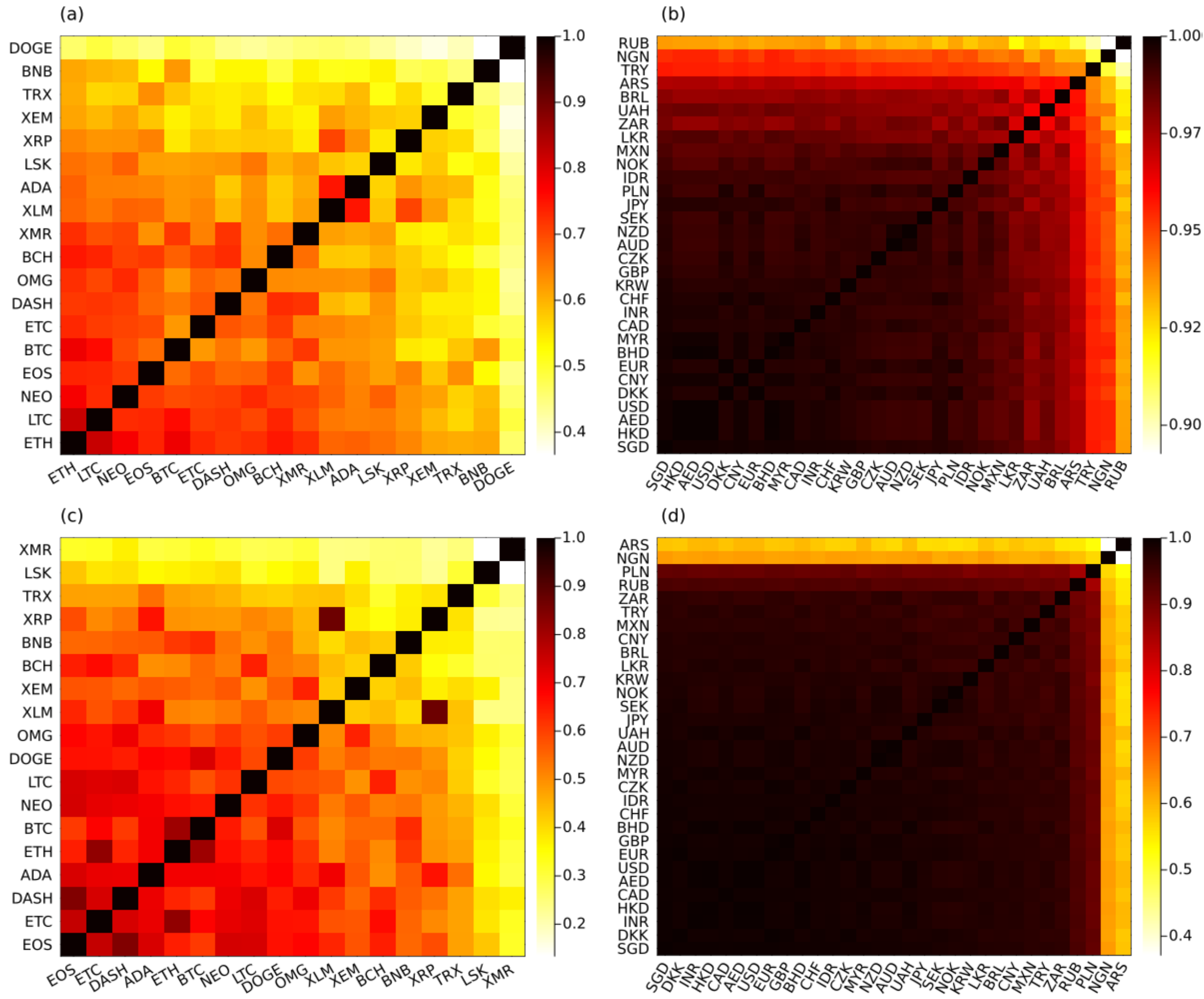

Fig. 1: Heatmap representations of the cross-correlation matrices computed from the daily returns of crypto–fiat pairs. Panels (a) and (b) correspond to the historical (“old”) dataset, with panel (a) displaying correlations among cryptocurrencies in the USA market and panel (b) showing correlations among fiat currencies for BTC. Panels (c) and (d) show the corresponding matrices for the recent (“new”) dataset.

3 Results

3.1 Cross-Correlation Analysis

3.1.1 Matrix Heatmap

The Pearson correlation coefficient (Eq. 1) quantifies linear relationships between pairs. Figure 1 presents heatmaps of cross-correlation matrices computed from daily returns of crypto(panels a and c) and fiat(panels b and d) pairs. The scores range from -1 (anti-correlation) to +1 (perfect correlation), which is shown by a color gradient (e.g., dark red for high positive correlations). During market stress, investors may sell riskier assets and buy safer ones, creating negative correlations. Wars, elections, or natural disasters can suddenly change market dynamics, causing temporary negative correlations. Due to this nature of markets, over short periods, correlations can fluctuate due to specific events or trends, and be negative, while long-term correlations are mostly stable and positive. For this reason, in Figure 1, which is calculated for a long period (2 to 4 years), negative correlation is not observed. Panels a and c elevated intra-crypto correlations (clustered red blocks) in USA market, and show that cryptocurrencies move together, driven by shared market drivers (e.g., ETH dominance). More recent data (panel c) may show higher correlations, reflecting clustered markets dynamics. In Panels b/d Fiat pairs exhibit high correlations, which can be powered by macro policy or geopolitical risk. Fiat pairs (like USD/EUR, JPY/PLN) tend to exhibit high correlation due to global economic conditions, High trading volumes, etc. The value of fiat currencies often moves in response to global economic con-

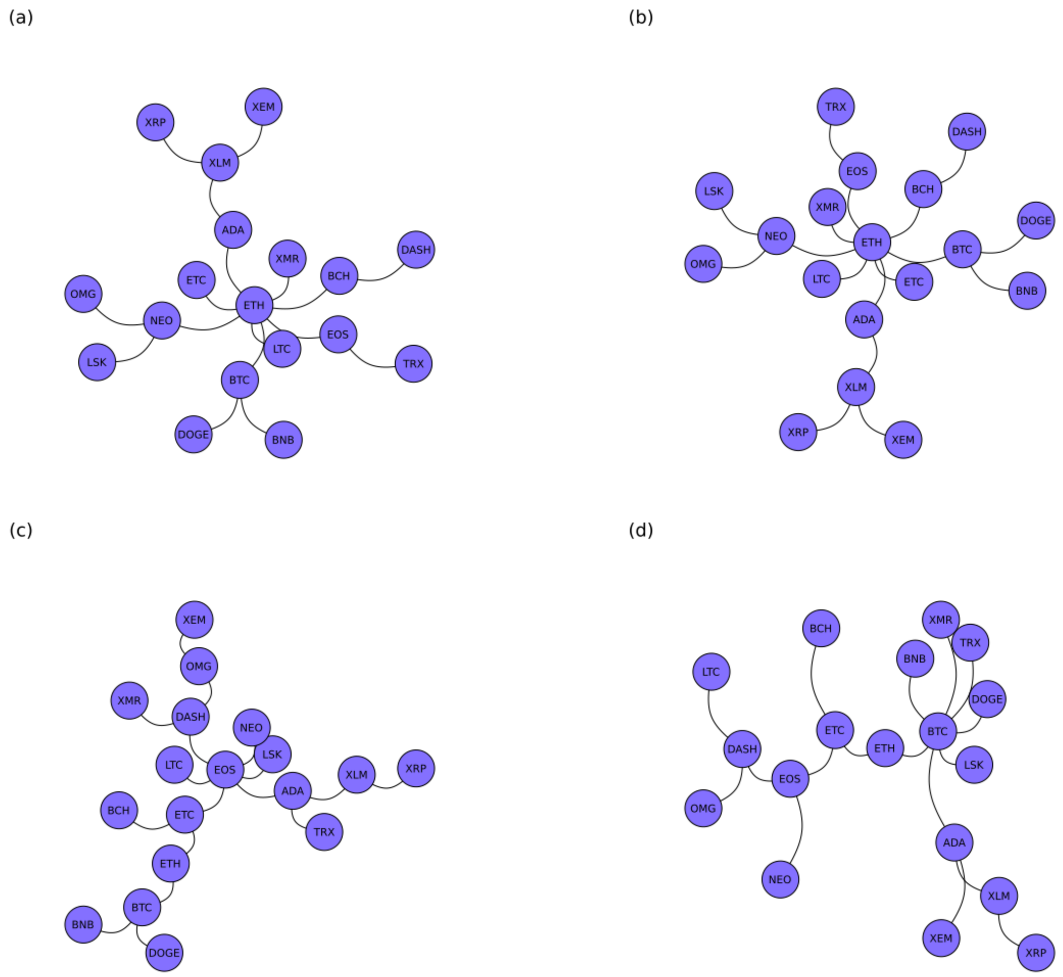

Fig. 2: Maximal spanning trees (MST) constructed from the correlation network of crypto–fiat pairs. Panels (a) and (c) depict the MST for the USD market based on the historical and recent datasets, respectively, while panels (b) and (d) show the MST for the ARS market. These trees highlight the strongest interdependencies and identify central nodes in each market.

ditions, which can affect the entire fiat market similarly. Also, high trading volumes in fiat pairs can lead to a more uniform price behavior across different fiats. These factors lead to more correlated fiats which demonstrated in panels b and d.

3.1.2 Correlation MST Graph

Figure 2 demonstrate MSTs for crypto–fiat correlation networks. Panels (a,c) and (b,d) represent USD and ARS markets for historical and recent datasets, respectively. Nodes in the graph represent crypto-USD or crypto-ARS (e.g., BTC-USD), while edges show the strongest correlations. Starting from the highest correlations, edges have been added one by one without loops (Section 2.6). The MST highlights the most important interdependencies. Recent data reflect decentralized structures when some altcoins, such as BNB, are in leading positions. Emerging markets like ARS (Argentine Peso) might exhibit the opposite in ETH-ARS, tight clustering around stablecoins as a hedge against hyperinflation. Ethereum(ETH) works its way to the central node of historical networks, given that it very often demonstrates high liquidity and hectic trading activity in a wide assortment of fiat currencies. The wide use of this cryptocurrency as the base or quote currency on many exchanges makes them have strong ETH-fiat pair statistical associations-a characteristic that simply makes them main candidates for primary links in any spanning tree. The central position of Ethereum(ETH) is further underlined by its more general use within the digital finance ecosystem, in particular in DeFi and tokenized asset markets. This wide utility, combined with market dynamics wherein ETH often plays the role of gateway

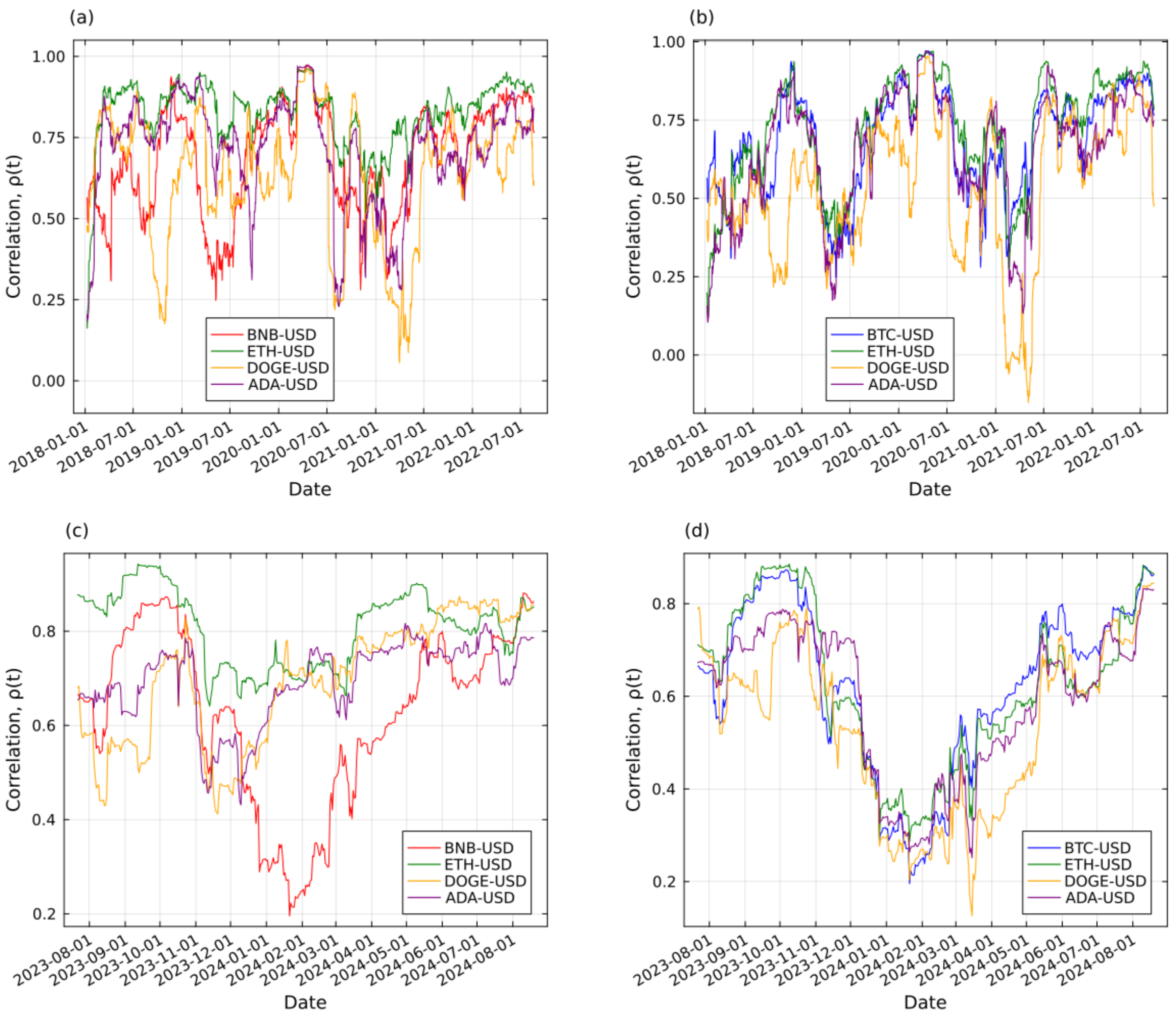

Fig. 3: Time evolution of moving correlations for key cryptocurrency pairs in the USD market. Panels (a) and (c) present the moving correlation of BTC–USD with a selection of other cryptocurrencies for the historical and recent datasets, respectively. Panels (b) and (d) show the analogous analysis for BNB–USD.

between traditional fiat systems and the wider crypto market, means it evidences consistently high connectivity across different fiat currencies. Because of that, when the maximal spanning tree algorithm underlines the most important relationships, ETH naturally takes the central position-actually bridging several fiat nodes and underlining its importance in the cryptocurrency ecosystem.

3.2 Dynamic Analysis

3.2.1 Moving Correlation Analysis of Crypto Currencies

Figures 3 (USD market) and 4 (CNY market) show time-evolving moving correlations between BTC/BNB and other cryptos. Panels (a/c) and (b/d) correspond to historical and recent datasets. Sliding window approach (Section 2.2) with window size w. Correlations fluctuate due to market events (e.g., Bitcoin halvings, regulatory announcements). In Figure 3 a and c, high correlations with major cryptos (ETH, LTC) persist but decline post-2023, indicating reduced Bitcoin hegemony. In Figure 3 b and d, increasing correlations with DeFi tokens (e.g., CAKE) reflect Binance Smart Chain's growth. In Figure 4 a and c, Sharp drops during China's crypto bans (2021) and rebounds after partial policy relaxations. In Figure 4 b and d, Lower correlations due to China's strict crypto regulations, isolating BNB from domestic markets.

3.2.2 Moving Correlation Analysis of Fiat Currencies

Figures 5 (BTC–fiat) and 6 (BNB–fiat) illustrate dynamic correlations between BTC or BNB and fiat currencies (USD, EUR, JPY, CNY). In Figure 5 a and c, low correlations during market crashes (e.g., COVID-19

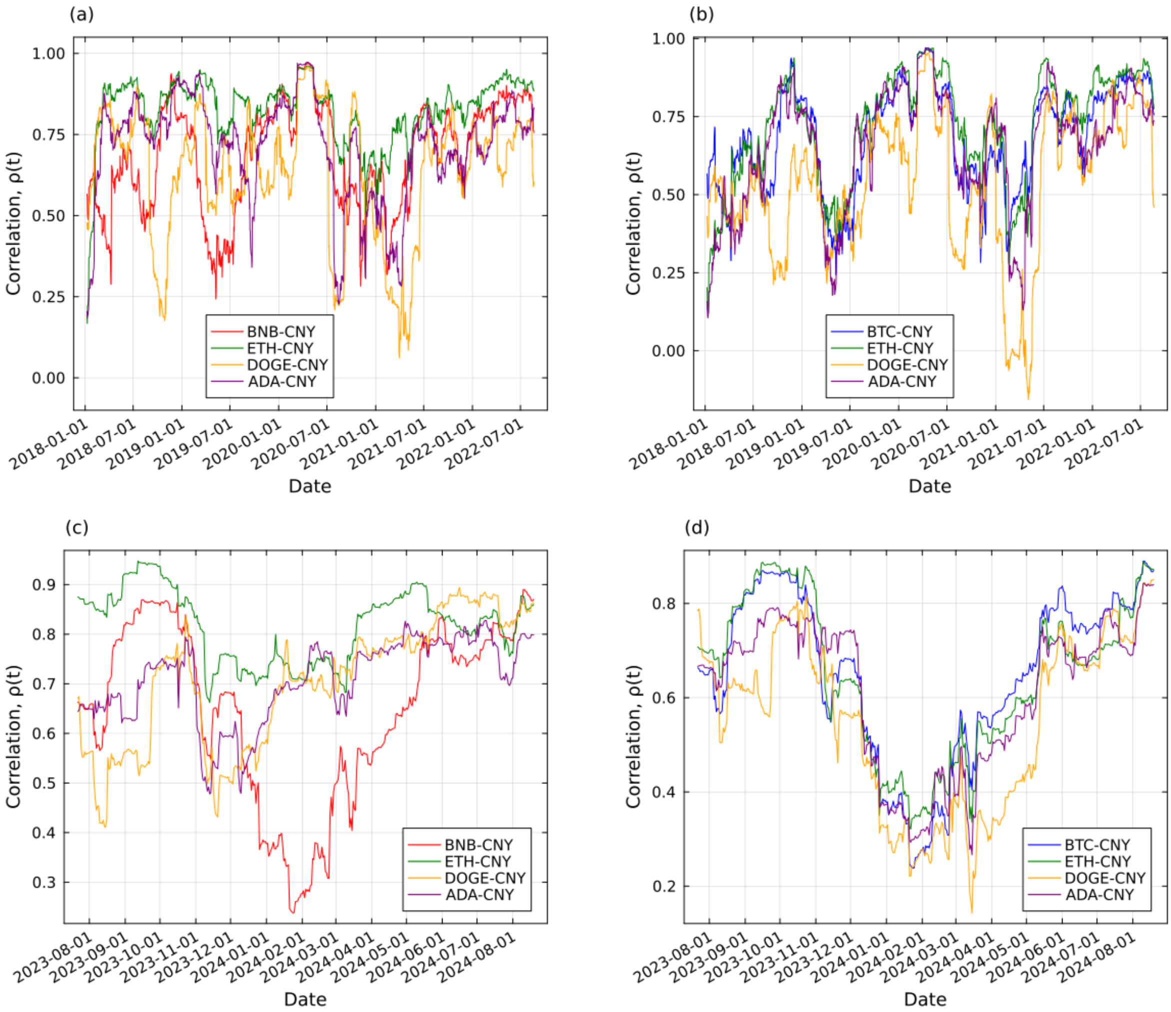

Fig. 4: Temporal moving correlations for crypto pairs in the CNY market. Panels (a) and (c) display the evolution of the moving correlation between BTC–CNY and selected cryptocurrencies for the historical and recent datasets, respectively, while panels (b) and (d) provide the corresponding analysis for BNB–CNY.

sell-off in 2020) as investors flee to USD. Recent data (panel c) shows weaker ties, suggesting decoupling from traditional markets. In Figure 5 b and d, anti-correlations during capital control tightening in China, with CNY devaluation driving crypto inflows. In Figure 6 a and c, rising correlations post-2023 mirror Binance's regulatory compliance in the U.S. In Figure 6 b and d, minimal linkage due to China's crypto bans, highlighting geographic fragmentation.

3.2.3 Eigenvalue Dynamics

In correlation matrices, eigenvalues quantify the variance (importance) of orthogonal factors. The largest eigenvalue (λ_{Max})(Figures 7 and 8) shows the market mode which explain the collective behavior of most assets. When λ_{Max} is large, it can be concluded that most assets are driven by a common force which can be anything like macroeconomic shocks or Bitcoin's price swings. In this situation a shock to one asset can propagate rapidly across the network.

During the 2018 crypto winter (long term market crash), Bitcoin's collapse dragged down altcoins which resulted to a spike in λ_{Max}. The 2021 Terra-Luna collapse increased panic selling cross crypto markets which further elevating λ_{Max}. Dampened (smaller) λ_{Max} in panel c shows reduction of dominance of a single market driver, this can be due to the increasing adoption of altcoins (e.g., Solana, Ripple) and decentralized finance(DeFi) tokens. Maturation of Regulation in these areas can also help to reduce the power of on dominant coin like Bitcoins. A spike in λ_{Min} (e.g., 2020-01-01) signals investors abandoning risky assets for safety. For example in 2020 COVID-19 Crash investors sold volatile cryptos like BTC and ETH and but stablecoins like USDT and USDC or fiats like

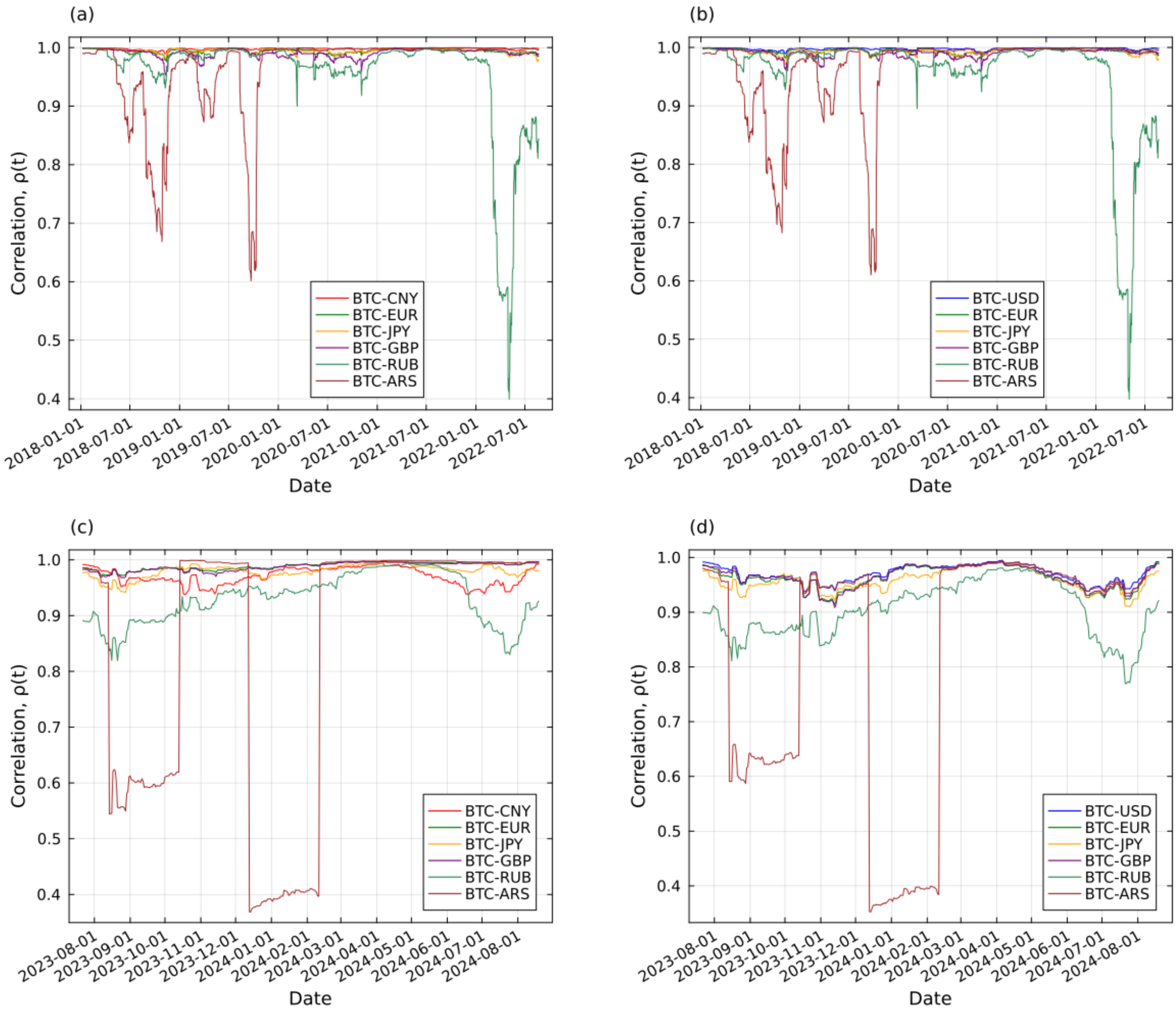

Fig. 5: Dynamic analysis of Bitcoin exchange rates with fiat currencies. Panels (a) and (c) illustrate the moving correlations between BTC–USD and a set of fiat currencies for the historical and recent datasets, respectively. Panels (b) and (d) show the moving correlation analysis for BTC–CNY with the same set of fiat currencies.

USD, creating anti correlation between risky and safe assts.

In the beginning of Russia-Ukraine war invasion (24th Feb 2022), Ruble (Russia's national currency) market experience a large increase in λ_{Max} and decrease in λ_{Min} which can demonstrate an unstable crypto-market in russia due to the uncertainty among people and selling cryptos to obtain safe money.

3.2.4 Spectral Analysis

In figure 9, panels a and c are for fixed crypto. The maximum of these frames are found in section 2.7 and for panels a and c are 31. The same value for the panels b and d (which are for fixed fiats and varying cryptos) is calculated to be 18. When the $\Delta\lambda$ is higher, it means that the overal system is more stable. When it reaches the maximum, this means that the system is effectively 1 dimentional, meaning that the mode of the network can be determined by only one time series. In section 2.7 we found that the 1 : 1 signal to noise ratio can be found by $2\sqrt{M}$ which for fixed crypto (a and c), this number is 11.14 and for fixed fiats (b and d) this number is 8.49. When the spectral gap is above that value, the network is signal dominated and when the spectral gap is less than that, the noise is dominant. We expect to see some phases where crypto market is noise dominated, due to the independencies and local values of different coins. This can be seen in panels b and d, but with the same previously mentioned reasons like Russia-Ukraine war, we can see deviations for different fiats. For panels a and c, where a coin is fixed, we expect the healthy market to have a near maximal spectral gap. The market has been found in signal zone, which is exactly what we expect from non-isolated nations. If the price of one coin in some market,

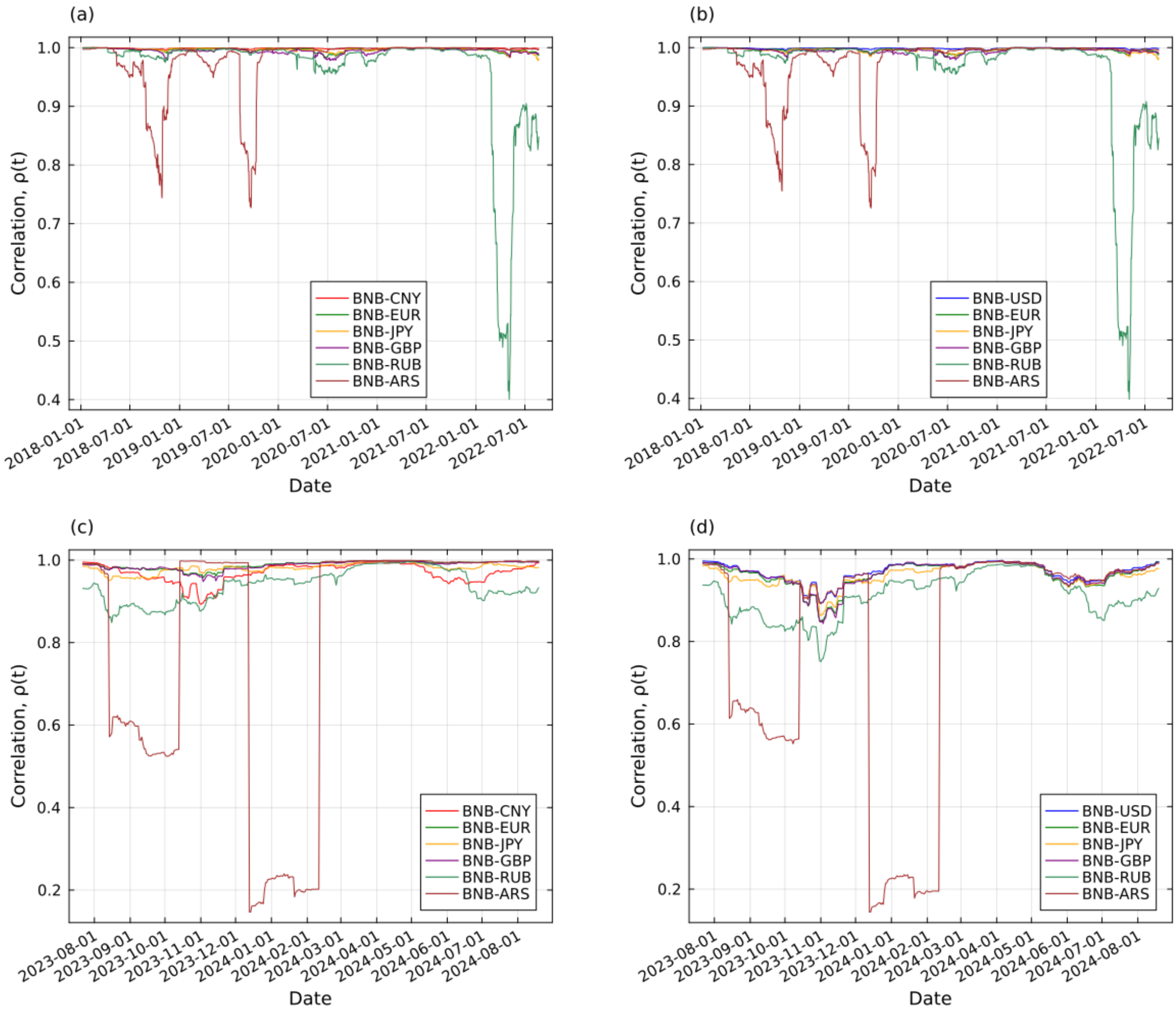

Fig. 6: Time evolution of moving correlations between Binance Coin (BNB) and fiat currencies. Panels (a) and (c) depict the moving correlations for BNB–USD for the historical and recent datasets, respectively, whereas panels (b) and (d) display the corresponding analysis for BNB–CNY.

for example BTC is totally independent from the price of the BTC in another market, we expect the spectral gap to get lower, hinting at possible separated communities, but since the change in the exchange rate between two fiats is slower that changes in cryptos, the independence of two exchange rates leads to different values of the same crypto in two fiats which easily gets exploited naturally. In figure 10, we can see the spectral gap of the overall network, consisted of all of the possible crypto-fiat pairs. The bound for being noise dominated can be calculated as 48.50, and the overall system is found to be always signal dominated, but it fluctuates.

3.2.5 Empirical Spectral Density

In figure 11, the histogram of eigenvalues are plotted in blue, and the redline is the eq 28 calculated for the dimension of the C. red line is only shifter vertically for better visualization. Since the scale of both frames are in Log-Log, vertical shift is a mere constant. In frame a, the value is calculated for older data, and in panel b this is calculated for newer data. The outlier λ_{max} is detected in both data sets, and for the newer data it has seen a notable shift to the right, meaning that the main cluster of the network is getting more fixed. There is another outlier in the small values that can be seen in the left part of both panels, which sets the fastest responses of the system. the smallest eigenvalue changes the eigenvector a lot (because there are many small eigenvalues and their respective eigenvector changes position as the pair of the leftmost eigenvalue). by changing eigenvectors, the fastest response of the system does not have a unique community.

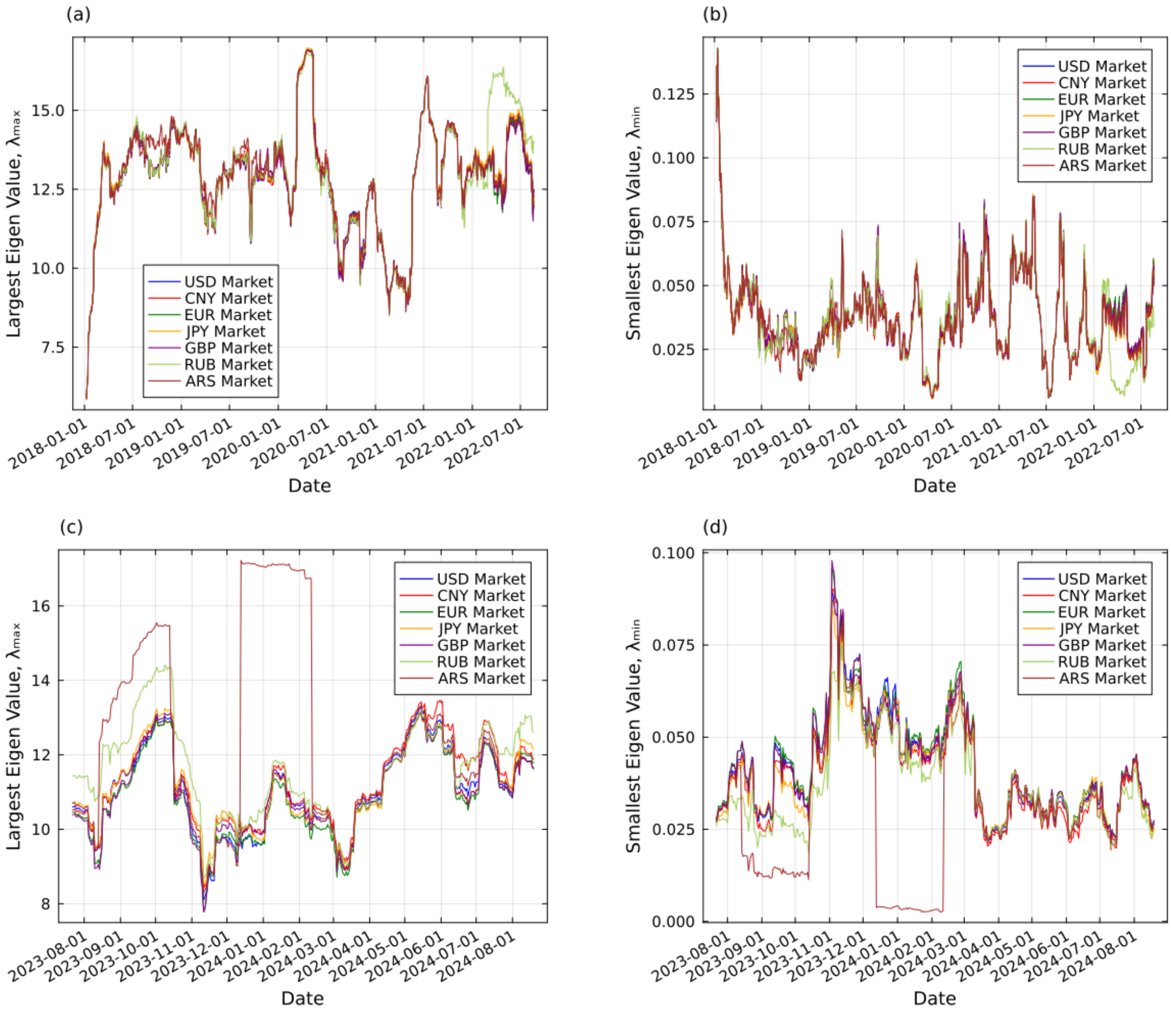

Fig. 7: Evolution of extreme eigenvalues from the fiat currency correlation matrices. Panels (a) and (c) show the time evolution of the largest eigenvalues (reflecting dominant market modes) for the historical and recent datasets, respectively, while panels (b) and (d) present the dynamics of the smallest eigenvalues (indicating fast responses of the system).

4 Discussion

The relationship between the cryptocurrencies and fiat currencies reflects the dynamic connection between them, shaped by investors, macroeconomic forces, regulatory frameworks, etc. In this article, we have shown how markets have evolved from a Bitcoin-centric ecosystem to a more interconnected network, reflecting broader changes in the maturity of the financial system and geopolitical strategies.[20] The shift in centrality from Bitcoin to Ethereum in the maximum spanning trees and Ethereum's high correlation in the markets underscores the historical shift in market power from digital gold to digital superpower in the cryptocurrency market.

During the early years (2017-2022), Bitcoin-USD acted as the main conduit for market sentiment.[21] However, in 2023-2024, Bitcoin's dominance declined significantly and Ethereum entered the game.[22] As the spectral gaps and maximum spanning trees indicate, this shift is aligned with the rise of the Ethereum DeFi ecosystem, regulatory scrutiny of Bitcoin mining (e.g., China's ban in 2021), and the emergence of seasonal cycles in which investors diversify into tokens like BNB and Solana.

The strong similarity between the USD and CNY markets illustrates how geopolitical tensions and regulatory policies make crypto-fiat networks well-suited to identifying arbitrage opportunities. China's tough stance on cryptocurrencies – which led to a mining and trading ban in 2021 – severed BTC-CNY correlations, directing capital flows toward stablecoins like USDT-ARS in inflation-prone economies (e.g., Argentina).[23] Emerging markets are increasingly prominent in the crypto-fiat network. In Turkey (TRY), crypto adoption increased during the 2023

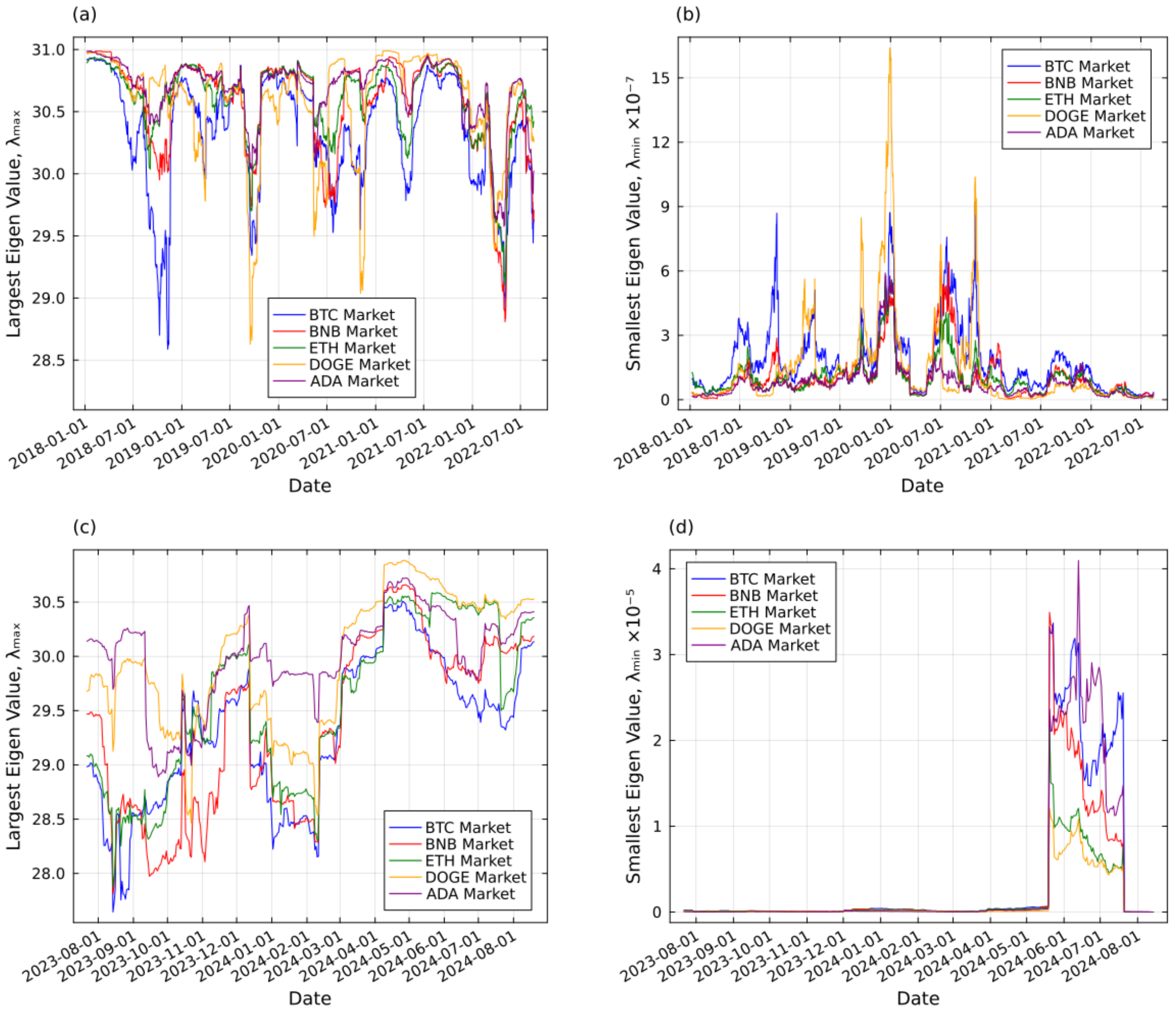

Fig. 8: Temporal evolution of extreme eigenvalues in the crypto currency correlation matrices. Panels (a) and (c) show the evolution of the largest eigenvalues, and panels (b) and (d) display the smallest eigenvalues, for the historical ("old") and recent ("new") datasets respectively, thus capturing shifts in overall market sentiment and specialized behavior.

currency crisis, and as citizens sought refuge in cryptocurrencies from the devaluation of the lira, the BTC-TRY correlation on the network increased.

The largest eigenvalue (λ_{Max}) peaked during economic crises (such as the 2018 crypto winter or the 2022 Terra-Luna collapse). The largest eigenvalue, which indicates the fast responses of the system to any noise, emphasizes the vulnerability of these sectors to leverage and bubbles. Notably, the deviation of the eigenvalue distributions from the Marchenko-Pastor rules (Section 2.3) highlights the inherent non-Gaussian nature of the crypto market.[24] The heavy-tailed returns and the microstructure-noise (created by retail buying and selling and their speculation in response to the media) make traditional portfolio models inadequate.[25]

While this paper greatly enhances our understanding of crypto-fiat networks, it has notable limitations. The inability to analyze causality is one of the problems with the Pearson correlation, which limits our ability to distinguish between leaders and followers in market movements. Relying on Pearson correlation also ignores nonlinear dependencies (due to group effects and the combination of multiple cryptocurrencies). We also encounter problems such as false edges or spurious edges when plotting the network using Pearson correlation.[26] Future research could address these problems by using mutual information or interaction-based models and polynomials.

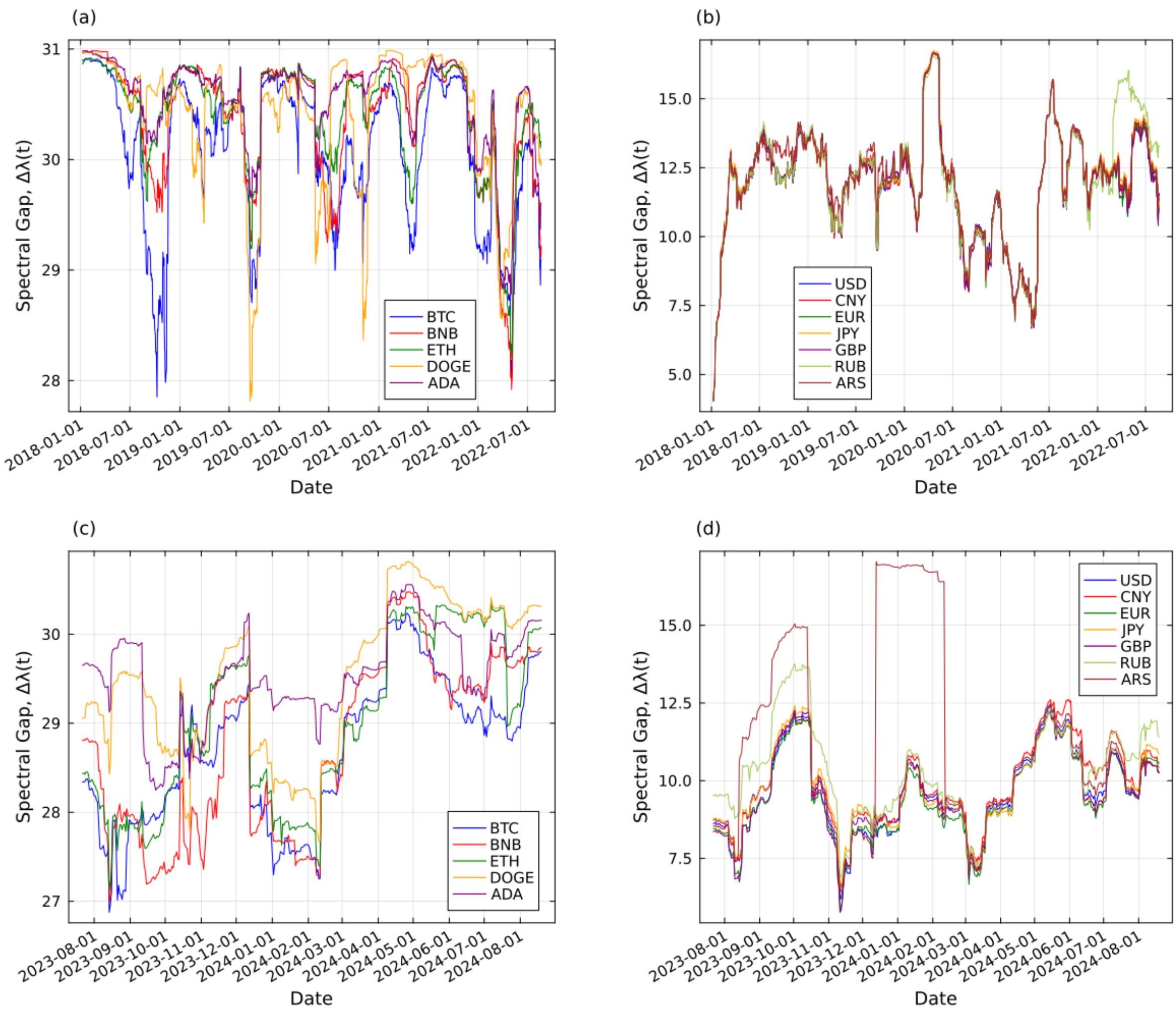

Fig. 9: Moving spectral gap analysis computed by fixing the reference (either crypto or fiat) in the correlation matrices. Panels (a) and (c) display the spectral gap when the crypto side is fixed, while panels (b) and (d) show the spectral gap when the fiat side is fixed, for the historical and recent datasets, respectively. (Ensure the labels match the actual panel content; adjust if file names imply the opposite.)

Appendix A: Appendix: List of Cryptocurrencies and Fiat Currencies Used

In this study, we analyze data from various cryptocurrencies and fiat currencies. The following is the complete list of assets considered in alphabetical order of their abbreviation.

- **Cryptocurrencies:** Cardano (ADA), Bitcoin Cash (BCH), Binance Coin (BNB), Bitcoin (BTC), Dash (DASH), Dogecoin (DOGE), EOS.IO (EOS), Ethereum Classic (ETC), Ethereum (ETH), Lisk (LSK), Litecoin (LTC), Neo (NEO), Omg network (OMG), TRON (TRX), NEM (XEM), Stellar (XLM), Monero (XMR), Ripple (XRP).
- **Fiat Currencies:** United Arab Emirates Dirham (AED), Argentina Peso (ARS), Australia Dollar (AUD), Bahrain Dinar (BHD), Brazil Real (BRL), Canada Dollar (CAD), Swiss Franc (CHF), Chinese Yuan (CNY), Czech Koruna (CZK), Denmark Krone (DKK), Euro (EUR), British Pound Sterling (GBP), Hong Kong Dollar (HKD), Indonesia Rupiah (IDR), India Rupee (INR), Japanese Yen (JPY), South Korea Won (KRW), Sri Lanka Rupee (LKR), Mexico Peso (MXN), Malaysia Ringgit (MYR), Nigeria Naira (NGN), Norway Kroner (NOK), New Zealand Dollar (NZD), Poland Zloty (PLN), Russia Rouble (RUB), Sweden Krona (SEK), Singapore Dollar (SGD), Turkish New Lira (TRY), Ukraine Hryvnia (UAH), United States Dollar (USD), and South Africa Rand (ZAR).

All currency symbols and abbreviations follow the standard ISO 4217 codes for fiat currencies and commonly accepted ticker symbols for cryptocurrencies.

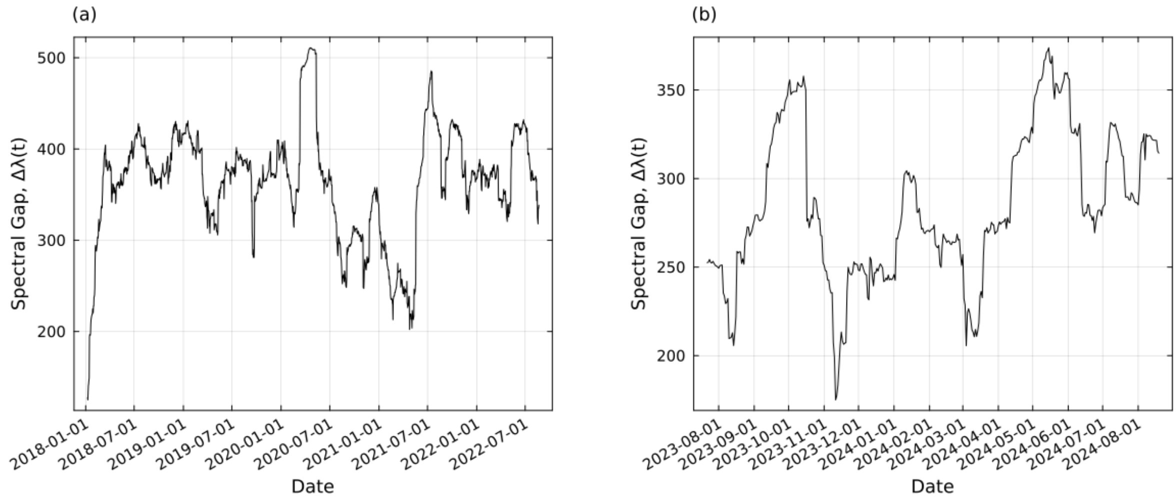

Fig. 10: Time evolution of the total spectral gap of the correlation matrices. Panel (a) shows the moving total spectral gap for the historical ("old") dataset, whereas panel (b) presents the corresponding evolution for the recent dataset. These plots highlight changes in the overall structure and collective dynamics of the crypto–fiat market over time.

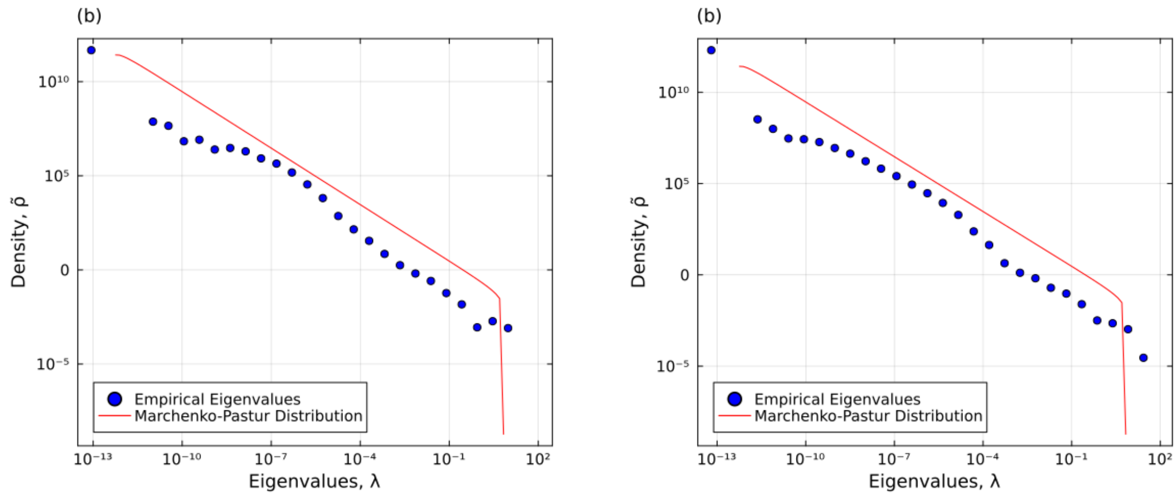

Fig. 11: Theoretical and Empirical eigenvalues of the random part of (a) old data, (b) newer data

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TTMP vol 2 (1), 2025
doi: 10.30511/ttmp.2025.2043936.1043

Transactions in Theoretical and Mathematical Physics

Controllability of fractional stochastic neutral integro-differential equations with state-dependent delay in Frechet spaces

R. Pradeepa[a,1,2], **R. Jayaraman**[b,3]

[1]Department of Mathematics, Periyar University, Salem - 636011, Tamil Nadu, India
[2]Department of Mathematics, Paavai Engineering College (Autonomous), Namakkal -637018, Tamil Nadu, India.
[3]Department of Mathematics, Government Arts and Science College, Pappireddipatti, Tamil Nadu, India.

Received: 21 October 2024 / Accepted: 25 February 2025 / Published: 27 February 2025

Abstract This paper investigates the existence and controllability of state-dependent delay functionally neutral stochastic fractional integro-differential equations within Fréchet spaces. By employing fixed-point techniques, the properties of solution operators, and tools from fractional calculus, we derive the conditions under which these systems are controllable. Our results extend the theoretical framework of fractional differential equations and provide new insights into the behavior of stochastic systems with memory effects. The findings have broad implications for various scientific fields, particularly those that model systems exhibiting complex dynamics, such as physics, biology, and engineering. To illustrate the practical application of our theoretical results, we provide a detailed example. This research contributes to the deeper understanding of fractional stochastic systems and lays the groundwork for future studies in the area of controlled fractional systems.

1 Introduction

Fractional order systems are helpful for concentrating the amazing behaviour of dynamical systems in a variety of scientific and engineering disciplines. The fractional differential equations offer a unique framework for debating the characteristics of real physical materials, such polymers. Recent research has shown that fractional order derivative-based differential models can be used to mathematically represent systems and processes in a variety of disciplines, including physics, chemistry, electrodynamics of complex media, polymer rheology, and aerodynamics. We recommend the monographs of Kilbas [1], Miller [2], Podlubny [3] and Zhou [4] as well as the works [5–12] and the references given therein to the readers for additional information. A specific kind of stochastic functional differential equations are stochastic differential equations with delay. Numerous academics have investigated the existence of outcomes for stochastic fractional differential equations with infinite delay and state-dependent delay (see [9, 10, 14, 15]). Many biological and physical applications of delay differential equations need us to think about variable or state-dependent delays. Controllability, one of the key concepts in mathematical control theory, is crucial to both deterministic and stochastic control theory.

Additionally, a number of authors have reported on the controllability of fractional differential equations and inclusions [6, 9, 12, 16–19]. Additionally, Benchohra et al. [20] evaluated the findings for fractional-order integro-differential inclusions in Frechet spaces in terms of existence and controllability. The approximate controllability of Hilfer fractional neutral integro-differential inclusions using almost sectorial operators has been explored, along with studies on impulsive ψ-Caputo fractional integrodifferential equations with boundary conditions. Additionally, recent research looked into the possibility of fractional neutral integro-differential inclusions in Fréchet spaces with state-dependent delay [11, 15, 21–24].

The following fractional order neutral integro-differential equations with state-dependent model delay are taken into consideration.

$$\begin{aligned} dD(\hbar,\zeta_{\sigma(\hbar,\zeta_\hbar)}) &= \int_0^\hbar \frac{(\hbar-\rho)^{\rho-2}}{\Gamma(\rho-1)} AD(\rho,\zeta_{\sigma(\rho,\zeta_\rho)})d\rho d\hbar \\ &\quad + F\left(\hbar,\zeta_{\sigma(\hbar,\zeta_\hbar)},\int_0^\hbar a(\hbar,\rho,\zeta_{\sigma(\rho,\zeta_\rho)})d\rho\right)dW(\hbar), \\ &\quad \hbar\in I=[0,1), \end{aligned} \tag{1}$$

$$\zeta_0 = \nu \in \mathbb{k}. \tag{2}$$

[a]e-mail: pradeepamahesh74@gmail.com
[b]e-mail: bjayarammaths@gmail.com

Here, Λ is a real separable Hilbert spaces, the state variable $\zeta(\cdot) \in \Lambda$ with norm $\|\cdot\|_\Lambda$ and inner product $(\cdot,\cdot)$, where $D(\hbar,\nu) = \nu(0) - G(\hbar,\nu)$, $1 < \rho < 2$, $G : I \times \mathbb{k} \to \Lambda$, $F : I \times \mathbb{k} \times \Lambda \to L_\varphi(\Xi,\Lambda)$ and $\sigma : I \times \mathbb{k} \to (-\infty,T]$ is a continuous function, $A : D(A) \subset \Lambda \to \Lambda$ is a linear densely defined operator of sectorial type on Λ.

Suppose $\{W(\hbar)\}_{\hbar\geq 0}$ is a given Ξ- valued Wiener process or Brownian motion defined on a complete probability space $(\mho_i,\mathfrak{F},P)$ with a finite trace nuclear covariance operator $\varphi > 0$ equipped with a normal filtration $\{\mathfrak{F}_\hbar\}_{\hbar\geq 0}$, which is generated by the Wiener process W.

We are also employing the norm $\|\cdot\|$ and $L(\Xi,\Lambda) : \Xi \to \Lambda$ denotes the space of all bounded linear operators. The time history $\zeta_\hbar : (-\infty,0] \to \Lambda$, $\zeta(\hbar+\vartheta), \vartheta \leq 0$, belongs to an axiomatically specified abstract phase space ($\mathbb{k}$). The initial data $\nu(\hbar) : -\infty < \hbar \leq 0$ is a random variable $\mathbb{k}$-valued, $\mathfrak{F}_0$-adapted and F, G, and σ are functions subject to further restrictions.

2 Preliminaries

Let $(\mho,\mathfrak{F},P,\mathbb{F})(\mathbb{F} = \{\mathfrak{F}_\hbar\}_{\hbar\geq 0})$ be a complete filtered probability space that satisfies the condition that all P-null sets of $\mathfrak{F}$ are contained in $\mathfrak{F}_0$. The stochastic process $S = \{\zeta(\hbar,W) : \mho \to \Lambda | \hbar \in I\}$ is a collection of random variables. An Λ-valued random variable is a $\mathfrak{F}$-measurable function $\zeta(\hbar) : \mho \to \Lambda$.

Generally, $\zeta(\hbar) : I \to \Lambda$ instead of $\zeta(\hbar,W)$ in the space of S. Let Ξ have a complete orthonormal basis with $\{e_i\}_{i=1}^\infty$. Denote $Tr(\varphi) = \sum_{i=1}^{\infty} \xi_i = \xi < 1$, which satisfies that $\varphi e_i = \xi_i e_i$. Assume that $W(\hbar) : \hbar \geq 0$ is a cylindrical Ξ-valued Wiener process with a finite. Therefore,

$$W(\hbar) = \sum_{i=1}^{\infty} \sqrt{\xi_i} W_i(\hbar) e_i, \tag{3}$$

where $\{W_i(\hbar)\}_{i=1}^\infty$ are truly mutually independent one dimensional standard Wiener processes. We suppose that the σ-algebra produced by W is $\mathfrak{F}_\hbar = \sigma\{W(\rho) : 0 \leq \rho \leq \hbar\}$. For $c_1, c_2 \in L(\Xi,\Lambda)$, we set

$$(c_1,c_2) = \mathrm{Tr}(c_1 \varphi c_2^*), \tag{4}$$

where c_2^* is the adjoint of c_2, and $\varphi \in L_\ell^+(\Xi)$ is the space of positive nuclear operators in Ξ.

For $\Psi \in L(\Xi,\Lambda)$, we set

$$\|\Psi\|_\varphi^2 = \mathrm{Tr}(\Psi\varphi\Psi^*) = \sum_{i=1}^{\infty} \|\sqrt{\xi_i}\Psi e_i\|^2. \tag{5}$$

If $\|\Psi\|_\varphi < \infty$, then Ψ is called a φ-Hilbert-Schmidt operator.

The space of all φ-Hilbert-Schmidt operators is denoted by $L_\varphi(\Xi,\Lambda)$. The completion $L_\varphi(\Xi,\Lambda)$ of $L(\Xi,\Lambda)$ in terms of the topology caused by the norm $\|\cdot\|_\varphi$, where $\|\Psi\|_\varphi^2 = (\Psi,\Psi)$ is a Hilbert space with the topology caused by the aforementioned norm [25].

The abstract phase space $\mathbb{k}$ is now on display. Assume that the phase space $(\mathbb{k}, \|\cdot\|_\mathbb{k})$ satisfies the following basic axioms and is a semi-normed linear space of functions translating $(-\infty,0]$ into Λ [26].

(A) If $\zeta : (-\infty,T] \to \Lambda$, $T > 0$ is continuous on I and $\zeta_0 \in \mathbb{k}$, then

(A1) $\zeta_\hbar \in \mathbb{k}, \ \forall \hbar \in I$

(A2) $\|\zeta(\hbar)\| \leq \tilde{\Lambda}\|\zeta_\hbar\|_\mathbb{k}, \ \forall \hbar \in I$

(A3) $\|\zeta_\hbar\|_\mathbb{k} \leq \Xi(\hbar) \sup_{0\leq\rho\leq\hbar} |\zeta(\rho)| + \Theta(\hbar)\|\zeta_0\|_\mathbb{k}, \ \forall \hbar \in I$,
where $\Xi, \Theta : [0,+\infty) \to [1,+\infty)$, Θ is locally bounded and $\tilde{\Lambda} \geq 0$- constant, Ξ is continuous, $\tilde{\Lambda}, K, \Theta$- independent of $\zeta(\cdot)$.

(B) $\zeta_\hbar$ is a $\mathbb{k}$- valued continuous functions on $[0,T]$, for the function $\zeta(\cdot)$ in (A),

(C) The space $\mathbb{k}$ is complete.

Definition 1 Assume that the domain $D(A)$ with a closed linear operator A in a Hilbert space Λ. In this case, A is the generator of a solution operator if and only if $\mu \in R$ and a strongly continuous function $S_\rho : R^+ \to L(\Lambda)$ exist, and if and only if $\{\xi^\rho : Re(\xi) > \mu\} \subset k(A)$, and if and only if $\xi^{\rho-1}(\xi^\rho - A)^{-1}\zeta = \int_0^\infty e^{\xi\hbar} S_\rho(\hbar) d\hbar$, $Re(\xi) > \mu$, $\zeta \in \Lambda$. In this instance, the solution operator produced by A is denoted as $S_\rho(\cdot)$.

The solution operator is characterised by

$$S_\rho(\hbar) = \frac{1}{2\pi} \int_\Sigma e^{-\xi\hbar} \xi^{\rho-1} (\xi^\rho - A)^{-1} d\xi. \tag{6}$$

where $0 < \vartheta < \pi\left(1 - \frac{\rho}{2}\right)$, A is sectorial type of μ and $\sum$ is a suitable path lying outside the sector $\mu + S_\rho$.

Cuesta [6] has proved that, there is $\chi > 0$, we have

$$\|S_\rho(\hbar)\| \leq \frac{\chi\Theta_0}{1 + |\mu|\hbar^\rho}, \quad \hbar \geq 0. \tag{7}$$

where A is a sectorial operator of type $\mu < 0$, for some $\Theta_0 > 0$ and $0 < \vartheta < \pi\left(1 - \frac{\rho}{2}\right)$.

Let $\Bbbk(\Lambda):\Lambda\to\Lambda$ be the space of all bounded linear operators, $\chi:[0,+\infty)\to\Lambda$ be the space of continuous functions and the norm $\|N\|=\sup\{\|N(\zeta)\|:\ \|\zeta\|=1\}$.

Bochner integrable is a measurable function $\zeta:[0,+\infty)\to\Lambda$, if $\|\zeta\|$ is Lebesgue integrable [39]. Let $L^1([0,+\infty),\Lambda)$ be the space of Bochner integrable measurable functions $\zeta:[0,+\infty)\to\Lambda$ with the norm

$$\|\zeta\|_{L^1}=\int_0^\infty\|\zeta(\hbar)\|d\hbar,\quad \text{for all}\quad \zeta\in L^1(I,\Lambda). \tag{8}$$

Recognize the space

$$\Bbbk_{+\infty}=\Big\{\zeta:(-\infty,+\infty)\to\Lambda\ |\quad \zeta|_I\in\chi_{\mathfrak{F}_\hbar}(I,\Lambda),\ \zeta_0\in L_0^2(\mho,\Lambda)\Big\}. \tag{9}$$

Let a family of semi-norms $\{\|\cdot\|_\ell\}_{\ell\in N}$ in a Frechet space Υ and $Y\subset\Upsilon$. We say that F is bounded if

$$\|z\|_\ell\le\overline{\Theta}_\ell,\quad \overline{\Theta}_\ell>0,\quad \text{for all } z\in Y. \tag{10}$$

for every $\ell\in N$.

Let Υ be a sequence of Banach spaces $\{(\Upsilon^\ell,\|\cdot\|_\ell)\}$ as follows:

1. The equivalence relation $\sim_\ell$ defined by $\zeta\sim_\ell z\Leftrightarrow\|\zeta-z\|_\ell=0$, for all $\zeta,z\in\Upsilon,\ell\in N$.

2. The completion of the quotient space $\Upsilon^\ell=(\Upsilon|_{\sim_\ell},\|\cdot\|_\ell)$ associated with $\|\cdot\|_\ell$.

   Let $Y\subset\Upsilon$, and a sequence $\{Y^\ell\}\subset\Upsilon^\ell$. Then

1. $[\zeta]_\ell$ be the equivalence class of $\zeta\subset\Upsilon^\ell$ and $Y^\ell=\{[\zeta]_\ell:\zeta\in Y\}$, $\forall\,\zeta\in\Upsilon$

2. We denote the closure (Y^ℓ), the interior $(int_\ell(Y^\ell))$ and boundary $(\partial_\ell Y^\ell)$ of Y^ℓ with respect to $\|\cdot\|_\ell\in\Upsilon^\ell$.

If $\{\|\cdot\|_\ell\}$ be the family of semi-norms, then

$$\|\zeta\|_1\le\|\zeta\|_2\le\|\zeta\|_3\le\cdots\ \text{for every }\zeta\in\Upsilon.$$

Definition 2 A function $F:I\times\Bbbk\times\Lambda\to L_\varphi(\Xi,\Lambda)$ is said to be an L^2- Caratheodory function if

(i) the function $F(\hbar,\cdot,\cdot):\Bbbk\times\Lambda\to L_\varphi(\Xi,\Lambda)$ is continuous, for each $\hbar\in I$

(ii) the function $F(\cdot,\zeta,z):I\to L_\varphi(\Xi,\Lambda)$ is $\mathfrak{F}_\hbar$-measurable, for each $\zeta\in\Bbbk$ and $z\in\Lambda$

(iii) if $F_k\in L^1_{loc}(I,R^+)$ exists, then

$$\aleph\|F(\hbar,\zeta,z)\|^2\le F_k(\hbar),\ \ \forall\ \aleph\|\zeta\|^2\le k\ \&\ \aleph\|z\|^2\le k,$$

for every positive integer k and for almost all $\hbar\in I$.

Lemma 1 *Let $\zeta:(-\infty,\ell]\to\Lambda$ be an $\mathfrak{F}_\hbar$-adapted measurable process such that $\zeta|_{I\in\Bbbk_{+\infty}}$ and $\mathfrak{F}_0$-adapted process $\zeta_0=\nu(\hbar)\in L_0^2(\mho,B)$. Then*

$$\|\zeta_\rho\|_\Bbbk\le\Theta_\ell\aleph\|\nu\|_\Bbbk+\Xi_\ell\sup_{0\le\rho\le n}\aleph\|\zeta(\rho)\|. \tag{11}$$

Lemma 2 *Let Y be a closed subset of a Frechet space Υ and $N:Y\to\Upsilon$ be a contraction such that $N(Y)$ is bounded. Then*

(a) N has a fixed point

(b) $\zeta\in\partial_\ell Y^\ell$ and $\xi\in[0,1),\ell\in N$ exists such that $\|\zeta-\xi N(\zeta)\|_\ell=0$.

3 Main results

Definition 3 An $\mathfrak{F}_\hbar$-adapted stochastic process $\zeta:(-\infty,+\infty)\to\Lambda$ is called a mild solution of the system (1) - (2) if the restriction of $\zeta(\cdot)$ to the interval I is continuous,

$$\zeta_0=\nu(\hbar),\quad \zeta_{\sigma(\rho,\zeta_\rho)}\in\Bbbk. \tag{12}$$

satisfying $\zeta_0\in L_0^2(\mho,\Lambda)$, and

$$\begin{aligned}\zeta(\hbar)=&\,S_\rho(\hbar)[\nu(0)-G(0,\nu)]+G(\hbar,\zeta_{\sigma(\hbar,\zeta_\hbar)})\\&+\int_0^\hbar S_\rho(\hbar-\rho)F\Bigg(\rho,\zeta_{\sigma(\rho,\zeta_\rho)},\\&\int_0^\rho a(\rho,\tau,\zeta_{\varsigma(\tau,\zeta_\tau)})d\tau\Bigg)dW(\rho),\qquad \hbar\in I.\end{aligned} \tag{13}$$

Let $\sigma:[0,\ell]\times\Bbbk\to(-\infty,\ell]$ is continuous. Then

(H1) $\Theta>0$ exists, and $\|S_\rho(\hbar)\|^2\le\Theta$, for all $\quad\hbar\ge0$.

(H2) There exists a bounded and continuous function $J^\nu: R(\sigma^-)\to(0,1)$ and the function $\hbar\to\nu_\hbar$ is continuous from $R(\sigma^-)=\{p(\rho,\Psi)\le0,\ (\rho,\Psi)\in[0,\ell]\times B\}$ into $\Bbbk$ such that $\|\nu_\hbar\|\le J^\nu(\hbar)\|\nu\|_\Bbbk$ for each $\hbar\in R(\sigma^-)$.

(H3) There exist a constants $\chi_1 \geq 0$, and $\chi_2 > 0$ such that

$$\aleph \|G(\hbar,\zeta)\|^2 \leq \chi_1 \|\zeta\|_{\Bbbk}^2 + \chi_2, \text{ for } \hbar \in I,\ \zeta \in \Bbbk. \quad (14)$$

(H4) There exists $\Gamma_\ell \in L^1_{loc}(I,R^+)$, $\ell > 0$ such that

$$\aleph \|G(\hbar,\upsilon) - G(\hbar,\zeta)\|^2 \leq \Gamma_\ell(\hbar) \aleph \|\upsilon - \zeta\|_{\Bbbk}^2, \quad (15)$$

for each $\hbar \in I$ and for all $\upsilon, \zeta \in \Bbbk$ with $\aleph \|\upsilon\|_{\Bbbk}^2 \leq \ell$ and $\aleph \|\zeta\|_{\Bbbk}^2 \leq \ell$.

(H5) The multifunction $F : I \times \Bbbk \times \Lambda \to P(L_\varphi(\Xi,\Lambda))$ is L^2_{loc}-Caratheodory with compact and convex values. If there exists $p \in L^1_{loc}(I,R^+)$ and $\Psi : I \to (0,1)$ be the continuous nondecreasing function such that

$$\aleph \|F(\hbar,\upsilon,\nu)\|^2 \leq p(\hbar)(\|\upsilon\|_{\Bbbk}^2 + \aleph \|\nu\|^2 \Lambda), \quad (16)$$

for every $\hbar \in I, \upsilon \in \Bbbk$ and $\nu \in \Lambda$.

(H6) There exists $\mathfrak{L}_\ell \in L^1_{loc}(I,R^+)$, $n > 0$ such that

$$\aleph \|F(\hbar,\upsilon_1,\zeta_1) - F(\hbar,\upsilon_2,\zeta_2)\|^2 \leq \mathfrak{L}_\ell(\hbar)(\aleph \|\upsilon_1 - \upsilon_2\|_{\Bbbk}^2 + \|\zeta_1 - \zeta_2\|^2 \Lambda), \quad (17)$$

for each $\hbar \in I, \upsilon_1, \upsilon_2 \in \Bbbk$ and $\zeta_1, \zeta_2 \in \Lambda$ with $\aleph \|\upsilon_1\|_{\Bbbk}^2 \leq \ell$, $\aleph \|\upsilon_2\|_{\Bbbk}^2 \leq \ell$, $\aleph \|\zeta_1\|_{\Bbbk}^2 \leq \ell$ and $\aleph \|\zeta_2\|_{\Bbbk}^2 \leq \ell$.

(H7) The function $a : D \times \Bbbk \to \Lambda$, where $D = \{(\hbar,\rho) \in I \times I,\ 0 \leq \rho \leq \hbar \leq T\}$ satisfies:

(i) The function $a(\hbar,\rho,\cdot) : \Bbbk \to \Lambda$, $(\hbar,\rho) \in D$ is continuous and the function $a(\cdot,\cdot,\upsilon) : D \to \Lambda$ is strongly measurable

$$\|a(\hbar,\rho,\upsilon)\|^2 \leq \Theta_a(1 + \|\upsilon\|_{\Bbbk}^2), \quad (18)$$

for each $\hbar, \rho \in D, \upsilon \in \Bbbk$.

(ii) There exist a constant $\tilde{\Theta}_a(r) > 0$, $r > 0$ such that

$$\aleph \|a(\hbar,\rho,\upsilon) - a(\hbar,\rho,\zeta)\|^2 \leq \tilde{\Theta}_a(r)\|\upsilon - \zeta\|_{\Bbbk}^2, \quad (19)$$

and $\hbar, \rho \in D, \upsilon, \zeta \in \Bbbk$.

(H8) There exists a constant $\beta_\ell > 0$, $\ell \in N$ such that

$$\frac{(1 - 6\Xi_\ell^2 \chi_1)\beta_\ell 1}{\Upsilon_1 + 6Tr(\varphi)\Theta \Xi_\ell^2 \Psi((1 + T\Theta_a)\beta_\ell)\|p\|_{L^1_{[0,\ell]}}} > 1, \quad (20)$$

where

$$\begin{aligned}\Upsilon =& 12M\Xi_\ell^2[\tilde{\Lambda}^2\|\nu\|_{\Bbbk}^2 + (\chi_1\|\nu\|_{\Bbbk}^2 + \chi_2)] \\ &+ 2[(\Theta_\ell + J_0^\nu)\|\nu\|_{\Bbbk}^2] + 6\Xi_\ell^2 \chi_2 \\ &+ 6\Xi_\ell^2 Tr(\varphi)\Theta \int_0^\ell p(\rho)(T\Theta_a)d\rho. \end{aligned} \quad (21)$$

Lemma 3 *[16, 17] Let $\zeta : (-\infty,\ell] \to \Lambda$ be continuous on $[0,\ell]$ and $\zeta_0 = \nu$. If (H2) is satisfied, then*

$$\|\zeta_\rho\|_{\Bbbk} \leq (\Theta_\ell + J_0^\nu)\|\nu\|_{\Bbbk} + \Xi_\ell \sup_{\vartheta \geq 0} \|\zeta(\vartheta)\|. \quad (22)$$

$\vartheta \in [0, \max\{0,\rho\}]\}$, $\rho \in R(\sigma^-) \cup [0,\ell]$, *where*

$$J_0^\nu = \sup_{\hbar \in R}(\sigma^-) J^\nu(\hbar). \quad (23)$$

Remark 1 [16] Let $\nu \in \Bbbk, \hbar \leq 0$ and the function $\nu_\hbar = \nu(\hbar + \vartheta)$, $\nu \in (-\infty,0]$ is well-defined for $\hbar < 0$. Consequently, if the function $\zeta(\cdot)$ in axiom (A) is such that $\zeta_0 = \nu$, then $\zeta_\hbar = \nu_\hbar$.

Theorem 1 *Let $\nu \in L_2^2(\mho,\Lambda)$, the assumptions (H1) - (H8) are satisfied and*

$$2\Xi_\ell^2 \sup_{\hbar \in [0,\ell]} \Gamma_\ell(\hbar) < 1, \quad \textit{for each} \quad \ell \in \mathbb{N}. \quad (24)$$

Then the problem (1) - (2) has a unique mild solution on I.

Proof Let us fix $\tau > 1$ and we define in $\Bbbk_{+\infty}$ the semi-norms $\|\zeta\|_\ell = \sup\{e^{\tau \mathfrak{L}_\ell^*}(\hbar) \aleph \|\zeta(\hbar)\|^2 : \hbar \in [0,\ell]\}$, for every $\ell \in N$, where

$$\mathfrak{L}_\ell^*(\hbar) = \int_0^\hbar \overline{\mathfrak{L}}_\ell(\rho)d\rho, \quad (25a)$$

$$\overline{\mathfrak{L}}_\ell(\hbar) = 2\,\mathrm{Tr}(\varphi)\Theta\Xi_\ell^2(1 + T\tilde{\Theta}_a(r))\mathfrak{L}_\ell(\rho). \quad (25b)$$

and $\mathfrak{L}_\ell$ is the function from (H6). Then $\Bbbk_{+\infty}$ is a Frechet space with the family of semi-norms $\|\cdot\|_{\ell \in N}$.

Consider the space $Y = \{\zeta \in \Bbbk_{+\infty} : \zeta(0) = \nu(0)\}$ endowed with the uniform convergence topology $(\|\cdot\|_\infty)$ and define $\Phi : Y \to Y$ by

$$(\Phi_y)(\hbar) = \begin{cases} 0, \\ S_\rho(\hbar)[\nu(0) - G(0,\nu)] + G(\hbar, \overline{\zeta}_{\sigma(\hbar,\overline{\zeta}_\hbar)}) \\ \qquad + \int_0^\hbar S_\rho(\hbar-\rho) \times F\left(\rho, \overline{\zeta}_{\sigma(\rho,\overline{\zeta}_\rho)}, \int_0^\rho a(\rho,\tau,\overline{\zeta}_{\varsigma(\tau,\overline{\zeta}_\tau)})d\tau\right) dW(\rho), \quad \hbar \in I. \end{cases} \tag{26}$$

where $\overline{\zeta} : (-\infty, 0] \to \Lambda$ such that $\overline{\zeta}_0 = \nu$ and $\overline{\zeta} = \zeta$ on $[0,\ell]$. Let $\overline{\nu} : (-\infty, 0) \to \Lambda$ be the extension of $(-\infty, 0]$ such that $\overline{\nu}(\vartheta) = \nu(0) = 0$ on $[0,\ell]$ and $J_0^\nu = \sup\{J^\nu(\rho) : \rho \in R(\sigma^-)\}$.

We show that Φ has a fixed point, which in turn is a mild solution of the problem (1) - (2). Let ζ be a possible solution of problem (1) - (2): Given $\ell \in N$ and $\hbar \in [0,\ell]$, then

$$\begin{aligned} \zeta(\hbar) &= S_\rho(\hbar)[\nu(0) - G(0,\nu)] + G(\hbar, \overline{\zeta}_{\sigma(\hbar,\overline{\zeta}_\hbar)}) \\ &\quad + \int_0^\hbar S_\rho(\hbar-\rho) \\ &\quad \times F\left(\rho, \overline{\zeta}_{\sigma(\rho,\overline{\zeta}_\rho)}, \int_0^\rho a(\rho,\tau,\overline{\zeta}_{\varsigma(\tau,\overline{\zeta}_\tau)})d\tau\right) dW(\rho). \end{aligned} \tag{27}$$

For each $\hbar \in [0,\ell]$ hypotheses (H3), (H5) and (H7) imply

$$\begin{aligned} \aleph\|\zeta(\hbar)\|^2 &\le 3\aleph\|S_\rho(\hbar)[\nu(0) - G(0,\nu)]\|^2 \\ &\quad + 3\aleph\|G(\hbar, \overline{\zeta}_{\sigma(\hbar,\overline{\zeta}_\hbar)})\|^2 + 3\aleph\left\|\int_0^\hbar S_\rho(\hbar-\rho)\right. \\ &\quad \left.\times F\left(\rho, \overline{\zeta}_{\sigma(\rho,\overline{\zeta}_\rho)}, \int_0^\rho a(\rho,\tau,\overline{\zeta}_{\varsigma(\tau,\overline{\zeta}_\tau)})d\tau\right) dW(\rho)\right\|^2 \\ &\le \sum_{i=1}^3 J_i. \end{aligned} \tag{28}$$

$$\begin{aligned} J_1 &= 3\aleph\|S_\rho(\hbar)[\nu(0) - G(0,\nu)]\|^2 \\ &\le 6\Theta[\tilde{\Lambda}^2\|\nu\|_{\Bbbk}^2 + (\chi_1\|\nu\|_{\Bbbk}^2 + \chi_2)] \end{aligned} \tag{29}$$

$$J_2 = 3\aleph\|G(\hbar, \overline{\zeta}_{\sigma(\hbar,\overline{\zeta}_\hbar)})\|^2 \le 3(\chi_1\|\overline{\zeta}_{\sigma(\hbar,\overline{\zeta}_\hbar)}\|_{\Bbbk}^2 + \chi_2) \tag{30}$$

$$\begin{aligned} J_3 &= 3\aleph\left\|\int_0^\hbar S_\rho(\hbar-\rho)\right. \\ &\quad \left.\times F\left(\rho, \overline{\zeta}_{\sigma(\rho,\overline{\zeta}_\rho)}, \int_0^\rho a(\rho,\tau,\overline{\zeta}_{\varsigma(\tau,\overline{\zeta}_\tau)})d\tau\right) dW(\rho)\right\|^2 \\ &\le 3\Theta\int_0^\hbar Tr(\varphi)\aleph \\ &\quad \times \left\|F\left(\rho, \overline{\zeta}_{\sigma(\rho,\overline{\zeta}_\rho)}, \int_0^\rho a(\rho,\tau,\overline{\zeta}_{\varsigma(\tau,\overline{\zeta}_\tau)})d\tau\right)\right\|^2 d\rho \\ &\le 3Tr(\varphi)\Theta\int_0^\hbar p(\rho) \\ &\quad \times \left(\|\overline{\zeta}_{\sigma(\rho,\overline{\zeta}_\rho)}\|_{\Bbbk}^2 + T\Theta_a(1 + \|\overline{\zeta}_{\sigma(\tau,\overline{\zeta}_\tau)}\|_{\Bbbk}^2)\right) d\rho. \end{aligned} \tag{31}$$

Substituting $(J_1) - (J_3)$ together with (28), we have

$$\begin{aligned} \aleph\|\zeta(\hbar)\|^2 &\le 6\Theta[\tilde{\Lambda}^2\|\nu\|_{\Bbbk}^2 + (\chi_1\|\nu\|_{\Bbbk}^2 + \chi_2)] \\ &\quad + 3\left(\chi_1\|\overline{\zeta}_{\sigma(\hbar,\overline{\zeta}_\hbar)}\|_{\Bbbk}^2 + \chi_2\right) \\ &\quad + 3Tr(\varphi)\Theta\int_0^\hbar p(\rho)\Psi\Bigg(\|\overline{\zeta}_{\sigma(\rho,\overline{\zeta}_\rho)}\|_{\Bbbk}^2 \\ &\quad + T\Theta_a(1 + \|\overline{\zeta}_{\sigma(\tau,\overline{\zeta}_\tau)}\|_{\Bbbk}^2)\Bigg) d\rho. \end{aligned} \tag{32}$$

By Lemma 1 and Lemma 3, $\Rightarrow$ $\varsigma(\rho, \overline{\zeta}_\rho) \le \rho$, $\rho \in [0,\ell]$ and

$$\begin{aligned} &\|\overline{\zeta}_{\sigma(\rho,\overline{\zeta}_\rho)}\|_{\Bbbk}^2 \\ &\quad \le 2[(\Theta_\ell + J_0^\nu)\|\nu\|_{\Bbbk}]^2 + 2\Xi_\ell^2 \sup_{\rho\in[0,\ell]} \aleph\|\zeta(\rho)\|^2. \end{aligned} \tag{33}$$

For each $\hbar \in [0,\ell]$, we get

$$\begin{aligned} \aleph\|\zeta(\hbar)\|^2 &\le 6\Theta[\tilde{\Lambda}^2\|\nu\|_{\Bbbk}^2 + (\chi_1\|\nu\|_{\Bbbk}^2 + \chi_2)] \\ &+ 3\left[\chi_1\left(2[(\Theta_\ell + J_0^\nu)\|\nu\|_{\Bbbk}]^2 + 2\Xi_\ell^2 \sup_{\rho\in[0,\ell]} \aleph\|\zeta(\rho)\|^2\right) + \chi_2\right] \\ &+ 3Tr(\varphi)\Theta\int_0^\hbar p(\rho)\Psi\Bigg\{\Bigg(2[(\Theta_\ell + J_0^\nu)\|\nu\|_{\Bbbk}]^2 \\ &\qquad + 2\Xi_\ell^2 \sup_{\rho\in[0,\ell]} \aleph\|\zeta(\rho)\|^2\Bigg) \\ &+ T\Theta_a\Bigg(1 + \Bigg(2[(\Theta_\ell + J_0^\nu)\|\nu\|_{\Bbbk}]^2 \\ &\qquad + 2\Xi_\ell^2 \sup_{\rho\in[0,\ell]} \aleph\|\zeta(\rho)\|^2\Bigg)\Bigg)\Bigg\} d\rho. \end{aligned} \tag{34}$$

Consider the norm of the function μ defined by

$$\mu(\hbar) := 2[(\Theta_\ell + J_0^\nu)\|\nu\|_{\Bbbk}]^2 + 2\Xi_\ell^2 \sup_{0\le\rho\le\hbar} \aleph\|\zeta(\rho)\|^2, \tag{35}$$

which $\|\mu\|_\infty = \sup_{0\le\hbar\le T}\mu(\hbar)$. By the previous inequality Eq. (34), we get

$$\|\mu\|_\infty \le 12M\Xi_\ell^2 \times \tilde{\Lambda}^2\|\nu\|_{\Bbbk}^2 + (\chi_1\|\nu\|_{\Bbbk}^2+\chi_2) + 2[(\Theta_\ell + J_0^\nu)\|\nu\|_{\Bbbk}]^2 + 6\Xi_\ell^2(\chi_1\|\mu\|_\infty+\chi_2) + 6Tr(\varphi)\Theta\Xi_\ell^2\int_0^\hbar p(\rho)\Psi\{(1+T\Theta_a)\|\mu\|_\infty + T\Theta_a\}d\rho, \quad (36)$$

for $\hbar \in [0,\ell]$. Therefore,

$$\|\mu\|_\infty \le 12M\Xi_\ell^2 \times \tilde{\Lambda}^2\|\nu\|_{\Bbbk}^2 + (\chi_1\|\nu\|_{\Bbbk}^2+\chi_2) + 2[(\Theta_\ell + J_0^\nu)\|\nu\|_{\Bbbk}]^2 + 6\Xi_\ell^2\chi_2 + 6Tr(\varphi)\Theta\Xi_\ell^2\int_0^\ell p(\rho)\Psi(T\Theta_a)d\rho + 6\Xi_\ell^2\chi_1\|\mu\|_\infty + 6Tr(\varphi)\Theta\Xi_\ell^2\Psi((1+T\Theta_a)\|\mu\|_\infty)\int_0^\ell p(\rho)d\rho. \quad (37)$$

Consequently,

$$\frac{(1-6\Xi_\ell^2\chi_1)\|\mu\|_\infty}{\Upsilon_1 + 6Tr(\varphi)\Theta\Xi_\ell^2\Psi((1+T\Theta_a)\|\mu\|_\infty)\int_0^\ell p(\rho)d\rho} \le 1. \quad (38)$$

Then by (H8), there exists β_ℓ such that $\|\mu\|_\infty \le \beta_\ell$.

Since $\|\zeta\|_{\Bbbk_{+\infty}} \le \|\mu\|_\infty \Rightarrow \|\zeta\|_\ell \le \beta_\ell$. Set:

$$\Upsilon = \Big\{\zeta \in \Bbbk_{+\infty} : \sup\{\aleph\|\zeta(\hbar)\|^2 : 0\le\hbar\le\ell\} \le \beta_\ell + 1, \text{ for all } \ell \in N\Big\}. \quad (39)$$

Clearly, $\Upsilon \subset \Bbbk_{+\infty}$ is closed. We shall show that $\Phi:\Upsilon \to \Bbbk_{+\infty}$ is a contraction operator.

Consider $\zeta^*, \zeta^{**} \in \Bbbk_{+\infty}$. By Lemma 1, Lemma 3 and (H4), (H6) and (H7), we get

$$\begin{aligned}
&\aleph\|\Phi\overline{\zeta}^*(\hbar) - \Phi\overline{\zeta}^{**}(\hbar)\|^2 \\
&\le 2\aleph\|G(\hbar,\overline{\zeta}^*_{\sigma(\hbar,\overline{\zeta}^*_\hbar)}) - G(\hbar,\overline{\zeta}^{**}_{\sigma(\hbar,\overline{\zeta}^{**}_\hbar)})\|^2 \\
&\quad + 2\aleph\Big\|\int_0^\hbar S_\rho(\hbar-\rho) \\
&\quad \times\Big[F\Big(\rho,\overline{\zeta}^*_{\sigma(\rho,\overline{\zeta}^*s)},\int_0^\rho a(\rho,\tau,\overline{\zeta}^*_{\varsigma(\tau,\overline{\zeta}^*_\tau)})d\tau\Big) \\
&\quad - F\Big(\rho,\overline{\zeta}^{**}_{\sigma(\rho,\overline{\zeta}^{**}_\rho)},\int_0^\rho a(\rho,\tau,\overline{\zeta}^{**}_{\varsigma(\tau,\overline{\zeta}^{**}_\tau)})d\tau\Big)\Big]dW(\rho)\Big\|^2 \\
&\le J_4 + J_5,
\end{aligned} \quad (40)$$

for each $\hbar \in [0,\ell]$ and $\ell \in N$. Here

$$\begin{aligned}
J_4 &\le 2\aleph\|G(\hbar,\overline{\zeta}^*_{\sigma(\hbar,\overline{\zeta}^*\hbar)}) - G(\hbar,\overline{\zeta}^{**}_{\sigma(\hbar,\overline{\zeta}^{**}_\hbar)})\|^2 \\
&\le 2\Gamma_\ell(\hbar)\aleph\|\overline{\zeta}^*_{\sigma(\hbar,\overline{\zeta}^*\hbar)} - \overline{\zeta}^{**}_{\sigma(\hbar,\overline{\zeta}^{**}_\hbar)}\|^2.
\end{aligned} \quad (41)$$

$$\begin{aligned}
J_5 &\le 2\aleph\Big\|\int_0^\hbar S_\rho(\hbar-\rho) \\
&\quad \times\Big[F\Big(\rho,\overline{\zeta}^*_{\sigma(\rho,\overline{\zeta}^*_\rho)},\int_0^\rho a(\rho,\tau,\overline{\zeta}^*_{\varsigma(\tau,\overline{\zeta}^*_\tau)})d\tau\Big) \\
&\quad - F\Big(\rho,\overline{\zeta}^{**}_{\sigma(\rho,\overline{\zeta}^{**}_\rho)},\int_0^\rho a(\rho,\tau,\overline{\zeta}^{**}_{\varsigma(\tau,\overline{\zeta}^{**}_\tau)})d\tau\Big)\Big]dW(\rho)\Big\|^2 \\
&\le 2\Theta Tr(\varphi)\int_0^\hbar \aleph\Big\|F\Big(\rho,\overline{\zeta}^*_{\sigma(\rho,\overline{\zeta}^*_\rho)},\int_0^\rho a(\rho,\tau,\overline{\zeta}^*_{\varsigma(\tau,\overline{\zeta}^*_\tau)})d\tau\Big) \\
&\quad - F\Big(\rho,\overline{\zeta}^{**}_{\sigma(\rho,\overline{\zeta}^{**}_\rho)},\int_0^\rho a(\rho,\tau,\overline{\zeta}^{**}_{\varsigma(\tau,\overline{\zeta}^{**}_\tau)})d\tau\Big)\Big\|^2 d\rho \\
&\le 2Tr(\varphi)\Theta\int_0^\hbar \mathfrak{L}_\ell(\rho)\Big((1+T\tilde{\Theta}_a(r))\aleph \\
&\quad \times\|\overline{\zeta}^*_{\sigma(\rho,\overline{\zeta}^*_\rho)} - \overline{\zeta}^{**}_{\sigma(\rho,\overline{\zeta}^{**}_\rho)}\|_{\Bbbk}^2\Big)d\rho.
\end{aligned} \quad (42)$$

Substituting (J_4) and (J_5) together with (40), we obtain

$$\begin{aligned}
&\aleph\|\Phi\overline{\zeta}^*(\hbar) - \Phi\overline{\zeta}^{**}(\hbar)\|^2 \\
&\le 2\Gamma_\ell(\hbar)\aleph\|\overline{\zeta}^*_{\sigma(\hbar,\overline{\zeta}^*_\hbar)} - \overline{\zeta}^{**}_{\sigma(\hbar,\overline{\zeta}^{**}_\hbar)}\|^2 + 2\Theta Tr(\varphi)\int_0^\hbar \mathfrak{L}_\ell(\rho) \\
&\quad \times\Big((1+T\tilde{\Theta}_a(r))\|\overline{\zeta}^*_{\sigma(\rho,\overline{\zeta}^*_\rho)} - \overline{\zeta}^{**}_{\sigma(\rho,\overline{\zeta}^{**}_\rho)}\|_{\Bbbk}^2\Big)d\rho \\
&\le 2\Gamma_\ell(\hbar)\Xi_\ell^2 \sup_{\hbar\in[0,\ell]}\aleph\|\overline{\zeta}^*(\hbar) - \overline{\zeta}^{**}(\hbar)\|^2 \\
&\quad + 2Tr(\varphi)\Theta\Xi_\ell^2 \\
&\quad \times\int_0^\hbar \mathfrak{L}_\ell(\rho)(1+T\tilde{\Theta}_a(r))\aleph\|\overline{\zeta}^*(\rho) - \overline{\zeta}^{**}(\rho)\|^2 d\rho \\
&\le 2\Xi_\ell^2\Gamma_\ell(\hbar)[e^{-\tau\overline{\mathfrak{L}}_\ell(\hbar)}] \\
&\quad \times\Big[e^{-\tau\overline{\mathfrak{L}}_\ell(\hbar)}\sup_{\hbar\in[0,\ell]}\aleph\|\overline{\zeta}^*(\hbar) - \overline{\zeta}^{**}(\hbar)\|^2\Big] \\
&\quad + \int_0^\hbar[\overline{\mathfrak{L}}_\ell(\rho)e^{\tau\overline{\mathfrak{L}}_\ell(\rho)}]\Big[e^{-\tau\overline{\mathfrak{L}}_\ell(\rho)}\aleph\|\overline{\zeta}^*(\rho) - \overline{\zeta}^{**}(\rho)\|^2\Big]d\rho \\
&\le 2\Xi_\ell^2[e^{\tau\overline{\mathfrak{L}}_\ell(\hbar)}]\sup_{\hbar\in[0,\ell]}\Gamma_\ell(\hbar)\|\overline{\zeta}^* - \overline{\zeta}^{**}\|_\ell \\
&\quad + \int_0^\hbar \frac{1}{\tau}[e^{\tau\overline{\mathfrak{L}}_\ell(\rho)}]' d\rho\|\overline{\zeta}^* - \overline{\zeta}^{**}\|_\ell \\
&\le e^{\tau\overline{\mathfrak{L}}_\ell(\hbar)}\Big[2\Xi_\ell^2\sup_{\hbar\in[0,\ell]}\Gamma_\ell(\hbar) + \frac{1}{\tau}\Big]\|\overline{\zeta}^* - \overline{\zeta}^{**}\|_\ell.
\end{aligned} \quad (43)$$

By using $\overline{\zeta} = \zeta$ on $[0,\ell]$ and taking supremum over $\hbar$,

$$\|\Phi\overline{\zeta}^{*}-\Phi\overline{\zeta}^{**}\|_{\ell} \leq \left[2\Xi_{\ell}^{2}\sup_{\hbar\in[0,\ell]}\Gamma_{\ell}(\hbar)+\frac{1}{\tau}\right]\|\overline{\zeta}^{*}-\overline{\zeta}^{**}\|_{\ell}, \quad (44)$$

for all $\ell\in N$ showing that the operator Φ is a contraction.

From the choice of Υ there is no $\zeta\in\partial\Upsilon^{\ell}$ such that $\zeta=\Phi(\zeta)$, for some $\xi\in(0,1)$, the problem (1)-(2) has a unique mild solution, ζ, which is the fixed point of the operator Φ. as a result of Frigon and Granas' nonlinear alternative. The proof is complete.

4 Applications to control theory

This part applies the reasoning from the preceding sections to the issue of whether a class of fractional neutral stochastic integro-differential equations in a Hilbert space Λ with state-dependent delay can be managed. We pay extra attention to the following problem.

$$dD(\hbar,\zeta_{\sigma(\hbar,\zeta_{\hbar})})=\int_{0}^{\hbar}\frac{(\hbar-\rho)^{\rho-2}}{\Gamma(\rho-1)}AD(\rho,\zeta_{\sigma(\rho,\zeta_{\rho})})d\rho + (\mathbb{k}u)(\hbar)d\hbar + F\left(\hbar,\zeta_{\sigma(\hbar,\zeta_{\hbar})},\int_{0}^{\hbar}a(\hbar,\rho,\zeta_{\sigma(\rho,\zeta_{\rho})})d\rho\right)dW(\hbar), \quad \hbar\in I=[0,1), \quad (45)$$

$$\zeta_{0}=\nu(\hbar)\in\mathbb{k}, \quad (46)$$

where A,F and D are as in Section 3. Additionally, the control function u is a member of the space $L^{2}(I,U)$, a Banach space of permissible control functions, which also contains the Banach space U. Additionally, $\mathbb{k}:U\to\Lambda$ is a bounded linear operator. Numerous authors have developed the controllability results for stochastic semi-linear differential and integro-differential systems in Hilbert spaces, including [6, 18, 19, 27, 28] and references thereto.

Definition 4 A mild solution of Eq. (46) is an $\mathfrak{F}_{\hbar}$-adapted stochastic process $\zeta:(-\infty,+\infty)\to\Lambda$, if the restriction of $\zeta(\cdot)$ to the interval I is continuous, $\zeta_{0}=\nu(\hbar)$, $\zeta_{\sigma(\rho,\zeta_{\rho})}\in\mathbb{k}$ satisfying $\zeta_{0}\in L_{2}^{0}(\mho,\Lambda)$ and

$$\zeta(\hbar)=S_{\rho}(\hbar)[\nu(0)-G(0,\nu)]+G(\hbar,\zeta_{\sigma(\hbar,\zeta_{\hbar})}) + \int_{0}^{\hbar}S_{\rho}(\hbar-\rho)(\mathbb{k}u)(\rho)d\rho + \int_{0}^{\hbar}S_{\rho}(\hbar-\rho)F \times\left(\rho,\zeta_{\sigma(\rho,\zeta_{\rho})},\int_{0}^{\rho}a(\rho,\tau,\zeta_{\varsigma(\tau,\zeta_{\tau})})d\tau\right)dW(\rho),\ \hbar\in I. \quad (47)$$

Definition 5 A stochastic control $u\in L^{2}(I,U)$, which is adapted to the filtration $\{\mathfrak{F}_{\hbar}\}_{\hbar\geq0}$, if for every initial random variable $\zeta_{0},\zeta_{1}\in L_{2}^{0}(\mho,\Lambda)$ such that the mild solution $\zeta(\hbar)$ of the system (45)-(46) satisfies $\zeta(\ell)=\zeta_{1}$, then the system (45)-(46) is said to be controllable on the interval I.

We make the ensuing presumptions:

(B1) Define $W:L^{2}([0,\ell],U)\to\Upsilon$ be the linear operator by

$$Wu=\int_{0}^{\ell}S_{\rho}(\ell-\rho)Bu(\rho)d\rho.$$

The inverse operator W^{-1} which takes values in $L^{2}([0,\ell],U)/Ker\ W$ and $\|BW^{-1}\|^{2}\leq\Theta_{\mathbb{k}}$, if a positive constants $\Theta_{\mathbb{k}}$ exist.

(B2) There exists a constant $\beta_{\ell}^{*}>0$ such that

$$\frac{(1-8\chi_{1}\Xi_{\ell}^{2}(1+4\Theta\Theta_{\mathbb{k}}\ell^{2}))\beta_{\ell}^{*}}{\Upsilon_{2}+8Tr(\varphi)\Theta\Xi_{\ell}^{2}(1+2\Theta\Theta_{\mathbb{k}}\ell^{2})(1+T\Theta_{a})\Psi(\beta_{\ell}^{*})\|p\|_{L^{1}_{[0,\ell]}}}>1, \qquad \ell\in N, \quad (48)$$

where

$$\Upsilon_{2}=16\Theta\Xi_{\ell}^{2}[\tilde{\Lambda}^{2}\|\nu\|_{\mathbb{k}}^{2}+(\chi_{1}\|\nu\|_{\mathbb{k}}^{2}+\chi_{2})] +2[(\Theta_{\ell}+J_{0}^{\nu})\|\nu\|_{\mathbb{k}}]^{2}+8\Xi_{\ell}^{2}\chi_{2} +32\Theta\Theta_{\mathbb{k}}\ell^{2}\Xi_{\ell}^{2}[\aleph\|\zeta_{1}\|^{2} +2\Theta[\tilde{\Lambda}^{2}\|\nu\|_{\mathbb{k}}^{2}+(\chi_{1}\|\nu\|_{\mathbb{k}}^{2}+\chi_{2})]+\chi_{2}].$$

Theorem 2 *Let $\nu\in L_{2}^{0}(\mho,\Lambda)$, the assumptions (H1) - (H8), (B1) and (B2) are holds and*

$$3\Xi_{\ell}^{2}(1+2\Theta\Theta_{\mathbb{k}}\ell^{2})\sup_{\hbar\in[0,\ell]}\Gamma_{\ell}(\hbar)<1,\ \ \ell\in N, \quad (49)$$

then the problem (45) - (46) has a unique mild solution on I.

Proof Let us fix $\tau>1$ and we define in $\mathbb{k}_{+\infty}$ the semi-norms

$$\|\zeta\|_\ell = \sup\left\{e^{\tau \mathfrak{L}^*_\ell}(\hbar)\aleph\|\zeta(\hbar)\|^2 : \hbar \in [0,\ell]\right\}, \quad \ell \in N, \quad (50)$$

where $\mathfrak{L}^*_\ell = \int_0^\hbar \overline{\mathfrak{L}}_\ell(\rho)d\rho$,

$$\overline{\mathfrak{L}}_\ell(\hbar) = 3Tr(\varphi)\Theta\Xi^2_\ell(1+T\tilde{\Theta}_a(r))(1+2\Theta\Theta_{\mathbb{k}}\ell^2)\mathfrak{L}_\ell(\rho) \quad (51)$$

where $\mathfrak{L}_\ell$ is the function from (H6). Then $\mathbb{k}_{+\infty}$ is a Frechet space with the family of semi-norms $\|\cdot\|_{\ell\in N}$.

Define the following control by using (B1), for each $\ell \in N$

$$u^\ell_\zeta(\hbar) = W^{-1}\Bigg[\zeta_1 - S_\rho(\ell)[\nu(0)-G(0,\nu)] - G(\ell,\overline{\zeta}_{\sigma(\ell,\overline{\zeta}_\ell)}) - \int_0^\ell S_\rho(\ell-\rho) \times F\left(\rho,\overline{\zeta}_{\sigma(\rho,\overline{\zeta}_\rho)},\int_0^\rho a(\rho,\tau,\overline{\zeta}_{\varsigma(\tau,\overline{\zeta}_\tau)})d\tau\right)dW(\rho)\Bigg](\hbar). \quad (52)$$

Consider the space $Y = \{\zeta \in \mathbb{k}_{+\infty} : \zeta(0) = \nu(0)\}$ endowed with the uniform convergence topology ($\|\cdot\|_\infty$) and define $\Phi : Y \to Y$ by

$$(\Phi_y)(\hbar) = \begin{cases} 0, & \hbar \in (-\infty,0], \\ S_\rho(\hbar)[\nu(0)-G(0,\nu)] + G(\hbar,\overline{\zeta}_{\sigma(\hbar,\overline{\zeta}_\hbar)}) + \int_0^\hbar S_\rho(\hbar-\rho)(\mathbb{k}u^\ell_\zeta)(\rho)d\rho + \int_0^\hbar S_\rho(\hbar-\rho) & \\ \qquad \times F\left(\rho,\overline{\zeta}_{\sigma(\rho,\overline{\zeta}_\rho)},\int_0^\rho a(\rho,\tau,\overline{\zeta}_{\varsigma(\tau,\overline{\zeta}_\tau)})d\tau\right)dW(\rho), & \hbar \in I, \end{cases} \quad (53)$$

where $\overline{\zeta} : (-\infty,0] \to \Lambda$ such that $\overline{\zeta}_0 = \nu$ and $\overline{\zeta} = \zeta$ on $[0,\ell]$. Let $\overline{\nu} : (-\infty,0) \to \Lambda$ be the extension of $(-\infty,0]$ such that $\overline{\nu}(\vartheta) = \nu(0) = 0$ on $[0,\ell]$ and $J^\nu_0 = \sup\{J^\nu_0 : \rho \in R(\varsigma^-)\}$. We show that Φ has a fixed point, which in turn is a mild solution of the problem (45) - (46).
Let ζ be a possible solution of problem (45) - (46). Given $\ell \in N$ and $\hbar \in [0,\ell]$, then

$$\zeta(\hbar) = S_\rho(\hbar)[\nu(0)-G(0,\nu)] + G(\hbar,\overline{\zeta}_{\sigma(\hbar,\overline{\zeta}_\hbar)}) + \int_0^\hbar S_\rho(\hbar-\rho)(\mathbb{k}u^\ell_\zeta)(\rho)d\rho + \int_0^\hbar S_\rho(\hbar-\rho) \times F\left(\rho,\overline{\zeta}_{\sigma(\rho,\overline{\zeta}_\rho)},\int_0^\rho a(\rho,\tau,\overline{\zeta}_{\varsigma(\tau,\overline{\zeta}_\tau)})d\tau\right)dW(\rho). \quad (54)$$

By (H3), (H5) and (H7) that, we obtain

$$\aleph\|\zeta(\hbar)\|^2 \le 4\aleph\|S_\rho(\hbar)[\nu(0)-G(0,\nu)]\|^2 + 4\aleph\|G(\hbar,\overline{\zeta}_{\sigma(\hbar,\overline{\zeta}_\hbar)})\|^2 + 4\aleph\left\|\int_0^\hbar S_\rho(\hbar-\rho)(\mathbb{k}u^\ell_\zeta)(\rho)d\rho\right\|^2 + 4\aleph\left\|\int_0^\hbar S_\rho(\hbar-\rho) \times F\left(\rho,\overline{\zeta}_{\sigma(\rho,\overline{\zeta}_\rho)},\int_0^\rho a(\rho,\tau,\overline{\zeta}_{\varsigma(\tau,\overline{\zeta}_\tau)})d\tau\right)dW(\rho)\right\|^2 \le \sum_{i=6}^9 J_i, \quad \hbar \in [0,\ell]. \quad (55)$$

$$J_6 = 4\aleph\|S_\rho(\hbar)[\nu(0)-G(0,\nu)]\|^2 \le 8\Theta[\tilde{\Lambda}^2\|\nu\|^2_{\mathbb{k}} + (\chi_1\|\nu\|^2_{\mathbb{k}} + \chi_2)] \quad (56)$$

$$J_7 = 4\aleph\|G(\hbar,\overline{\zeta}_{\sigma(\hbar,\overline{\zeta}_\hbar)})\|^2 \le 4(\chi_1\|\overline{\zeta}_{\sigma(\hbar,\overline{\zeta}_\hbar)}\|^2_{\mathbb{k}} + \chi_2) \quad (57)$$

$$\begin{aligned} J_8 &\le 4\aleph\left\|\int_0^\hbar S_\rho(\hbar-\rho)(\mathbb{k}u^\ell_\zeta)(\rho)d\rho\right\|^2 \\ &\le 4\aleph\Bigg\|\int_0^\hbar S_\rho(\hbar-\rho)BW^{-1}\Bigg[\zeta_1 - S_\rho(\ell)[\nu(0) - G(0,\nu)] - G(\ell,\overline{\zeta}_{\sigma(\ell,\overline{\zeta}_\ell)}) - \int_0^\ell S_\rho(\ell-\rho) \\ &\qquad \times F\left(\rho,\overline{\zeta}_{\sigma(\rho,\overline{\zeta}_\rho)},\int_0^\rho a(\rho,\tau,\overline{\zeta}_{\varsigma(\tau,\overline{\zeta}_\tau)})d\tau\right) \times dW(\rho)\Bigg](\hbar)\Bigg\|^2 \\ &\le 16\Theta\Theta_{\mathbb{k}}\ell\int_0^\hbar\Bigg\{\aleph\|\zeta_1\|^2 + 2\Theta[\tilde{\Lambda}\|\nu\|^2_{\mathbb{k}} + (\chi_1\|\nu\|^2_{\mathbb{k}} + \chi_2)] + (\chi_1\|\overline{\zeta}_{\sigma(\ell,\overline{\zeta}_\ell)}\|^2_{\mathbb{k}} + \chi_2) + Tr(\varphi)\Theta\int_0^\ell p(\tau) \\ &\qquad \times \Psi\left(\|\overline{\zeta}_{\sigma(\rho,\overline{\zeta}_\rho)}\|^2_{\mathbb{k}} + T\Theta_a(1+\|\overline{\zeta}_{\sigma(\tau,\overline{\zeta}_\tau)}\|^2_{\mathbb{k}})\right)d\tau\Bigg\}d\rho. \end{aligned} \quad (58)$$

$$J_9 = 4\aleph\Big\|\int_0^{\hbar} S_\rho(\hbar-\rho) \times F\left(\rho,\overline{\zeta}_{\sigma(\rho,\overline{\zeta}_\rho)},\int_0^{\rho} a(\rho,\tau,\overline{\zeta}_{\varsigma(\tau,\overline{\zeta}_\tau)})d\tau\right)dW(\rho)\Big\|^2 v \leq 4Tr(\varphi)\Theta\int_0^{\hbar}\aleph\Big\| \times F\left(\rho,\overline{\zeta}_{\sigma(\rho,\overline{\zeta}_\rho)},\int_0^{\rho} a(\rho,\tau,\overline{\zeta}_{\varsigma(\tau,\overline{\zeta}_\tau)})d\tau\right)\Big\|^2 d\rho \leq 4Tr(\varphi)\Theta\int_0^{\hbar} p(\rho) \times\left(\|\overline{\zeta}_{\sigma(\rho,\overline{\zeta}_\rho)}\|_{\mathbb{k}}^2 + T\Theta_a(1+\|\overline{\zeta}_{\sigma(\tau,\overline{\zeta}_\tau)}\|_{\mathbb{k}}^2)\right)d\rho. \quad (59)$$

Substituting (J_6) - (J_9) together with (55), we get

$$\aleph\|\zeta(\hbar)\|^2 \leq 8\Theta[\tilde{\Lambda}^2\|v\|_{\mathbb{k}}^2 + (\chi_1\|v\|_{\mathbb{k}}^2+\chi_2)] + 4(\chi_1\|\overline{\zeta}_{\sigma(\hbar,\overline{\zeta}_\hbar)}\|_{\mathbb{k}}^2+\chi_2) + 16\Theta\Theta_{\mathbb{k}}\ell\int_0^{\hbar}\Big\{\aleph\|\zeta_1\|^2 + 2\Theta[\tilde{\Lambda}\|v\|_{\mathbb{k}}^2 + (\chi_1\|v\|_{\mathbb{k}}^2+\chi_2)] + (\chi_1\|\overline{\zeta}_{\sigma(\ell,\overline{\zeta}_\ell)}\|_{\mathbb{k}}^2+\chi_2) + Tr(\varphi)\Theta\int_0^{\ell} p(\tau)\Psi\Big(\|\overline{\zeta}_{\sigma(\rho,\overline{\zeta}_\rho)}\|_{\mathbb{k}}^2 + T\Theta_a(1+\|\overline{\zeta}_{\sigma(\tau,\overline{\zeta}_\tau)}\|_{\mathbb{k}}^2)\Big)d\tau\Big\}d\rho + 4Tr(\varphi)\Theta\int_0^{\hbar} p(\rho) \times\Psi\left(\|\overline{\zeta}_{\sigma(\rho,\overline{\zeta}_\rho)}\|_{\mathbb{k}}^2 + T\Theta_a(1+\|\overline{\zeta}_{\sigma(\tau,\overline{\zeta}_\tau)}\|_{\mathbb{k}}^2)\right)d\rho$$
$$\leq 8\Theta[\tilde{\Lambda}^2\|v\|_{\mathbb{k}}^2 + (\chi_1\|v\|_{\mathbb{k}}^2+\chi_2)] + 4\chi_2 + 16\Theta\Theta_{\mathbb{k}}\ell^2\Big[\aleph\|\zeta 1\|^2 + 2\Theta[\tilde{\Lambda}\|v\|_{\mathbb{k}}^2 + (\chi_1\|v\|_{\mathbb{k}}^2+\chi_2)] + \chi_2\Big] + 4\chi_1\|\overline{\zeta}_{\sigma(\hbar,\overline{\zeta}_\hbar)}\|_{\mathbb{k}}^2 + 16\Theta\Theta_{\mathbb{k}}\ell^2\chi_1\|\overline{\zeta}_{\sigma(\ell,\overline{\zeta}_\ell)}\|_{\mathbb{k}}^2 + 16\Theta\Theta_{\mathbb{k}}\ell^2 Tr(\varphi)\Theta\int_0^{\ell} p(\tau)\Psi\Big(\|\overline{\zeta}_{\sigma(\rho,\overline{\zeta}_\rho)}\|_{\mathbb{k}}^2 + T\Theta_a(1+\|\overline{\zeta}_{\sigma(\tau,\overline{\zeta}_\tau)}\|_{\mathbb{k}}^2)\Big)d\tau + 4Tr(\varphi)\Theta\int_0^{\hbar} p(\rho) \times\Psi\left(\|\overline{\zeta}_{\sigma(\rho,\overline{\zeta}_\rho)}\|_{\mathbb{k}}^2 + T\Theta_a(1+\|\overline{\zeta}_{\sigma(\tau,\overline{\zeta}_\tau)}\|_{\mathbb{k}}^2)\right)d\rho. \quad (60)$$

It follows that

$$\aleph\|\zeta(\hbar)\|^2 \leq 8\Theta[\tilde{\Lambda}^2\|v\|_{\mathbb{k}}^2 + (\chi_1\|v\|_{\mathbb{k}}^2+\chi_2)] + 4\chi_2 + 16\Theta\Theta_{\mathbb{k}}\ell^2\left[\aleph\|\zeta 1\|^2 + 2\Theta[\tilde{\Lambda}\|v\|_{\mathbb{k}}^2 + (\chi_1\|v\|_{\mathbb{k}}^2+\chi_2)] + \chi_2\right] + 4\chi_1\left(2[(\Theta_\ell + J_0^v)\|v\|_{\mathbb{k}}] + 2\Xi_\ell^2\sup_{\rho\in[0,\ell]}\aleph\|\zeta(\rho)\|^2\right) + 16\Theta\Theta_{\mathbb{k}}\ell^2\chi_1\left(2[(\Theta_\ell + J_0^v)\|v\|_{\mathbb{k}}]^2 + 2\Xi_\ell^2\sup_{\rho\in[0,\ell]}\aleph\|\zeta(\rho)\|^2\right) + 16\Theta\Theta_{\mathbb{k}}\ell^2 Tr(\varphi)\Theta \times\int_0^{\ell} p(\tau)\Psi\Bigg[\left(2[(\Theta_\ell + J_0^v)\|v\|_{\mathbb{k}}]^2 + 2\Xi_\ell^2\sup_{\rho\in[0,\ell]}\aleph\|\zeta(\rho)\|^2\right) + T\Theta_a \times\left(1+\left(2[(\Theta_\ell + J_0^v)\|v\|_{\mathbb{k}}]^2 + 2\Xi_\ell^2\sup_{\rho\in[0,\ell]}\aleph\|\zeta(\rho)\|^2\right)\right)\Bigg]d\tau + 4Tr(\varphi)\Theta\int_0^{\hbar} p(\rho) \times\Psi\Bigg[\left(2[(\Theta_\ell + J_0^v)\|v\|_{\mathbb{k}}]^2 + 2\Xi_\ell^2\sup_{\rho\in[0,\ell]}\aleph\|\zeta(\rho)\|^2\right) + T\Theta_a\left(1+\left(2[(\Theta_\ell + J_0^v)\|v\|_{\mathbb{k}}]^2 + 2\Xi_\ell^2\sup_{\rho\in[0,\ell]}\aleph\|\zeta(\rho)\|^2\right)\right)\Bigg]d\rho. \quad (61)$$

By Theorem 1, we defined $\|\mu\|_\infty = \sup_{0\leq\hbar\leq n}\mu(\hbar)$, and the previous inequality, we obtain

$$\|\mu\|_\infty \leq 16\Theta\Xi_\ell^2\left[\tilde{\Lambda}^2\|v\|_{\mathbb{k}}^2 + (\chi_1\|v\|_{\mathbb{k}}^2+\chi_2)\right] + 2[(\Theta_\ell + J_0^v)\|v\|_{\mathbb{k}}]^2 + 8\Xi_\ell^2\chi_2 + 32\Theta\Theta_{\mathbb{k}}\ell^2\Xi_\ell^2\Big[\aleph\|\zeta_1\|^2 + 2\Theta[\tilde{\Lambda}^2\|v\|_{\mathbb{k}}^2 + (\chi_1\|v\|_{\mathbb{k}}^2+\chi_2)] + \chi_2\Big] + 8\chi_1\Xi_\ell^2\|\mu\|_\infty + 32\Theta\Theta_{\mathbb{k}}\ell^2\chi_1\Xi_\ell^2\|\mu\|_\infty + 32\Theta\Theta_{\mathbb{k}}\ell^2 Tr(\varphi)\Theta\Xi_\ell^2 \times\int_0^{\ell} p(\rho)\Psi\Big\{(1+T\Theta_a)\|\mu\|_\infty + T\Theta_a\Big\}d\rho + 8Tr(\varphi)\Theta\Xi_\ell^2 \times\int_0^{\hbar} p(\rho)\Psi\Big\{(1+T\Theta_a)\|\mu\|_\infty + T\Theta_a\Big\}d\rho, \quad (62)$$

i.e.,

$$\|\mu\|_\infty \le \Upsilon_2 + 8\chi_1 \Xi_\ell^2 (1+4\Theta\Theta_{\Bbbk}\ell^2)\|\mu\|_\infty + 8\Theta Tr(\varphi)\Xi_\ell^2[(1+T\Theta_a)(1+4\Theta\Theta_{\Bbbk}\ell^2)]\Psi(\|\mu\|_\infty) \times \int_0^\ell p(\rho)d\rho + 8Tr(\varphi)\Theta\Xi_\ell^2\int_0^\ell p(\rho)\Psi(T\Theta_a)d\rho. \tag{63}$$

Consequently,

$$\frac{(1-8\chi_1\Xi_\ell^2(1+4\Theta\Theta_{\Bbbk}\ell^2))\|\mu\|_\infty}{\Upsilon_2 + 8Tr(\varphi)\Theta\Xi_\ell^2(1+4\Theta\Theta_{\Bbbk}\ell^2)\Psi((1+T\Theta_a)\|\mu\|_\infty)\int_0^\ell p(\rho)d\rho} \le 1. \tag{64}$$

Then by (B2), $\|\mu\|_\infty \le \beta_\ell$, if β_ℓ^* exists. Since

$$displaystyle\|\zeta\|_{\Bbbk_{+\infty}} \le \|\mu\|_\infty \;\Rightarrow\; \|\zeta\|_\ell \le \beta_\ell^*. \tag{65}$$

Set $\Upsilon = \Big\{\zeta \in \Bbbk_{+\infty}:\ \sup\{\aleph\|\zeta(\hbar)\|^2:\ 0\le\hbar\le n\} \le \beta_\ell + 1, \text{ for all } \ell\in N\Big\}$. Clearly, $\Upsilon \subset \Bbbk_{+\infty}$ is closed.
We shall show that $\Phi:\Upsilon\to\Bbbk_{+\infty}$ is a contraction operator. Consider $\zeta^*,\zeta^{**}\in\Bbbk_{+\infty}$. By using Lemma 1, Lemma 3, and (H4), (H6) and (H7), for each $\hbar\in[0,\ell]$ and $\ell\in N$, we have

$$\begin{aligned}
&\aleph\|\Phi\overline{\zeta}^*(\hbar) - \Phi\overline{\zeta}^{**}(\hbar)\|^2 \\
&\le 3\aleph\|G(\hbar,\overline{\zeta}^*_{\sigma(\hbar,\overline{\zeta}^*_\hbar)}) - G(\hbar,\overline{\zeta}^{**}_{\sigma(\hbar,\overline{\zeta}^{**}_\hbar)})\|^2 \\
&\quad + 3\aleph\Big\|\int_0^\hbar S_\rho(\hbar-\rho) \\
&\quad \times BW^{-1}\Bigg[\zeta_1 - S_\rho(\ell)[\nu(0)-G(0,\nu)] - G(\ell,\overline{\zeta}^*_{\varsigma(\ell,\overline{\zeta}^*_\ell)}) \\
&\quad - \int_0^\ell S_\rho(\ell-\eta) \\
&\quad \times F\left(\eta,\overline{\zeta}^*_{\varsigma(\eta,\overline{\zeta}^*_\eta)},\int_0^\eta a(\eta,\tau,\overline{\zeta}^*_{\varsigma(\tau,\overline{\zeta}^*_\tau)})d\tau\right)dW(\eta)\Bigg] \\
&\quad - \Bigg[\zeta_1 - S_\rho(\ell)[\nu(0)-G(0,\nu)] - G(\ell,\overline{\zeta}^{**}_{\varsigma(\ell,\overline{\zeta}^{**}_\ell)}) \\
&\quad - \int_0^\ell S_\rho(\ell-\eta) \\
&\quad \times F\left(\eta,\overline{\zeta}^{**}_{\varsigma(\eta,\overline{\zeta}^{**}_\eta)},\int_0^\eta a(\eta,\tau,\overline{\zeta}^{**}_{\varsigma(\tau,\overline{\zeta}^{**}_\tau)})d\tau\right)dW(\eta)\Bigg]d\rho\Big\|^2 \\
&\quad + 3\aleph\Big\|\int_0^\hbar S_\rho(\hbar-\rho) \\
&\quad \times\Bigg[F\left(\rho,\overline{\zeta}^*_{\sigma(\rho,\overline{\zeta}^*_\rho)},\int_0^\rho a(\rho,\tau,\overline{\zeta}^*_{\varsigma(\tau,\overline{\zeta}^*_\tau)})d\tau\right) \\
&\quad - F\left(\rho,\overline{\zeta}^{**}_{\sigma(\rho,\overline{\zeta}^{**}_\rho)},\int_0^\rho a(\rho,\tau,\overline{\zeta}^{**}_{\varsigma(\tau,\overline{\zeta}^{**}_\tau)})d\tau\right)\Bigg]dW(\rho)\Big\|^2 \\
&= \sum_{i=10}^{12} J_i.
\end{aligned} \tag{66}$$

$$\begin{aligned}
J_{10} &\le 3\aleph\|G(\hbar,\overline{\zeta}^*_{\sigma(\hbar,\overline{\zeta}^*_\hbar)}) - G(\hbar,\overline{\zeta}^{**}_{\sigma(\hbar,\overline{\zeta}^{**}_\hbar)})\|^2 \\
&\le 3\aleph\Gamma_\ell(\hbar)\|\overline{\zeta}^*_{\sigma(\hbar,\overline{\zeta}^*_\hbar)} - \overline{\zeta}^{**}_{\sigma(\hbar,\overline{\zeta}^{**}_\hbar)}\|^2
\end{aligned} \tag{67}$$

$$\begin{aligned}
J_{11} &\le 3\aleph\Big\|\int_0^\hbar S_\rho(\hbar-\rho) \\
&\quad \times BW^{-1}\Bigg[\zeta_1 - S_\rho(\ell)[\nu(0)-G(0,\nu)] - G(\ell,\overline{\zeta}^*_{\varsigma(\ell,\overline{\zeta}^*_\ell)}) \\
&\quad - \int_0^\ell S_\rho(\ell-\eta) \\
&\quad \times F\left(\eta,\overline{\zeta}^*_{\varsigma(\eta,\overline{\zeta}^*_\eta)},\int_0^\eta a(\eta,\tau,\overline{\zeta}^*_{\varsigma(\tau,\overline{\zeta}^*_\tau)})d\tau\right)dW(\eta)\Bigg] \\
&\quad - \Bigg[\zeta_1 - S_\rho(\ell)[\nu(0)-G(0,\nu)] - G(\ell,\overline{\zeta}^{**}_{\varsigma(\ell,\overline{\zeta}^{**}_\ell)}) \\
&\quad - \int_0^\ell S_\rho(\ell-\eta) \\
&\quad \times F\left(\eta,\overline{\zeta}^{**}_{\varsigma(\eta,\overline{\zeta}^{**}_\eta)},\int_0^\eta a(\eta,\tau,\overline{\zeta}^{**}_{\varsigma(\tau,\overline{\zeta}^{**}_\tau)})d\tau\right)dW(\eta)\Bigg]d\rho\Big\|^2 \\
&\le 6\Theta\Theta_{\Bbbk}\ell\int_0^\hbar\Big[\aleph\|G(\ell,\overline{\zeta}^*_{\varsigma(\ell,\overline{\zeta}^*_\ell)}) - G(\ell,\overline{\zeta}^{**}_{\varsigma(\ell,\overline{\zeta}^{**}_\ell)})\|^2 \\
&\quad + Tr(\varphi)\Theta_\ell\int_0^\ell\aleph \\
&\quad \times\Big\|F\left(\eta,\overline{\zeta}^*_{\varsigma(\eta,\overline{\zeta}^*_\eta)},\int_0^\eta a(\eta,\tau,\overline{\zeta}^*_{\varsigma(\tau,\overline{\zeta}^*_\tau)})d\tau\right) \\
&\quad - F\left(\eta,\overline{\zeta}^{**}_{\varsigma(\eta,\overline{\zeta}^{**}_\eta)},\int_0^\eta a(\eta,\tau,\overline{\zeta}^{**}_{\varsigma(\tau,\overline{\zeta}^{**}_\tau)})d\tau\right)\Big\|d\eta d\rho \\
&\le 6\Theta\Theta_{\Bbbk}\ell^2\Gamma_\ell(\hbar)\|\overline{\zeta}^*_{\varsigma(\ell,\overline{\zeta}^*_\ell)} - \overline{\zeta}^{**}_{\varsigma(\ell,\overline{\zeta}^{**}_\ell)}\|^2 \\
&\quad + 6\Theta^2\Theta_{\Bbbk}\ell^2 Tr(\varphi) \\
&\quad \times\int_0^\hbar \mathfrak{L}_\ell(\rho)\left((1+T\tilde{\Theta}_a(r))\|\overline{\zeta}^*_{\varsigma(\eta,\overline{\zeta}^*_\eta)} - \overline{\zeta}^{**}_{\varsigma(\eta,\overline{\zeta}^{**}_\eta)}\|^2_{\Bbbk}\right)d\rho
\end{aligned} \tag{68}$$

$$J_{12} \leq 3\aleph \left\| \int_0^{\hbar} S_\rho(\hbar-\rho)\left[F\left(\rho, \overline{\zeta}^*_{\sigma(\rho,\overline{\zeta}^*_\rho)}, \int_0^\rho a(\rho,\tau,\overline{\zeta}^*_{\varsigma(\tau,\overline{\zeta}^*_\tau)})d\tau\right) F\left(\rho, \overline{\zeta}^{**}_{\sigma(\rho,\overline{\zeta}^{**}_\rho)}, \int_0^\rho a(\rho,\tau,\overline{\zeta}^{**}_{\varsigma(\tau,\overline{\zeta}^{**}_\tau)})d\tau\right)\right] dW(\rho)\right\|^2$$
$$\leq 3\Theta Tr(\varphi)\int_0^{\hbar} \aleph \left\| F\left(\rho, \overline{\zeta}^*_{\sigma(\rho,\overline{\zeta}^*_\rho)}, \int_0^\rho a(\rho,\tau,\overline{\zeta}^*_{\varsigma(\tau,\overline{\zeta}^*_\tau)})d\tau\right) - F\left(\rho, \overline{\zeta}^{**}_{\sigma(\rho,\overline{\zeta}^{**}_\rho)}, \int_0^\rho a(\rho,\tau,\overline{\zeta}^{**}_{\varsigma(\tau,\overline{\zeta}^{**}_\tau)})d\tau\right)\right\|^2 d\rho$$
$$\leq 3\Theta Tr(\varphi)\int_0^{\hbar} \mathfrak{L}_\ell(\rho)\left((1+T\tilde{\Theta}_a(r))\|\overline{\zeta}^*_{\sigma(\rho,\overline{\zeta}^*_\rho)} - \overline{\zeta}^{**}_{\sigma(\rho,\overline{\zeta}^{**}_\rho)}\|^2_{\mathbb{k}}\right)d\rho. \tag{69}$$

Substituting (J_{10}) - (J_{12}) together with (66), we arrive

$$\aleph\|\Phi\overline{\zeta}^*(\hbar) - \Phi\overline{\zeta}^{**}(\hbar)\|^2$$
$$\leq 3\aleph\Gamma_\ell(\hbar)\|\overline{\zeta}^*_{\sigma(\hbar,\overline{\zeta}^*_\hbar)} - \overline{\zeta}^{**}_{\sigma(\hbar,\overline{\zeta}^{**}_\hbar)}\|^2 + 6\Theta\Theta_{\mathbb{k}}\ell^2\Gamma_\ell(\hbar)\|\overline{\zeta}^*_{\varsigma(\ell,\overline{\zeta}^*_\ell)} - \overline{\zeta}^{**}_{\varsigma(\ell,\overline{\zeta}^{**}_\ell)}\|^2 + 6\Theta^2\Theta_{\mathbb{k}}\ell^2 Tr(\varphi)\int_0^{\hbar}\mathfrak{L}_\ell(\rho)$$
$$\times\left((1+T\tilde{\Theta}_a(r))\|\overline{\zeta}^*_{\varsigma(\eta,\overline{\zeta}^*_\eta)} - \overline{\zeta}^{**}_{\varsigma(\eta,\overline{\zeta}^{**}_\eta)}\|^2_{\mathbb{k}}\right)d\rho + 3\Theta Tr(\varphi)\int_0^{\hbar}\mathfrak{L}_\ell(\rho)\left((1+T\tilde{\Theta}_a(r))\|\overline{\zeta}^*_{\sigma(\rho,\overline{\zeta}^*_\rho)} - \overline{\zeta}^{**}_{\sigma(\rho,\overline{\zeta}^{**}_\rho)}\|^2_{\mathbb{k}}\right)d\rho$$
$$\leq 3\Gamma_\ell(\hbar)\Xi^2_\ell \sup_{\hbar\in[0,\ell]}\aleph\|\overline{\zeta}^*(\hbar) - \overline{\zeta}^{**}(\hbar)\|^2 + 6\Theta\Theta_{\mathbb{k}}\ell^2\Gamma_\ell(\hbar)\Xi^2_\ell\sup_{\hbar\in[0,\ell]}\aleph\|\overline{\zeta}^*(\hbar) - \overline{\zeta}^{**}(\hbar)\|^2$$
$$+6\Theta^2\Theta_{\mathbb{k}}\ell^2 Tr(\varphi)\Xi^2_\ell\int_0^{\hbar}\mathfrak{L}_\ell(\rho)(1+T\tilde{\Theta}_a(r))\aleph\|\overline{\zeta}^*(\rho) - \overline{\zeta}^{**}(\rho)\|^2 d\rho + 3Tr(\varphi)\Theta\Xi^2_\ell$$
$$\times\int_0^{\hbar}\mathfrak{L}_\ell(\rho)(1+T\tilde{\Theta}_a(r))\aleph\|\overline{\zeta}^*(\rho) - \overline{\zeta}^{**}(\rho)\|^2 d\rho$$
$$\leq 3\Gamma_\ell(\hbar)\Xi^2_\ell(1+2\Theta\Theta_{\mathbb{k}}\ell^2)[e^{\tau\overline{\mathfrak{L}}_\ell}(\hbar)]$$
$$\times\left[e^{-\tau\overline{\mathfrak{L}}_\ell}(\hbar)\sup_{\rho\in[0,\ell]}\aleph\|\overline{\zeta}^*(\hbar) - \overline{\zeta}^{**}(\hbar)\|^2 *\right]\int_0^{\hbar}\left[\overline{\mathfrak{L}}_\ell(\rho)e^{\tau\overline{\mathfrak{L}}_\ell}(\rho)\right]\left[e^{-\tau\overline{\mathfrak{L}}_\ell}(\rho)\aleph\|\overline{\zeta}^*(\rho) - \overline{\zeta}^{**}(\rho)\|^2\right]d\rho$$
$$\leq 3\Gamma_\ell(\hbar)\Xi^2_\ell(1+2\Theta\Theta_{\mathbb{k}}\ell^2)[e^{\tau\overline{\mathfrak{L}}_\ell}(\rho)]\|\overline{\zeta}^* - \overline{\zeta}^{**}\|_\ell + \int_0^{\hbar}\frac{1}{\tau}[e^{\tau\overline{\mathfrak{L}}_\ell}(\rho)]' d\rho\|\overline{\zeta}^* - \overline{\zeta}^{**}\|_\ell$$
$$\leq e^{\tau\overline{\mathfrak{L}}_\ell}(\hbar)\left[3\sup_{\hbar\in[0,\ell]}\Gamma_\ell(\hbar)\Xi^2_\ell(1+2\Theta\Theta_{\mathbb{k}}\ell^2) + \frac{1}{\tau}\right]\|\overline{\zeta}^* - \overline{\zeta}^{**}\|_\ell. \tag{70}$$

By using $\overline{\zeta} = \zeta$ on $[0,\ell]$ and taking supremum over $\hbar$, implies

$$\|\Phi\overline{\zeta}^* - \Phi\overline{\zeta}^{**}\|_\ell \leq \left[3\sup_{\hbar\in[0,\ell]}\Gamma_\ell(\hbar)\Xi^2_\ell(1+2\Theta\Theta_{\mathbb{k}}\ell^2) + \frac{1}{\tau}\right]\|\overline{\zeta}^* - \overline{\zeta}^{**}\|_\ell, \tag{71}$$

showing that the operator Φ is a contraction.

From the choice of Υ there is no $\zeta \in \partial\Upsilon^\ell$ such that $\zeta = \Phi(\zeta)$ for some $\xi \in (0,1)$.

The operator Φ has a unique fixed point ζ, which is the unique mild solution of the problem (45) - (46), as a result of Frigon and Granas' nonlinear alternative. The proof is complete.

Conclusion

In this work, we have effectively investigated the controllability in Frechet spaces of fractional stochastic neutral integro-differential equations with state-dependent delay. We have proven the existence of unique mild solutions under certain conditions by utilizing sophisticated mathematical approaches including fractional calculus, fixed point theory, and the characteristics of characteristic solution operators. Our results integrate the fractional aspect and state dependent delays, which are essential for many real-world applications in science and engineering, and so expand on the literature that already exists. To further expand the application of these results, future research should try to investigate more generalized systems and loosen up some of the assumptions utilized in this study.

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