Transactions in Theoretical and Mathematical Physics

Publisher: Qom University of Technology

ISSN: 3041-8984

Volume**:** 3, Issue 3, 2026

Available online: https://ttmp.qut.ac.ir

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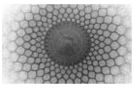
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TTMP vol 3 (3), 2026
doi: 10.30511/TTMP.2026.2091721.1081

Transactions in Theoretical and Mathematical Physics



Geometric Leakage versus Dissipation: Open Boundaries in Chern–Simons Fluid Dynamics

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Received: 15 May 2026 / Accepted: 23 June 2026 / Published: 23 June 2026

Abstract When a topological quantity satisfies a local conservation law, its apparent non-conservation inside a finite region need not signal dissipation: it may instead reflect a geometric leakage of information through the boundary. We make this precise in the Chern–Simons formulation of compressible fluid dynamics recently proposed by Bustamante, Andrianopoli, Trigiante and Zanelli, where the proto-helicity $H(t)$ – a helicity-type invariant of topological origin – satisfies a local divergence-form law $\partial_{\hat{\mu}} h^{\hat{\mu}} = 0$. On an open domain $\Omega(t)$, this local law gives rise not to conservation but to the balance $dH/dt = F_{\partial\Omega(t)}$, where $F_{\partial\Omega(t)}$ is the flux of a geometric current through the boundary. The bulk equations are unchanged; what changes is entirely the boundary. A graviton–fluid correspondence emerges from the comparison with the AdS_3 Chern–Simons setting: the radiative term in the boundary flux plays the role of boundary gravitons carrying topological information toward the conformal boundary, and the spectral character of $F_{\partial\Omega(t)}$ determines three distinct decay regimes for $H(t)$ – stable ($O(1)$, bound state), transiently localized ($O(t^{-1/3})$, caviton-like), and dispersive ($O(t^{-1/2})$, pure radiation) – providing a spectroscopic diagnostic of the interior geometry from boundary measurements alone.

Keywords: Chern-Simons, fluid dynamics, proto-helicity, open boundary conditions, geometric leakage.

1 Introduction

In open systems, the apparent loss of a conserved topological quantity is often mistakenly attributed to bulk dissipation. We show that in the Chern–Simons formulation of compressible fluid dynamics, such loss is instead a geometric leakage through the boundary. The proto-helicity $H(t) = \int_{\Omega(t)} A_a B_a \, d^3x$ satisfies a local conservation law $\partial_{\hat{\mu}} h^{\hat{\mu}} = 0$ identically, but its integrated value over an open domain $\Omega(t)$ evolves solely through a boundary flux:

$$\frac{dH}{dt} = F_{\partial\Omega(t)}, \tag{1}$$

where $F_{\partial\Omega(t)}$ is the flux of the geometric current $J_{\text{eff}} = (A_b B^b)V + A_0 B + A \times E$ through $\partial\Omega(t)$. No dissipative term appears in the bulk equations; the distinction between conservation and leakage is purely a matter of boundary conditions.

This balance law follows from the five-dimensional Abelian Chern–Simons action introduced by Bustamante, Andrianopoli, Trigiante and Zanelli [1], whose dimensional reduction yields the topological current $h^{\hat{\mu}}$. On a closed or periodic domain, the flux vanishes and $H(t)$ is exactly conserved. On an open domain, the three terms of J_{eff} – advection of helicity-carrying parcels, potential-driven transport, and radiative propagation of the gauge field – can each drain $H(t)$ from $\Omega(t)$ without any irreversible bulk process.

The radiative term $A \times E$ suggests a structural analogy with the AdS_3 Chern–Simons setting of [2], where boundary gravitons carry stress-tensor information to the conformal boundary through a flux of the linearised connection. Both systems realise the same abstract open-boundary balance

$$\frac{d\mathcal{Q}}{dt} = \mathscr{F}[\varphi|_{\partial}], \tag{2}$$

where $\mathcal{Q}$ is a topological charge and $\mathscr{F}$ depends only on the boundary values of the fundamental field. The correspondence is structural rather than dynamical: the bulk evolution equations differ (KdV hierarchy with eigenfunction forcing versus compressible fluid equations), but the radiative sector shares the same dispersive mechanisms. This motivates a spectral classification of the decay regimes of $H(t)$: a discrete bound state gives $O(1)$; a quasi-resonance near a criti-

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cal wavenumber yields $O(t^{-1/3})$; and generic incoherent radiation produces $O(t^{-1/2})$. These regimes are unified by the balance law (11) and differentiated entirely by the spectral character of $F_{\partial\Omega(t)}$.

We deliberately restrict attention to smooth configurations; singularities such as shocks are not considered here and would require a distributional formulation. The paper is organized as follows. Section 2 derives the open-boundary balance law (11) from the local Chern–Simons identity, discusses its gauge dependence and the gauge-invariant relative proto-helicity, and establishes the closed-flux conservation and open-domain leakage regimes. Section 3 develops the physical interpretation via the Poynting analogy, the three transport mechanisms, and the graviton–fluid correspondence, including a discussion of effective irreversibility. Section 4 illustrates the independence of bulk and boundary dynamics using the inviscid Burgers equation. Section 5 provides the spectral classification of decay regimes and outlines perspectives for a quantitative open-boundary geometric hydrodynamics.

2 Chern–Simons formulation and the open-boundary proto-helicity balance law

We review the geometric balance law arising in the Chern–Simons formulation of compressible fluid dynamics developed in [1], and derive its form on an open, moving domain. The starting point is the five-dimensional Abelian Chern–Simons action

$$I[A] = \int_{M_5} \frac{1}{3!} A \wedge F \wedge F, \tag{3}$$

where $F = dA$ is the curvature two-form associated with the gauge connection A. The Euler–Lagrange equations are

$$F \wedge F = 0. \tag{4}$$

Following the dimensional reduction of [1], we decompose the connection as $A = A_{\hat{\mu}} dx^{\hat{\mu}} + A_4 dx^4$ with $\hat{\mu} = 0,1,2,3$ and assume all fields are independent of x^4. Introducing the electric and magnetic fields

$$E_a = -F_{0a}, \qquad B_a = \frac{1}{2}\varepsilon_{abc} F_{bc}, \tag{5}$$

the reduced system gives rise to the topological current

$$h^{\hat{\mu}} = \frac{1}{2}\varepsilon^{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}} A_{\hat{\nu}} F_{\hat{\rho}\hat{\sigma}}, \tag{6}$$

whose temporal and spatial components are $h^0 = A_a B_a$ and $h^a = -A_0 B^a - (A \times E)^a$. Using (4) together with the Bianchi identity $dF = 0$, one finds $\partial_{\hat{\mu}} h^{\hat{\mu}} = 0$, which in $(3+1)$ form reads

$$\partial_0 (A_a B_a) + \nabla \cdot (-A_0 B - A \times E) = 0. \tag{7}$$

This is an exact local identity: it holds pointwise, regardless of the geometry of the domain on which the fluid evolves. What changes with the domain is its integrated consequence.

Let $\Omega(t) \subset \mathbb{R}^3$ be a smooth domain transported by a velocity field V – the boundary $\partial\Omega(t)$ may be at rest, comoving with the fluid, or moving independently – and define the proto-helicity functional

$$H(t) = \int_{\Omega(t)} A_a B_a \, d^3x. \tag{8}$$

On an open domain, $H(t)$ is in general gauge dependent: under $A \mapsto A + d\lambda$, $H(t)$ changes by a surface term $\oint_{\partial\Omega(t)} \lambda B \cdot n \, dS$. Throughout this work we either restrict to gauge transformations for which λ vanishes sufficiently fast on $\partial\Omega(t)$, or treat $H(t)$ as a quantity defined relative to a fixed boundary gauge condition. A gauge-invariant alternative, familiar from open-domain helicity in MHD [3, 4], is the *relative* proto-helicity

$$H_{\mathrm{rel}}(t) = \int_{\Omega(t)} \left(A_a B_a - A^{\mathrm{ref}}_a B^{\mathrm{ref}}_a \right) d^3x, \tag{9}$$

where A^{ref} is any fixed reference connection sharing the same boundary values $B^{\mathrm{ref}} \cdot n = B \cdot n$ on $\partial\Omega(t)$. The gauge ambiguities cancel in $H_{\mathrm{rel}}(t)$, which satisfies the same balance law (11) with J_{eff} replaced by $J_{\mathrm{eff}} - J^{\mathrm{ref}}_{\mathrm{eff}}$. All statements below apply equally to $H_{\mathrm{rel}}(t)$; we use $H(t)$ for notational simplicity.

Applying the Reynolds transport theorem to (8) and using (7), the bulk terms combine into a pure divergence,

$$\frac{dH}{dt} = \int_{\Omega(t)} \nabla \cdot \Big[(A_b B^b) V + A_0 B + A \times E \Big] \, d^3x, \tag{10}$$

so that the divergence theorem gives the central result of this work:

$$\begin{aligned} \frac{dH}{dt} &= F_{\partial\Omega(t)} \equiv \oint_{\partial\Omega(t)} J_{\mathrm{eff}} \cdot n \, dS, \\ J_{\mathrm{eff}} &= (A_b B^b) V + A_0 B + A \times E. \end{aligned} \tag{11}$$

The evolution of the proto-helicity inside $\Omega(t)$ is entirely controlled by the flux of the geometric current J_{eff} through $\partial\Omega(t)$. No term has been added to or removed from the bulk identity (7): the boundary term in (11) is exactly the divergence term in (7), integrated over $\Omega(t)$ and converted to a surface integral via the divergence theorem.

Two regimes follow immediately from (11). If

$$F_{\partial\Omega(t)} = 0 \tag{12}$$

– as on a periodic domain, on $\mathbb{R}^3$ for fields decaying fast enough, or whenever incoming and outgoing fluxes cancel exactly – then $dH/dt = 0$ and H is exactly conserved. Such configurations define geometrically closed transport sectors:

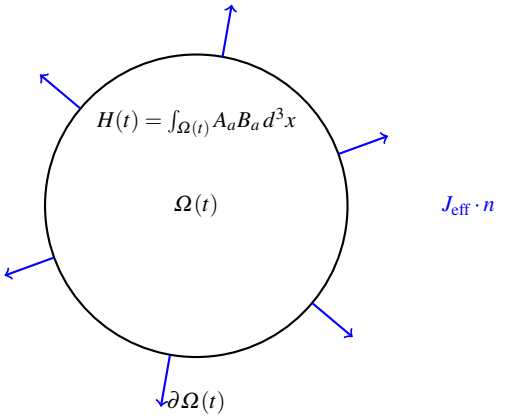


Fig. 1: An open region $\Omega(t)$ carrying proto-helicity $H(t)$. The outward flux of $J_{\text{eff}} = (A_b B^b)V + A_0 B + A \times E$ through $\partial\Omega(t)$ entirely controls dH/dt: a vanishing total flux gives exact conservation (12), while a one-signed outward flux gives the monotonic decay (15) – geometric leakage rather than bulk dissipation.

the current J_{eff} is confined to $\Omega(t)$ for all time. If instead $F_{\partial\Omega(t)} \neq 0$, integrating (11) between t_1 and t_2 gives

$$H(t_2) - H(t_1) = \int_{t_1}^{t_2} F_{\partial\Omega(t)} \, dt, \tag{13}$$

so that every unit of H lost or gained inside $\Omega(t)$ is accounted for by the time-integrated flux through the moving boundary. In particular, if

$$F_{\partial\Omega(t)} \leq 0 \quad \text{for all } t, \tag{14}$$

then $dH/dt \leq 0$ and

$$H(t_2) \leq H(t_1) \qquad \text{for all } t_2 \geq t_1, \tag{15}$$

so the proto-helicity decreases monotonically inside $\Omega(t)$. The distinction between the two regimes is thus encoded entirely in the boundary data, not in the bulk equations.

3 Physical interpretation

The balance law (11) is structurally analogous to the Poynting theorem,

$$\partial_t \left(\frac{E^2 + B^2}{2} \right) + \nabla \cdot (E \times B) = -J \cdot E, \tag{16}$$

whose integrated form shows that even with $J = 0$, the field energy inside Ω can decrease purely through radiation leaving the box. Equation (11) is the exact analogue with $H \leftrightarrow \int (E^2 + B^2)/2$ and $J_{\text{eff}} \leftrightarrow E \times B$: the proto-helicity plays the role of a radiated geometric quantity, and J_{eff} is its Poynting vector. Unlike energy or mass conservation, the proto-helicity originates directly from the Chern–Simons geometric structure of the theory; its flux therefore measures the transport of topological information encoded in the gauge connection and its associated vorticity field.

The three terms of J_{eff} correspond to three physically distinct mechanisms. The advective term $(A_b B^b)V$ carries helicity out of $\Omega(t)$ simply because helicity-carrying fluid parcels cross $\partial\Omega(t)$: this is what happens when a twisted vortex tube is advected through the wall of an observation volume, carrying its linkage with it. The potential-driven term $A_0 B$ is weighted by the temporal component A_0 of the connection, playing the role of a pressure-like contribution to the transport of the vorticity field B. The radiative term $A \times E$ is structurally identical to the Poynting vector $E \times B$ [5]: it can carry H across $\partial\Omega(t)$ even for a boundary at rest and in the absence of fluid crossing it, through the propagation of the gauge field itself. A concrete picture: take $\Omega(t)$ to be a fixed laboratory volume containing a knotted vortex tube. As the tube is stretched and part of it is advected out of $\Omega(t)$, or as the underlying gauge-field configuration radiates linkage information through the walls, an observer measuring only $H(t)$ inside $\Omega(t)$ sees it decay and might attribute this to an effective viscosity or hyper-resistivity. Equation (11) shows that no such mechanism needs to be invoked: the decay is fully accounted for by $F_{\partial\Omega(t)}$.

A structural correspondence with the AdS_3 Chern–Simons setting of [2] clarifies the role of the radiative term $A \times E$ and provides a concrete spectral framework for $F_{\partial\Omega(t)}$. Both systems originate from a Chern–Simons action – Abelian in five dimensions for the fluid, non-Abelian in three dimensions for AdS_3 gravity – and both produce, after dimensional reduction, a local conservation law $\partial_\mu J^\mu = 0$ for a current quadratic in the gauge connection, together with the abstract open-boundary balance (2) in which a topological charge evolves only through the boundary values of the fundamental field. The structural dictionary between the two realisations is the following:

Table 1: Dictionary between the fluid Chern–Simons formulation and the AdS_3 Chern–Simons description.

Fluid CS (5D → 4D)	**AdS$_3$ CS (3D → 2D)**
Gauge connection $A_{\hat{\mu}}$	Dreibein and spin connection
Proto-helicity $H(t) = \int A_a B_a \, d^3x$	Virasoro charge $\mathscr{L}_0$
Boundary flux $F_{\partial\Omega(t)}$	Conformal-boundary stress flux $T^{01}\|_{\partial}$
Radiative term $A \times E$	Boundary graviton δe^a, $\delta\omega^{ab}$
$J_{\text{eff}} \cdot n$ on $\partial\Omega(t)$	$T^{0i} n_i$ on $\partial(\text{AdS}_3)$
Localised vortex (B-field soliton)	KdV soliton (reflectionless potential)
Caviton (quasi-resonance in $F_{\partial\Omega}$)	Forced KdV quasi-resonance
Dispersive decay $O(t^{-1/2})$	Dispersive KdV decay $O(t^{-1/2})$

The radiative row deserves comment. In the AdS$_3$ setting, boundary gravitons are linearised perturbations $(\delta e^a, \delta\omega^{ab})$ of the CS connection whose boundary flux takes the schematic form $\langle \delta e \wedge \delta\omega \rangle|_{\partial}$, carrying stress-tensor information to the conformal boundary without any bulk matter crossing it. In the fluid setting, $A \times E = A \times (-\partial_t A - \nabla A_0)$ is the cross-product of the spatial connection with its own time derivative – structurally the same object: the flux of a linearised connection perturbation to a boundary, generated by the field itself. The identification $A \times E \leftrightarrow \langle \delta e \wedge \delta\omega \rangle|_{\partial}$ is therefore a consequence of the shared Chern–Simons origin of both currents, not a loose analogy. The correspondence is structural rather than dynamical: the abstract form (2) and the radiative identification are common to both systems, but the bulk evolution equations differ – KdV hierarchy with eigenfunction forcing in the gravitational case, compressible fluid equations here. The solitonic sector of [2] corresponds, in the fluid picture, to localised vortex structures that preserve their helicity content and propagate without dispersion; the radiative sector corresponds to the dispersive leakage encoded in $F_{\partial\Omega(t)}$.

If $\Omega(t)$ is fixed and the flux $F_{\partial\Omega(t)}$ is predominantly outgoing – because the exterior acts as an effectively unbounded reservoir into which J_{eff} escapes and does not return on observable timescales – then $H(t)$ decreases monotonically (cf. (15)). This produces an effective irreversibility for an interior observer, analogous to radiation damping in open classical systems, even though the underlying dynamics is time-reversal symmetric. The boundary is therefore not a passive device but an active ingredient that decides whether a geometric quantity is conserved, balanced, or effectively dissipated.

A caveat is in order: the Poynting analogy is heuristic, not a physical identity. In fluid dynamics, the proto-helicity flux is not directly measurable; only its integrated effect on $H(t)$ is accessible. The distinction between geometric leakage and genuine dissipation would require independent boundary measurements of $J_{\text{eff}} \cdot n$, which are challenging in practice.

4 Exact helical vortex solution and explicit flux computation

We exhibit a family of exact static solutions of the reduced Chern–Simons system on which the balance law (11) can be evaluated in closed form, and from which a sharp functional bound on $|F_{\partial\Omega(t)}|$ is derived.

Working in cylindrical coordinates (r, φ, z), consider a gauge connection with no φ-dependence and no r-component,

$$A = A_\varphi(r)\,\hat{\varphi} + A_z(r,z)\,\hat{z}, \qquad A_0 = 0, \tag{17}$$

with all fields time-independent ($\partial_t A = 0$, so $E = 0$). The magnetic field $B = \nabla \times A$ then has components

$$B_r = 0, \qquad B_\varphi = -\partial_r A_z, \qquad B_z = \frac{1}{r}\partial_r\big(r A_\varphi\big). \tag{18}$$

Since $E = 0$ and $\partial_t B = 0$, the Bianchi identity $\partial_t B + \nabla \times E = 0$ is satisfied trivially, and $\nabla \cdot B = 0$ follows from (18) (as $B_r = 0$ and B_z is z-independent). The condition $B \cdot E = 0$ required by $F \wedge F = 0$ holds since $E = 0$. The ansatz (17) therefore defines an exact solution of the reduced Chern–Simons system for any choice of $A_\varphi(r)$ and $A_z(r,z)$.

We specify the potentials by a *regularised helical vortex*:

$$A_\varphi(r) = \frac{\Gamma}{4\pi}\,\frac{r}{r^2 + a^2}, \qquad A_z(r,z) = \alpha\, e^{-z/\lambda}, \tag{19}$$

where $\Gamma > 0$ is the total circulation, $a > 0$ a vortex-core regularisation length, $\alpha \in \mathbb{R}$ the axial amplitude, and $\lambda > 0$ an axial decay length. The resulting field strengths are

$$B_\varphi = 0, \qquad B_z(r) = \frac{\Gamma}{2\pi}\,\frac{a^2}{(r^2 + a^2)^2}, \tag{20}$$

a Lorentzian profile centred on the axis satisfying $\int_0^\infty B_z \cdot 2\pi r\, dr = \Gamma$. The proto-helicity density (8) is

$$A_a B_a = A_z B_z = \frac{\alpha\Gamma a^2}{2\pi}\,\frac{e^{-z/\lambda}}{(r^2 + a^2)^2}, \tag{21}$$

which is separable in r and z and decays axially on the scale λ.

Let $\Omega = \{(r,\varphi,z)\colon 0 \le r \le R,\ 0 \le z \le L\}$ be a fixed cylindrical domain, with a uniform axial flow $V = V_z\hat{z}$, $V_z > 0$ (outflow at $z = L$). With $E = 0$ and $A_0 = 0$, the effective current reduces to $J_{\text{eff}} = (A_b B^b)V$, and only the two axial caps contribute to the surface integral:

$$F_{\partial\Omega} = \int_0^{2\pi}\int_0^R \Big[(A_z B_z)\big|_{z=L} - (A_z B_z)\big|_{z=0}\Big] V_z\, r\, dr\, d\varphi. \tag{22}$$

Substituting (21) and using the radial integral $\int_0^R r\,dr/(r^2+a^2)^2 = R^2/[2a^2(R^2+a^2)]$, one obtains the exact closed-form result

$$F_{\partial\Omega} = \frac{\alpha\Gamma V_z R^2}{2(R^2+a^2)}\left(e^{-L/\lambda} - 1\right). \tag{23}$$

Since $e^{-L/\lambda} - 1 < 0$ for all $L, \lambda > 0$, one has $F_{\partial\Omega} < 0$ whenever $\alpha\Gamma > 0$: the proto-helicity leaks out of Ω through the downstream cap at $z = L$. The proto-helicity inventory is likewise computable,

$$H = \int_\Omega A_z B_z\, d^3x = \frac{\alpha\Gamma\lambda R^2}{2(R^2+a^2)}\left(1 - e^{-L/\lambda}\right), \tag{24}$$

and satisfies $dH/dt = F_{\partial\Omega} < 0$ in accordance with (11): the domain loses proto-helicity at a rate determined entirely by the two boundary values $A_z|_{z=0,L}$ and the velocity V_z, while the full interior profile (21) plays no role in the sign or magnitude of $F_{\partial\Omega}$.

Two limiting regimes of (23) are instructive. When $L \gg \lambda$ the exponential vanishes and the flux saturates at $-\alpha\Gamma V_z R^2/[2(R^2+a^2)]$: the upstream cap dominates and further lengthening Ω does not increase the leakage rate, while H approaches $\alpha\Gamma\lambda R^2/[2(R^2+a^2)]$, showing that only the first axial decay length contributes to the inventory. When $L \ll \lambda$ one has $e^{-L/\lambda} \approx 1 - L/\lambda$, so $F_{\partial\Omega} \approx -\alpha\Gamma V_z R^2 L/[2\lambda(R^2+a^2)]$: the flux is linear in L, the entire column participates, and the leakage rate grows with the length of the domain.

The exact result (23) is specific to the ansatz (19). We now derive a general upper bound valid for any solution with $E, B, V \in L^\infty(\partial\Omega)$ and $A \in H^1(\Omega)$. Setting $M_B = \|B\|_{L^\infty(\partial\Omega)}$, $M_E = \|E\|_{L^\infty(\partial\Omega)}$, $M_V = \|V\|_{L^\infty(\partial\Omega)}$, the pointwise estimate $|J_{\text{eff}}| \le |A|(|B||V| + |E|) + |A_0||B|$, integration over $\partial\Omega$, and the Cauchy–Schwarz inequality give

$$\begin{aligned} |F_{\partial\Omega(t)}| \;\le\; & (M_B M_V + M_E)\,|\partial\Omega|^{1/2}\,\|A\|_{L^2(\partial\Omega)} \\ & + M_B\,\|A_0\|_{L^1(\partial\Omega)}. \end{aligned} \tag{25}$$

Applying the Sobolev trace inequality $\|A\|_{L^2(\partial\Omega)} \le C_{\text{tr}}\|A\|_{H^1(\Omega)}$, where $C_{\text{tr}} > 0$ depends only on the geometry of Ω, yields

$$\begin{aligned} |F_{\partial\Omega(t)}| \;\le\; & C_{\text{tr}}\,(M_B M_V + M_E)\,|\partial\Omega|^{1/2}\,\|A\|_{H^1(\Omega)} \\ & + M_B\,\|A_0\|_{L^1(\partial\Omega)}. \end{aligned} \tag{26}$$

Three consequences follow. If $B|_{\partial\Omega} = E|_{\partial\Omega} = 0$ (magnetically and electrically screened boundary), then $M_B = M_E = 0$ and $F_{\partial\Omega(t)} = 0$: proto-helicity is exactly conserved regardless of the interior dynamics, extending (12) to screened open domains. When $M_V M_B \gg M_E$ (advection dominates radiation at the boundary), the first term scales as $M_V M_B\|A\|_{H^1(\Omega)}$: large-scale, slowly varying configurations (small $\|A\|_{H^1}$ relative to $\|A\|_{L^2}$) leak less than rapidly varying ones of the same L^2-energy. Finally, for the helical ansatz (19) with $A_0 = E = 0$, one has $M_B = \Gamma/(2\pi a^2)$ (attained at $r = 0$), and (26) gives $|F_{\partial\Omega}| \lesssim C_{\text{tr}}\Gamma V_z \alpha R(2\pi RL)^{1/2}/(2\pi a^2)$, which exceeds the exact value (23) by a factor growing as $(RL/a^2)^{1/2}$: the bound is not saturated because the r^{-4} fall-off of B_z is not captured by the L^∞ estimate M_B.

The static ansatz (19) deliberately sets $E = 0$ and $A_0 = 0$, thereby suppressing the radiative term $A \times E$ in the flux. A full time-dependent solution would be required to probe the dispersive regimes discussed in Section 6. Nevertheless, the explicit evaluation of $F_{\partial\Omega}$ for this restricted family confirms that the balance law (11) yields quantitative, testable predictions for boundary-driven helicity leakage.

5 Nonlinear transport: the Burgers equation

We illustrate how the open-boundary balance law of Section 2 interacts with genuinely nonlinear transport, using the inviscid Burgers equation [6]

$$\partial_t u + u\,\partial_x u = 0, \tag{27}$$

on a non-periodic domain $\Omega(t) \subset \mathbb{R}$ as the simplest tractable model. Along characteristics $dx/dt = u(x(t),t)$ one has $du/dt = 0$, so u is constant along $x(t) = \xi + t\,u_0(\xi)$ with $u_0(\xi) = u(\xi,0)$. If $\min_\xi u_0'(\xi) < 0$, the characteristic map ceases to be invertible at the finite time $t_* = -1/\min_\xi u_0'(\xi)$, at which the gradient of u diverges – the standard nonlinear steepening mechanism.

To make contact with Section 2, consider $Q(t) = \int_{\Omega(t)} u(x,t)\,dx$ with $\Omega(t) = [a(t),b(t)]$. Using $\partial_t u = -\partial_x(u^2/2)$ and the Reynolds transport theorem,

$$\begin{aligned} \frac{dQ}{dt} = & \underbrace{\tfrac{1}{2}u(a,t)^2 - \tfrac{1}{2}u(b,t)^2}_{\text{flux through } \partial\Omega} \\ & + \underbrace{u(b,t)b'(t) - u(a,t)a'(t)}_{\text{moving boundary advection}}. \end{aligned} \tag{28}$$

This is precisely the structure of (11): $Q(t)$, the analogue of $H(t)$, evolves only through quantities evaluated at the two endpoints. If $\Omega(t)$ is comoving with the fluid, $a'(t) = u(a,t)$ and $b'(t) = u(b,t)$ (a material interval), the two contributions in (28) cancel exactly[1], giving $dQ/dt = 0$: the closed-flux regime (12). If instead $\Omega = [a,b]$ is a fixed laboratory

[1] $u(b)b' - \frac{1}{2}u(b)^2 = \frac{1}{2}u(b)^2$ and similarly at a with opposite sign, so the net result is zero.

interval and u is a decreasing front with $u(a,t) > u(b,t) > 0$ before shock formation, (28) reduces to

$$\frac{dQ}{dt} = \tfrac{1}{2}\left[u(a,t)^2 - u(b,t)^2\right] > 0, \tag{29}$$

so $Q(t)$ grows, with the sign fixed entirely by the boundary values of u – exactly the mechanism of (11), where the sign of $F_{\partial\Omega(t)}$ controls whether H grows or decays. The two mechanisms are independent: a shock can form anywhere strictly inside $(a(t), b(t))$ without affecting (28), which depends only on the boundary values of u, while $Q(t)$ can vary substantially through (29) even when u remains smooth throughout $\Omega(t)$. This mirrors the proto-helicity case: the bulk dynamics and the boundary transport (11) are independent ingredients that must both be specified to determine the evolution of any integrated quantity.

6 Spectral classification of the proto-helicity decay regimes

The three decay regimes $H(t) \sim O(1)$, $O(t^{-1/3})$, $O(t^{-1/2})$ announced in the introduction are here derived directly from the balance law (11) and the spectral properties of the linearised Chern–Simons field on $\partial\Omega(t)$.

Spectral representation of $H(t)$

Fix Ω and decompose the gauge connection as $A = A^{(0)} + \delta A$, where $A^{(0)}$ is a time-independent background satisfying the static CS equations ($E^{(0)} = 0$, $\partial_t B^{(0)} = 0$) and $\delta A(x,t)$ is a small outgoing perturbation. To leading order in $|\delta A|$, the proto-helicity splits as

$$\begin{aligned} H(t) = &\underbrace{\int_\Omega A_a^{(0)} B_a^{(0)}\, d^3x}_{H_0 \text{ (constant)}} \\ &+ \underbrace{\int_\Omega \delta A_a(x,t)\, B_a^{(0)}(x)\, d^3x}_{H_{\rm rad}(t)} + O(|\delta A|^2). \end{aligned} \tag{30}$$

The static contribution H_0 satisfies $dH_0/dt = 0$ (no boundary flux when $E = 0$ and $V = 0$). The radiating part $H_{\rm rad}(t)$ obeys $dH_{\rm rad}/dt = F_{\partial\Omega}$ and controls all non-trivial time dependence.

The linearised equation for δA arising from $F \wedge F = 0$ is a hyperbolic system admitting outgoing solutions with a spectral decomposition

$$\delta A(x,t) = \int_{\mathbb{R}} a(\omega)\, \psi_\omega(x)\, e^{-i\omega t}\, d\omega + \text{c.c.}, \tag{31}$$

where ψ_ω are outgoing eigenmodes of the linearised CS operator, normalised so that $\psi_\omega \cdot n|_{\partial\Omega}$ is the outgoing amplitude at frequency ω, and $a(\omega)$ encodes the initial data. Substituting (31) into (30):

$$\begin{aligned} H_{\rm rad}(t) &= \int_{\mathbb{R}} a(\omega)\, M(\omega)\, e^{-i\omega t}\, d\omega + \text{c.c.}, \\ M(\omega) &= \int_\Omega \psi_\omega(x) \cdot B^{(0)}(x)\, d^3x. \end{aligned} \tag{32}$$

where $M(\omega)$ is the *spectral overlap* between the outgoing mode and the background vorticity. The large-time behaviour of $H_{\rm rad}(t)$ is therefore controlled by the singularity structure of $a(\omega)M(\omega)$ in the complex ω-plane.

Three regimes

Regime 1: $H(t) \sim O(1)$ (stable topological structure). If the linearised CS operator admits a discrete eigenvalue $\omega_0 = \omega_R - i\gamma$ with $\gamma > 0$ (a quasi-normal mode), then $a(\omega)M(\omega)$ has a simple pole in the lower half-plane and the residue theorem gives

$$H_{\rm rad}(t) \;\sim\; C\, e^{-\gamma t} \cos(\omega_R t + \varphi) \qquad (t \to \infty), \tag{33}$$

for constants C, φ determined by initial data. Since $e^{-\gamma t}$ is integrable on $[0,\infty)$, the total integral $\int_0^\infty F_{\partial\Omega}(s)\, ds$ converges and $H(t) \to H_0 + \text{const} = O(1)$. The physical realisation is a localised vortex structure – a *vortex soliton* – whose B-field profile is supported in Ω and whose helicity content does not disperse: $\gamma \to 0$ in the reflectionless (soliton) limit, and $H(t)$ remains bounded away from zero for all time.

Regime 2: $H(t) \sim O(t^{-1/2})$ (generic dispersive radiation). Suppose the spectrum of the linearised CS operator is purely continuous, so that $a(\omega)M(\omega)$ is analytic away from a branch cut along $\omega \in [0,\infty)$. For generic initial data, the spectral weight $a(\omega)M(\omega)$ behaves as

$$a(\omega)M(\omega) \;\sim\; c_0\, \omega^{-1/2} \qquad (\omega \to 0^+) \tag{34}$$

near the branch point (corresponding to long-wavelength modes with $k \to 0$). The Laplace-transform Tauberian theorem then gives

$$H_{\rm rad}(t) \;\sim\; c_0\, \Gamma(1/2)\, t^{-1/2} = c_0 \sqrt{\pi}\, t^{-1/2} \qquad (t \to \infty), \tag{35}$$

and hence $H(t) \sim O(t^{-1/2})$. In terms of the boundary flux: $F_{\partial\Omega}(t) = dH_{\rm rad}/dt \sim -c_0\sqrt{\pi}/2 \cdot t^{-3/2}$, which is integrable on $[1,\infty)$, confirming that the total helicity leaks to zero over infinite time but at a rate given precisely by the half-integer power (35). The underlying mechanism is standard dispersive spreading: in the mode decomposition (31), modes with different ω arrive at $\partial\Omega$ at different times and interfere destructively, with the interference integral producing the $t^{-1/2}$ envelope via the one-dimensional stationary-phase formula $\int_0^\infty f(u) e^{-iut}\, du \sim f(0)\,(\pi/t)^{1/2}$ for smooth f.

Regime 3: $H(t) \sim O(t^{-1/3})$ (Airy quasi-resonance). A qualitatively different regime arises when the dispersion relation $\omega = \omega(k)$ of the outgoing modes has an *inflection point* at a critical wavenumber k^*: $\omega''(k^*) = 0$, $\omega'''(k^*) \neq 0$. Near k^*, the phase of the integrand in (32) is

$$\phi(k,t) = k\cdot x - \omega(k)t \approx k^*\cdot x - \omega^* t - \frac{\omega'''(k^*)}{6}(k-k^*)^3 t, \quad (36)$$

and the integral over k near k^* evaluates by the Airy-function identity $\int_{-\infty}^{\infty} e^{i(su-u^3/3)}\,du = 2\pi\,\mathrm{Ai}(-s)$ with $s \sim (x - x_{\mathrm{g}}(t))/(\omega''' t)^{1/3}$, where $x_{\mathrm{g}}(t) = \omega'(k^*)t$ is the group velocity ray. As $t \to \infty$ along this ray,

$$|\delta A(x_{\mathrm{g}}(t),t)| \;\sim\; \frac{C}{(\omega''' t)^{1/3}}, \quad (37)$$

and the boundary contribution to $H_{\mathrm{rad}}(t)$ inherits this decay:

$$H_{\mathrm{rad}}(t) \;\sim\; \frac{C_*}{(\omega'''(k^*))^{1/3}}\, t^{-1/3} \qquad (t \to \infty), \quad (38)$$

giving $H(t) \sim O(t^{-1/3})$. In terms of the spectral weight: near the critical frequency $\omega^* = \omega(k^*)$, the density $a(\omega)M(\omega)$ accumulates as $(\omega - \omega^*)^{-2/3}$ – an integrable singularity weaker than (34) – producing a *quasi-resonance*: the inverse Fourier transform is spatially concentrated near the group velocity ray without any discrete eigenvalue being present. This is the spectral signature of a *caviton* – a transiently localised structure entrained by the boundary flux rather than self-sustained – whose helicity content decays precisely as $t^{-1/3}$ rather than the generic $t^{-1/2}$.

The three estimates (33)–(38) are derived directly from the spectral representation (32) of the proto-helicity integral in the linearised Chern–Simons fluid. They do not presuppose the KdV structure of [2]; the same Airy and stationary-phase arguments apply to any dispersive system whose boundary flux admits the decomposition (32). The structural correspondence of Section 3 then explains *a posteriori* why the same three rates appear in both systems: they are consequences of the singularity structure of the spectral measure, regardless of whether the bulk dynamics is governed by KdV or by the compressible CS fluid equations.

We emphasize that the rates discussed above should be interpreted as representative spectral scenarios rather than universal predictions of the Chern–Simons fluid system. Establishing which of these regimes is actually realized requires a spectral analysis of the outgoing boundary dynamics, which lies beyond the scope of the present work.

7 Conclusion

We have examined the consequences of replacing periodic or closed boundary conditions by open ones in the Chern–Simons formulation of compressible fluid dynamics of [1]. The local conservation law $\partial_{\hat{\mu}} h^{\hat{\mu}} = 0$ is unaffected by the choice of domain, but its integral over a moving region $\Omega(t)$ becomes the balance law (11): dH/dt equals the flux of the geometric current J_{eff} through $\partial\Omega(t)$. Periodic domains recover exact conservation, $dH/dt = 0$; open domains replace this by a transport relation in which $H(t)$ can grow, decay, or oscillate according to the sign of $F_{\partial\Omega(t)}$, including the monotonic decay (15) when the flux is one-signed. The apparent non-conservation is geometric leakage rather than dissipation, and produces an effective irreversibility for an observer confined to $\Omega(t)$, analogous to radiation damping in open classical systems. The Burgers example shows that this boundary mechanism coexists with, and is independent of, internal nonlinear effects such as shock formation.

The structural correspondence with [2] identified in Section 3 reinforces the conclusions above. Both theories realise the abstract balance (2) and share the same radiative subsector – $A \times E$ in the fluid, boundary gravitons $(\delta e^a, \delta\omega^{ab})$ in AdS_3 gravity – that propagates topological or stress-tensor information to a boundary through the gauge connection itself, independently of matter transport. The identification rests on the shared Chern–Simons origin of both currents: the correspondence is structural rather than dynamical.

The spectral analysis of Section 6 provides a possible classification of decay regimes: $O(1)$, $O(t^{-1/3})$, and $O(t^{-1/2})$, depending on whether the spectral measure has poles, a generic branch cut, or an Airy quasi-resonance. These are spectral scenarios, not universal predictions; establishing which actually occurs for the compressible fluid requires a detailed spectral analysis of the boundary dynamics.

Several open questions remain. A gauge-invariant formulation via relative proto-helicity (9) would put (11) on a fully rigorous footing. An explicit radiation condition at $\partial\Omega(t)$ would make the graviton–fluid analogy quantitative. If the open-boundary dynamics can be cast as an eigenfunction-forced integrable hierarchy, the Gelfand–Levitan–Marchenko formalism could yield an exact expression for $F_{\partial\Omega(t)}$. These questions define a concrete programme toward an open, non-periodic geometric hydrodynamics built on the Chern–Simons formulation.

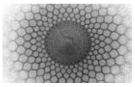
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TTMP vol 3 (3), 2026
doi: 10.30511/TTMP.2026.2093603.1082

Transactions in Theoretical and Mathematical Physics



On New Divisibility Properties of Generalized Central Trinomial Coefficients and Legendre Polynomials

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Received: 06 July 2026 / Accepted: 31 July 2026 / Published: 01 August 2026

Abstract We present a new formula for the highest power of $a+b$ that divides the sum $B(n,m,a,b)=\sum_{k=0}^{n}\binom{n}{k}^m a^{n-k}b^k$ for the case $m=2$. By using this formula, we give a complete 3-adic valuation for central Delannoy numbers. Also, we find the highest power of an odd integer x that divides Legendre's polynomial $P_n(x)$. By using the same idea, generalized trinomial coefficients and generalized Motzkin numbers are treated. As a result, we give a complete 3-adic valuation for little Schröder numbers and restricted hexagonal numbers. By using a new class of binomial sums, we examine the divisibility of $B(n,m,a,b)$ by powers of $a+b$ for $m>2$.

Keywords: Legendre polynomials, central Delannoy numbers, generalized central trinomial coefficients, Legendre's theorem, Kummer's theorem.

1 Introduction

Let us consider the following sum:

$$B(n,m,a,b)=\sum_{k=0}^{n}\binom{n}{k}^m a^{n-k}b^k, \tag{1}$$

where n and m are natural numbers such that $m\geq 2$, and a, b are integers.

Recently [15, Section 2, Proposition 1, p. 4], it is shown that the sum $B(n,2,a,b)$ represents the number of weighted free Dyck paths of semilength n from $(0,0)$ to $(2n,0)$ with up steps $(1,1)$ and down steps $(1,-1)$; where the weight a is assigned to up steps in the peaks, and the weight b to other up steps. Also, the sum $B(n,2,a,b)$ represents the number of the same weighted free Dyck paths of semilength n, where the weight a is assigned to even ascents and the weight b to even descents.

The sum $B(n,2,a,b)$ is closely related to well-known Legendre polynomials $P_n(x)$. It is well-known [5, Introduction] that Legendre polynomials have various explicit representations. We refer readers to [9, pp. 1–3] for seven different definitions of these polynomials. For example,

$$P_n(x)=\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}^2(x+1)^k(x-1)^{n-k}. \tag{2}$$

It is readily verified that the two following formulae are true:

$$P_n(x)=B(n,2,\frac{x-1}{2},\frac{x+1}{2}), \tag{3}$$

$$B(n,2,a,b)=(a-b)^nP_n(\frac{a+b}{a-b}),\ a\neq b. \tag{4}$$

The sum $B(n,2,1,1)$ is equal to the central binomial coefficient $\binom{2n}{n}$. It is well-known that the integer $\binom{2n}{n}$ counts all lattice paths from $(0,0)$ to (n,n) by using steps $(1,0)$ and $(0,1)$. Note that sum $B(n,3,1,1)$ is equal to the n-th Franel number $f_n=\sum_{k=0}^{n}\binom{n}{k}^3$.

The sum $B(n,2,1,2)$ is equal to the central Delannoy number D_n. It is well-known that D_n counts all lattice paths from $(0,0)$ to (n,n) by using steps $(1,0)$, $(0,1)$, and $(1,1)$. Such paths are called royal paths [27, Introduction]. Sulanke [29] gave, in a recreational spirit, a collection of 29 configurations counted by these numbers. Banderier and Schwer [1] gave additional information about origin and use of this number sequence. Central Delannoy numbers have a close relation to Legendre polynomials due to the fact that $D_n=P_n(3)$. See also [8, 11, 25]. Furthermore, central Delannoy numbers D_n (see [28, Section 6.4] and [**?**]) satisfy the second-order recurrence:

$$(n+2)D_{n+2}=3(2n+3)D_{n+1}-(n+1)D_n, \tag{5}$$

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where n is a non-negative integer.

The large Schröder numbers S_n count all lattice paths from $(0,0)$ to (n,n) by using steps $(1,0)$, $(0,1)$, and $(1,1)$ such that never go above the diagonal $y=x$. Large Schröder numbers bear the same relationship to the central Delannoy numbers as the Catalan numbers do to the central binomial coefficients. Recall that the n-th Catalan number $C_n=\frac{1}{n+1}\binom{2n}{n}$ counts all lattice paths from $(0,0)$ to (n,n) by using steps $(1,0)$ and $(0,1)$ such that never go above the diagonal $y=x$. Stanley [28, Exercise 6.39] gave 11 combinatorial objects counted by large Schröder numbers S_n. See also [27, Introduction].

It is known that:

$$S_n=\frac{1}{2}\left(-D_{n-1}+6D_n-D_{n+1}\right). \tag{6}$$

where n is a natural number.

The little Schröder numbers s_n count the number of plane trees with a given set of leaves, the number of ways of inserting parentheses into a sequence, and the number of ways of dissecting a convex polygon into smaller polygons by inserting diagonals. Let n be a natural integer. Then $s_n=\frac{1}{2}S_n$.

Furthermore, for $m=2$, it is known that:

$$B(n,2,a,b)=\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n}{k}\binom{n-k}{k}a^k b^k(a+b)^{n-2k}. \tag{7}$$

For $m=3$, there is MacMahon's identity:

$$B(n,3,a,b)=\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n}{2k}\binom{2k}{k}\binom{n+k}{k}a^k b^k(a+b)^{n-2k}. \tag{8}$$

Recently, new combinatorial proofs of Eqns. (7) and (8) are presented, as well as combinatorial proofs of some known representations of Legendre polynomials based on weighted enumeration of free lattice paths [15].

Let us consider [31] the following sum:

$$T_n(a,b)=\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n}{k}\binom{n-k}{k}a^k\cdot b^{n-2k}, \tag{9}$$

where a and b are integers. The number $T_n(a,b)$ represents the number of weighted free Motzkin's paths of the length n form $(0,0)$ to (n,n) by using up steps $(1,1)$, down steps $(1,-1)$, and level steps $(1,0)$; where the weight a is assigned to up steps and the weight b to level steps. The number $T_n(a,b)$ is also known [4] as a generalized central trinomial coefficient. They can be defined as coefficients of x^n in the expansion of $(x^2+bx+a)^n$. See also [26].

By the Eq. (7), it follows that

$$B(n,2,a,b)=T_n(ab,a+b). \tag{10}$$

Also let us consider [31] the following sum:

$$M_n(a,b)=\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n}{2k}C_k\cdot a^k\cdot b^{n-2k}, \tag{11}$$

where a and b are integers. The number $M_n(a,b)$ is known as a generalized Motzkin number. Particularly, it is known that:

$$s_{n+1}=M_n(2,3), \tag{12}$$

where n is a non-negative integer.

Our goal is to examine divisibility of $B(n,m,a,b)$ by powers of $a+b$, for $m\geq 2$. For $m=2$, we shall find the highest power of $a+b$ such that divides the sum $B(n,2,a,b)$.

2 Main results

Let $\omega_x(y)$ denote the highest power of an integer x that divides an integer y; where $x\neq 0$ and $x\neq\pm 1$. If an integer x is a prime number, then we shall use a notation $v_p(y)$ instead of $\omega_p(y)$. For calculating $\omega_x(y)$, we use the following formula:

$$\omega_x(y)=\min\left\{\left\lfloor\frac{v_p(y)}{v_p(x)}\right\rfloor:\ p \text{ is a prime such that } p\mid x\right\}. \tag{13}$$

Our first main result is:

Theorem 1 *Let n be a non-negative integer. Let a and b be integers such that $\gcd(a,b)=1$, $a+b\neq 0$, and $a+b\neq\pm 1$. For $m=2$, we have:*

$$\omega_{a+b}\left(B(2n,2,a,b)\right)=\omega_{a+b}\left(\binom{2n}{n}\right), \tag{14}$$

$$\omega_{a+b}\left(B(2n+1,2,a,b)\right)=1+\omega_{a+b}\left((2n+1)\times\binom{2n}{n}\right). \tag{15}$$

Our second main result is:

Theorem 2 *Let n be a non-negative integer. Let a and b be integers such that $\gcd(a,b)=1$, $a+b\neq 0$, and $a+b\neq\pm 1$. Let m be an arbitrary natural number greater than 2. Then*

$$\omega_{a+b}\left(B(2n,m,a,b)\right)\geq\omega_{a+b}\left(\binom{2n}{n}\right), \tag{16}$$

$$\omega_{a+b}\left(B(2n+1,m,a,b)\right)\geq 1+\omega_{a+b}\left((2n+1)\times\binom{2n}{n}\right). \tag{17}$$

We present several applications of Theorems 1 and 2.

The first application is for calculating the highest power of two that divides central binomial coefficient.

Corollary 1 *Let n be a non-negative integer.Then*

$$v_2\left(\binom{4n+2}{2n+1}\right) = 1 + v_2\left(\binom{2n}{n}\right), \tag{18}$$

$$v_2\left(\binom{4n}{2n}\right) = v_2\left(\binom{2n}{n}\right). \tag{19}$$

By using Eqns. (18) and (19), it can be shown the well-known fact that $v_2(\binom{2n}{n})$ is equal to the number of ones in binary representation of the number n.

The second application is for estimation for the highest power of two that divides Franel number f_n.

Corollary 2 *Let n be a non-negative integer.Then*

$$v_2(f_{2n+1}) \geq 1 + v_2\left(\binom{2n}{n}\right), \tag{20}$$

$$v_2(f_{2n}) \geq v_2\left(\binom{2n}{n}\right). \tag{21}$$

The third application is for calculating the highest power of three that divides central Delannoy number D_n.

Theorem 3 *Let n be a non-negative integer.Then*

$$v_3(D_{2n+1}) = 1 + v_3(2n+1) + v_3\left(\binom{2n}{n}\right), \tag{22}$$

$$v_3(D_{2n}) = v_3\left(\binom{2n}{n}\right). \tag{23}$$

The fourth application is for calculating the highest power of three that divides large Schröder number S_n.

Theorem 4 *Let n be a non-negative integer.Then*

$$v_3(S_{2n+1}) = v_3(C_n), \tag{24}$$

$$v_3(S_{2n+2}) = 1 + v_3(2n+1) + v_3(C_n). \tag{25}$$

Note that $v_3(s_n) = v_3(S_n)$.

Recently, Lengyel, among other, gave the formula [14, Theorem 10, p. 8] for 3-adic valuation of central Dellanoy numbers. Note that it is asserted that the formula [14, Theorem 10, p. 8] is true for any n sufficiently large. Also, Lengyel gave the formula [14, Theorem 17, p. 19] for 3-adic valuation of large Schröder numbers S_n for the case $n \equiv 1 \pmod 3$.

The fifth application is for calculating the highest power of an odd integer x that divides Legendre polynomial $P_n(x)$.

Corollary 3 *Let n be a non-negative integer, and let x be an odd integer such that $x \neq \pm 1$. Then*

$$\omega_x\big(P_{2n+1}(x)\big) = 1 + \omega_x\left((2n+1) \times \binom{2n}{n}\right), \tag{26}$$

$$\omega_x\big(P_{2n}(x)\big) = \omega_x\left(\binom{2n}{n}\right). \tag{27}$$

Furthermore, from our proof of Theorem 1, we can calculate the highest power of an integer b that divides generalized central trinomial coefficient $T_n(a,b)$.

Theorem 5 *Let n be a non-negative integer. Let a and b be integers such that* $\gcd(a,b) = 1$, *$b \neq \pm 1$, and $b \neq 0$. Then:*

$$\omega_b\big(T_{2n}(a,b)\big) = \omega_b\left(\binom{2n}{n}\right), \tag{28}$$

$$\omega_b\big(T_{2n+1}(a,b)\big) = 1 + \omega_b\left((2n+1) \times \binom{2n}{n}\right). \tag{29}$$

Also, by using the same idea from our proof of Theorem 1, we can calculate the highest power of an integer b that divides generalized Motzkin number $M_n(a,b)$.

Theorem 6 *Let n be a non-negative integer. Let a and b be integers such that* $\gcd(a,b) = 1$, *$b \neq 0$, and $b \neq \pm 1$. Then:*

$$\omega_b\big(M_{2n}(a,b)\big) = \omega_b(C_n), \tag{30}$$

$$\omega_b\big(M_{2n+1}(a,b)\big) = 1 + \omega_b\big((2n+1) \times C_n\big). \tag{31}$$

For a proof of Theorem 1, we use Eq. (7) and the following lemma:

Lemma 1 *Let n be a natural number, and let p be a prime number. Then*

$$v_p\left((2n+1)\binom{2n}{n}\right) \leq n.$$

Furthermore, for a proof of Theorem 2, we use Lemma 1 and a new class of binomial sums [20, Eqns. (27) and (28)] that we call M sums.

3 A New Class of Binomials Sums

Definition 1 Let n and m be positive integers, and let a and b be integers. Let $S(n,m,a,b) = \sum_{k=0}^{n} \binom{n}{k}^m F(n,k,a,b)$, where $F(n,k,a,b)$ is an integer-valued function. Then M sums for the sum $S(n,m,a,b)$ are, as follows:

$$M_S(n,j,t;a,b) = \binom{n-j}{j} \sum_{v=0}^{n-2j} \binom{n-2j}{v} \times \binom{n}{j+v}^t F(n,j+v,a,b); \tag{32}$$

where j and t are non-negative integers such that $j \leq \lfloor \frac{n}{2} \rfloor$.

Obviously, the following equation [20, Eq. (29)],

$$S(n,m,a,b) = M_S(n,0,m-1;a,b) \tag{33}$$

holds.

By setting $t := 0$ in Eq. (32), it follows that

$$M_S(n,j,0;a,b) = \binom{n-j}{j} \sum_{v=0}^{n-2j} \binom{n-2j}{v} \times F(n,j+v,a,b). \tag{34}$$

Let n, j, t, a, and b be same as in Definition 1.

It is known [20, Th. 8] that M sums satisfy the following recurrence relation:

$$M_S(n,j,t+1;a,b) = \binom{n}{j} \sum_{u=0}^{\lfloor \frac{n-2j}{2} \rfloor} \binom{n-j}{u} \times M_S(n,j+u,t;a,b). \tag{35}$$

In one particular situation, Relation (35) has a simple consequence [20, Section 4] which is important for us.

Corollary 4 *Let n be a fixed natural number, and let a,b be fixed integers. Let us suppose that, for some non-negative integer t', $M_S(n,j,t';a,b)$ is divisible by an integer $q(n,a,b)$ for all $0 \le j \le \lfloor \frac{n}{2} \rfloor$. Then $M_S(n,j,t;a,b)$ is divisible by $q(n,a,b)$ for all $t \ge t'$ and for all $0 \le j \le \lfloor \frac{n}{2} \rfloor$. Furthermore, $S(n,m,a,b)$ is divisible by $q(n,a,b)$ for all $m \ge t'+1$.*

The idea is to calculate M sums for the sum $B(n,m,a,b)$ by using Eq. (34) and Relation (35). By using binomial theorem, we show that:

$$M_B(n,j,0;a,b) = \binom{n-j}{j} (ab)^j (a+b)^{n-2j}. \tag{36}$$

Furthermore, we prove a slight generalization of Eq. (7):

$$M_B(n,j,1;a,b) = \binom{n}{j} \sum_{k=0}^{\lfloor \frac{n-2j}{2} \rfloor} \binom{n-j}{k} \times \binom{n-j-k}{j+k} (ab)^{j+k} \times (a+b)^{n-2j-2k}. \tag{37}$$

By setting $j := 0$ in Eq. (37), we obtain Eq. (7).

M sums give an elementary proof of Calkin's result [2, Thm. 1, p. 1]. It is known [20, Eqns. (22) and (25)] that:

$$M_B(2n,j,0;1,-1) = \begin{cases} 0, & \text{if } 0 \le j < n; \\ (-1)^n, & \text{if } j = n. \end{cases}$$

$$M_B(2n,j,1;1,-1) = (-1)^n \binom{2n}{n} \binom{n}{j},$$

where $0 \le j \le n$. It follows that $q(2n,1,-1) = \binom{2n}{n}$ and $t' = 1$.

By using M sums, we can derive a slight generalization of, the MacMahon's identity, Eq. (8). Also we can derive formulas for $B(n,m,a,b)$ for the cases $m = 4$ and $m = 5$.

Note that there are several other applications ([19, Background],[20], and [22]) of M sums.

The rest of the paper is structured as follows. In Section 4, we give a proof of Lemma 1 by using Legendre's theorem. In Section 5, we give a proof of Theorem 1. In Section 6, we prove Theorem 2 by using a new class of binomials sums. In Section 7, we give a proof of Corollary 1. In Section 8, we give a proof of Corollary 2. In Section 9, we prove Theorem 3. In Section 10, we give a sketch of proofs of Theorem 5 and Corollary 3. In Section 11, we give a proof of Theorem 6. In Section 12, we give a proof of Theorem 4 by using Theorem 6. In Section 13, we give concluding remarks.

4 A Proof of Lemma 1

Let $n!!$ denote the double factorial, i. e., $n \cdot (n-2) \cdot (n-4) \cdots$.

The first case:

Let $p = 2$.

We have gradually:

$$\begin{aligned} v_2((2n+1)\binom{2n}{n}) &= v_2(\binom{2n}{n}) \\ &= v_2(\frac{(2n)!! \cdot (2n-1)!!}{(n!)^2}) \\ &= v_2(\frac{2^n \cdot n! \cdot (2n-1)!!}{(n!)^2}) \\ &= v_2(2^n) + v_2((2n-1)!!) - v_2(n!). \end{aligned} \tag{38}$$

From the last equation above, it follows that

$$v_2((2n+1)\binom{2n}{n}) = n - v_2(n!). \tag{39}$$

By the Eq. (39), it follows that $v_2((2n+1)\binom{2n}{n}) \le n$. This proves the first case.

The second case:

Let p be an odd prime. By Legendre's formula, it follows that:

$$v_p(n!) = \sum_{k=1}^{\infty} \lfloor \frac{n}{p^k} \rfloor \tag{40}$$

Let $n = \sum_{l=0}^{k} a_l \cdot p^l$ be the expansion of n in the base p, where a_l are non-negative integers such that $a_l < p$ for all $l = 1,k$. Let $s_p(n)$ denote the sum $\sum_{l=0}^{k} a_l$.

By Legendre's formula, it follows Legendre's theorem [16, Theorem 1.1]:

$$v_p(n!) = \frac{n - s_p(n)}{p-1}. \tag{41}$$

Let us consider the number $c(n,p) = \frac{(pn)!}{(n!)^p}$. Obviously, $c(n,p)$ is always an integer due to the fact that $c(n,p)$ is a central multinomial coefficient.

It is known that:

$$c(n,p)=\prod_{k=2}^{p}\binom{kn}{n}. \tag{42}$$

By (41), it follows that:

$$\begin{aligned} v_p(c(n,p)) &= v_p(\frac{(pn)!}{(n!)^p}) \\ &= v_p((pn)!)-pv_p(n!) \\ &= \frac{pn-s_p(pn)}{p-1}-p\cdot\frac{n-s_p(n)}{p-1} \\ &= \frac{p\cdot s_p(n)-s_p(pn)}{p-1}. \end{aligned} \tag{43}$$

By using the fact $s_p(pn)=s_p(n)$, from the Eq. (43), it follows that:

$$v_p(c(n,p))=s_p(n). \tag{44}$$

From the expansion of n in the base p, using the obvious ineqality $p^l\geq 1$, it follows that $s_p(n)\leq n$. By using the Eq. (44), we obtain that:

$$v_p(c(n,p))\leq n. \tag{45}$$

Let $F(n,k)=\frac{1}{(k-1)n+1}\binom{kn}{n}$ denote generalized Catalan numbers or Fuss-Catalan numbers, where n and k are natural numbers. It is well-known that Fuss-Catalan number is always an integer [10, Eq. (17.1), p. 375]. Obviously, $F(n,2)=C_n$ and $F(n,3)=\frac{1}{2n+1}\binom{3n}{n}$. See also [24, Introduction] and [22].

Now we use assumption that p is an odd prime. By the Eq. (42), it follows that $c(n,p)$ is divisible by $\binom{3n}{n}\binom{2n}{n}$. Note that $\binom{3n}{n}\binom{2n}{n}=F(n,3)\cdot(2n+1)\binom{2n}{n}$. It follows that $c(n,p)$ is divisible by $(2n+1)\binom{2n}{n}$. Therefore,

$$v_p((2n+1)\binom{2n}{n})\leq v_p(c(n,p)). \tag{46}$$

Eqns. (45) and (46) prove the second case of Lemma 1. This completes the proof of Lemma 1.

Remark 1 Let p be an odd prime. Then, by using the same idea from the second case of the proof of Lemma 1, it can be shown that:

$$v_p((\prod_{k=2}^{p-1}(kn+1))\binom{2n}{n})\leq n. \tag{47}$$

Remark 2 The Eq. (44) is true for all prime numbers p. Particulary, for $p=2$, it follows that $v_2(\binom{2n}{n})$ is equal to number of ones in the binary representation of the number n.

5 A Proof of Theorem 1

Firstly, we give a proof of the Eq. (14).

5.1 A Proof of the Eq. (14)

By setting $n:=2n$ in Eq. (7), we obtain that:

$$\begin{aligned} B(2n,2,a,b)=&\sum_{k=0}^{n}\binom{2n}{k}\binom{2n-k}{k} \\ &\times(ab)^k(a+b)^{2n-2k}. \end{aligned} \tag{48}$$

By using the substitution $k=n-s$, Eq. (48) becomes gradually:

$$\begin{aligned} &B(2n,2,a,b)= \\ &\sum_{s=0}^{n}\binom{2n}{n-s}\binom{n+s}{n-s}(a+b)^{2s}(ab)^{n-s}, \\ &=\sum_{s=0}^{n}\binom{2n}{n+s}\binom{n+s}{2s}(a+b)^{2s}\cdot(ab)^{n-s}. \end{aligned} \tag{49}$$

It is readily verified that [10, Eq. (1.4), p. 5]:

$$\binom{2n}{n+s}\binom{n+s}{2s}=\frac{\binom{2n}{n}\binom{n}{s}^2}{\binom{2s}{s}}. \tag{50}$$

By using Eq. (50), Eq. (49) becomes, as follows:

$$B(2n,2,a,b)=\sum_{s=0}^{n}(\frac{\binom{2n}{n}\binom{n}{s}^2}{\binom{2s}{s}})(a+b)^{2s}\cdot(ab)^{n-s}. \tag{51}$$

Let p be a prime number such that divides $a+b$. For $s=0$, the first summand in Eq. (51) is $\binom{2n}{n}\cdot(ab)^n$. Since $a+b$ and ab are relatively prime, it follows that

$$v_p(\binom{2n}{n}\cdot(ab)^n)=v_p(\binom{2n}{n}). \tag{52}$$

Let $1\leq s\leq n$. Note that by Lemma 1, it follows that

$$v_p(\binom{2s}{s})\leq s. \tag{53}$$

Then, for the $(s+1)$-th summand Eq. (51), we obtain that:

$$\begin{aligned} &v_p\left(\frac{\binom{2n}{n}\binom{n}{s}^2}{\binom{2s}{s}}(a+b)^{2s}(ab)^{n-s}\right) \\ &\quad=v_p\left(\binom{2n}{n}\right)+2v_p\left(\binom{n}{s}\right) \\ &\quad\quad-v_p\left(\binom{2s}{s}\right)+2s\,v_p(a+b). \end{aligned} \tag{54}$$

By using Ineq. (53), from Eq. (54), we obtain that:

$$\begin{aligned} &v_p\left(\frac{\binom{2n}{n}\binom{n}{s}^2}{\binom{2s}{s}}(a+b)^{2s}(ab)^{n-s}\right)\geq \\ &\quad v_p\left(\binom{2n}{n}\right)+s\big(2v_p(a+b)-1\big). \end{aligned} \tag{55}$$

Obviously, for $1 \le s \le n$,

$$v_p\left(\binom{2n}{n}\right) + s\big(2v_p(a+b)-1\big) \ge v_p\left(\binom{2n}{n}\right) + s \ge v_p\left(\binom{2n}{n}\right) + 1. \tag{56}$$

By Ineqs. (55) and (56), for $1 \le s \le n$, it follows that:

$$v_p\Big(\frac{\binom{2n}{n}\binom{n}{s}^2}{\binom{2s}{s}}(a+b)^{2s}\cdot(ab)^{n-s}\Big) \ge v_p\Big(\binom{2n}{n}\Big) + 1. \tag{57}$$

By using Eqns. (52) and (57), we obtain that:

$$v_p\big(B(2n,2,a,b)\big) = v_p\left(\binom{2n}{n}\right). \tag{58}$$

where p is a prime number that divides $a+b$.

Now, by using Relation (13), it follows that:

$$\begin{aligned}\omega_{a+b}\big(B(2n,2,a,b)\big) &= \min_{\substack{p \text{ prime}\\ p|a+b}}\left\lfloor \frac{v_p\big(B(2n,2,a,b)\big)}{v_p(a+b)}\right\rfloor \\ &= \min_{\substack{p \text{ prime}\\ p|a+b}}\left\lfloor \frac{v_p\left(\binom{2n}{n}\right)}{v_p(a+b)}\right\rfloor \\ &= \omega_{a+b}\left(\binom{2n}{n}\right).\end{aligned} \tag{59}$$

The Eq. (59) completes the proof of Eq. (14).

5.2 A Proof of Eq. (15)

By setting $n := 2n+1$ in Eq. (7), we obtain that:

$$\begin{aligned}B(2n+1,2,a,b) &= \sum_{k=0}^{n}\binom{2n+1}{k}\binom{2n+1-k}{k} \\ &\quad\times (ab)^k(a+b)^{2n+1-2k} \\ &= (a+b)\sum_{k=0}^{n}\binom{2n+1}{k}\binom{2n+1-k}{k} \\ &\quad\times (ab)^k(a+b)^{2n-2k}.\end{aligned} \tag{60}$$

By using the substitution $k = n-s$, Eq. (60) becomes gradually:

$$\begin{aligned}B(2n+1,2,a,b) &= (a+b)\sum_{s=0}^{n}\binom{2n+1}{n-s}\binom{n+s+1}{n-s} \\ &\quad\times (a+b)^{2s}(ab)^{n-s} \\ &= (a+b)\sum_{s=0}^{n}\binom{2n+1}{n+s+1}\binom{n+s+1}{2s+1} \\ &\quad\times (a+b)^{2s}(ab)^{n-s}.\end{aligned} \tag{61}$$

It is readily verified that [10, Eq. (1.4), p. 5]

$$\binom{2n+1}{n+s+1}\binom{n+s+1}{2s+1} = \frac{(2n+1)\binom{2n}{n}\binom{n}{s}^2}{(2s+1)\binom{2s}{s}}. \tag{62}$$

By using Eq. (62), Eq. (61) becomes, as follows:

$$\begin{aligned}B(2n+1,2,a,b) &= (a+b)\sum_{s=0}^{n}\left[\frac{(2n+1)\binom{2n}{n}\binom{n}{s}^2}{(2s+1)\binom{2s}{s}}\right] \\ &\quad\times (a+b)^{2s}(ab)^{n-s}.\end{aligned} \tag{63}$$

By using Eq. (63), it follows that

$$\begin{aligned}&\omega_{a+b}\big(B(2n+1,2,a,b)\big) = \\ &\quad 1+\omega_{a+b}\left(\sum_{s=0}^{n}\left[\frac{(2n+1)\binom{2n}{n}\binom{n}{s}^2}{(2s+1)\binom{2s}{s}}\right]\right. \\ &\quad\left.\times (a+b)^{2s}(ab)^{n-s}\right).\end{aligned} \tag{64}$$

Let $P(2n+1,2,a,b)$ denote $\sum_{s=0}^{n}\big[\frac{(2n+1)\binom{2n}{n}\binom{n}{s}^2}{(2s+1)\binom{2s}{s}}\big](a+b)^{2s}\cdot(ab)^{n-s}$.

Then, Eq. (64) can be rewritten as:

$$\omega_{a+b}(B(2n+1,2,a,b)) = 1+\omega_{a+b}(P(2n+1,2,a,b)). \tag{65}$$

Let p be a prime number such that divides $a+b$. For $s=0$, the first summand in the sum $P(2n+1,2,a,b)$ is $(2n+1)\binom{2n}{n}\cdot(ab)^n$. Since $a+b$ and ab are relatively prime, it follows that

$$v_p\Big((2n+1)\binom{2n}{n}\cdot(ab)^n\Big) = v_p\Big((2n+1)\binom{2n}{n}\Big). \tag{66}$$

Let $1 \le s \le n$.

Then, for the $(s+1)$-th summand the sum $P(2n+1,2,a,b)$, it follows that $v_p\big(\frac{(2n+1)\binom{2n}{n}\binom{n}{s}^2}{(2s+1)\binom{2s}{s}}\big)(a+b)^{2s}\cdot(ab)^{n-s}\big)$ is equal to

$$\begin{aligned}&v_p\left((2n+1)\binom{2n}{n}\right) + 2v_p\left(\binom{n}{s}\right) \\ &\quad - v_p\left((2s+1)\binom{2s}{s}\right) + 2s\,v_p(a+b).\end{aligned} \tag{67}$$

By using Lemma 1, from Eq. (67), it follows that:

$$\begin{aligned}&v_p\left(\frac{(2n+1)\binom{2n}{n}\binom{n}{s}^2}{(2s+1)\binom{2s}{s}}\right. \\ &\quad\left.\times (a+b)^{2s}(ab)^{n-s}\right) \\ &\quad\ge v_p\left((2n+1)\binom{2n}{n}\right) + s\big(2v_p(a+b)-1\big).\end{aligned} \tag{68}$$

Obviously, for $1 \le s \le n$,

$$\begin{aligned} & v_p\left((2n+1)\binom{2n}{n}\right) + s\big(2v_p(a+b)-1\big) \\ & \quad \ge v_p\left((2n+1)\binom{2n}{n}\right) + s \\ & \quad \ge v_p\left((2n+1)\binom{2n}{n}\right) + 1. \end{aligned} \tag{69}$$

By Ineqs. (68) and (69), for $1 \le s \le n$, it follows that:

$$\begin{aligned} & v_p\left(\frac{(2n+1)\binom{2n}{n}\binom{n}{s}^2}{(2s+1)\binom{2s}{s}} \right. \\ & \qquad \left. \times (a+b)^{2s}(ab)^{n-s}\right) \\ & \quad \ge v_p\left((2n+1)\binom{2n}{n}\right) + 1. \end{aligned} \tag{70}$$

By using Eq. (66) and Ineq. (70), we obtain that:

$$v_p(P(2n+1,2,a,b)) = v_p((2n+1)\binom{2n}{n}), \tag{71}$$

where p is a prime number that divides $a+b$.

Now, by using Relation (13), it follows that:

$$\begin{aligned} & \omega_{a+b}\big(P(2n+1,2,a,b)\big) \\ & \qquad = \min_{\substack{p \text{ prime} \\ p \mid a+b}} \left\lfloor \frac{v_p\big(P(2n+1,2,a,b)\big)}{v_p(a+b)} \right\rfloor \\ & \qquad = \min_{\substack{p \text{ prime} \\ p \mid a+b}} \left\lfloor \frac{v_p\left((2n+1)\binom{2n}{n}\right)}{v_p(a+b)} \right\rfloor \\ & \qquad = \omega_{a+b}\left((2n+1)\binom{2n}{n}\right). \end{aligned} \tag{72}$$

By using Eqns. (65) and (72), Eq. (15) follows. This completes the proof of Theorem 1.

6 A Proof of Theorem 2

Firstly, we give a proof of Eq. (36).

6.1 A Proof of Eq. (36)

Obviously, the sum $B(n,m,a,b)$ is an instance of the sum $S(n,m,a,b)$, where $F(n,k,a,b) = a^{n-k}b^k$.

By Eq. (34), it follows that

$$\begin{aligned} M_B(n,j,0;a,b) &= \binom{n-j}{j} \sum_{v=0}^{n-2j} \binom{n-2j}{v} a^{n-j-v}b^{j+v}, \\ &= \binom{n-j}{j} a^j \cdot b^j \sum_{v=0}^{n-2j} \binom{n-2j}{v} a^{n-2j-v}b^v, \\ &= \binom{n-j}{j} (ab)^j (a+b)^{n-2j}, \end{aligned} \tag{73}$$

where $0 \le j \le \lfloor \frac{n}{2} \rfloor$. This proves Eq. (36).

Secondly, we give a proof of Eq. (37).

6.2 A Proof of Eq. (37)

By setting $t := 0$ and $S := B$ in Eq. (35), it follows that

$$\begin{aligned} M_B(n,j,1;a,b) = \binom{n}{j} & \sum_{u=0}^{\lfloor \frac{n-2j}{2} \rfloor} \binom{n-j}{u} \\ & \times M_B(n,j+u,0;a,b). \end{aligned} \tag{74}$$

By using Eq. (36), Eq. (74) becomes:

$$\begin{aligned} M_B(n,j,1;a,b) = \binom{n}{j} & \sum_{u=0}^{\lfloor \frac{n-2j}{2} \rfloor} \binom{n-j}{u} \\ & \times \binom{n-j-u}{j+u} (ab)^{j+u} \\ & \times (a+b)^{n-2j-2u}. \end{aligned} \tag{75}$$

The Eq. (75) leads to Eq. (37).

6.3 A Proof of Eq. (16)

By setting $n := 2n$ in Eq. (75), it follows that:

$$\begin{aligned} M_B(2n,j,1;a,b) = \binom{2n}{j} & \sum_{u=0}^{n-j} \binom{2n-j}{u} \\ & \times \binom{2n-j-u}{j+u} (ab)^{j+u} \\ & \times (a+b)^{2n-2j-2u}. \end{aligned} \tag{76}$$

By using the substitution $u = n-j-s$, Eq. (76) becomes:

$$\begin{aligned} M_B(2n,j,1;a,b) = & \sum_{s=0}^{n-j} \binom{2n}{j} \binom{2n-j}{n+s} \\ & \times \binom{n+s}{n-s} (ab)^{n-s}(a+b)^{2s}. \end{aligned} \tag{77}$$

It is readily verified that

$$\binom{2n}{j}\binom{2n-j}{n+s} = \binom{2n}{n+s}\binom{n-s}{j}. \tag{78}$$

By using Eq. (78), Eq. (77) becomes:

$$\begin{aligned} M_B(2n,j,1;a,b) = & \sum_{s=0}^{n-j} \binom{2n}{n+s} \binom{n-s}{j} \\ & \times \binom{n+s}{2s} (ab)^{n-s}(a+b)^{2s}. \end{aligned} \tag{79}$$

By using Eq. (50) in Eq. (79), it follows that

$$M_B(2n,j,1;a,b) = \sum_{s=0}^{n-j} \left[\frac{\binom{2n}{n}\binom{n}{s}^2}{\binom{2s}{s}} \right] \times (ab)^{n-s}(a+b)^{2s}\binom{n-s}{j}. \tag{80}$$

Let p be a prime number such that divides $a+b$. Due to the fact that ab and $a+b$ are relatively prime, it follows that p does not divide ab. Let j be a fixed non-negative integer such that $j \leq n$.

Obviously,

$$v_p\left(\left[\frac{\binom{2n}{n}\binom{n}{s}^2}{\binom{2s}{s}} \right] (ab)^{n-s}(a+b)^{2s}\binom{n-s}{j} \right) \geq v_p\left(\left[\frac{\binom{2n}{n}\binom{n}{s}^2}{\binom{2s}{s}} \right] \times (ab)^{n-s}(a+b)^{2s} \right). \tag{81}$$

By using Eqns. (52) and (57), we know that

$$v_p([\frac{\binom{2n}{n}\binom{n}{s}^2}{\binom{2s}{s}}](ab)^{n-s}(a+b)^{2s}) \geq v_p(\binom{2n}{n}), \tag{82}$$

where $0 \leq s \leq n-j$.

By using Eqns. (81) and (82), it follows that:

$$v_p([\frac{\binom{2n}{n}\binom{n}{s}^2}{\binom{2s}{s}}](ab)^{n-s}(a+b)^{2s}\binom{n-s}{j}) \geq v_p(\binom{2n}{n}), \tag{83}$$

where $0 \leq s \leq n-j$.

By using Eqns. (80) and (83), it follows that:

$$v_p\big(M_B(2n,j,1;a,b)\big) \geq v_p\left(\binom{2n}{n}\right). \tag{84}$$

where $0 \leq j \leq n$.

Now, by using Relation (13) and the Eq. (84), it follows that:

$$\omega_{a+b}\big(M_B(2n,j,1;a,b)\big) = \min_{\substack{p \text{ prime} \\ p|a+b}} \left\lfloor \frac{v_p\big(M_B(2n,j,1;a,b)\big)}{v_p(a+b)} \right\rfloor \geq \min_{\substack{p \text{ prime} \\ p|a+b}} \left\lfloor \frac{v_p\left(\binom{2n}{n}\right)}{v_p(a+b)} \right\rfloor. \tag{85}$$

By using Eqns. (13) and (85), it follows that:

$$\omega_{a+b}(M_B(2n,j,1;a,b)) \geq \omega_{a+b}(\binom{2n}{n}), \tag{86}$$

where $0 \leq j \leq n$. By Eq. (86), it follows that:

$$(a+b)^{\omega_{a+b}(\binom{2n}{n})} \text{ divides } M_B(2n,j,1;a,b), \tag{87}$$

for any j such that $0 \leq j \leq n$.

Note that $q(2n,a,b) = (a+b)^{\omega_{a+b}(\binom{2n}{n})}$. By Corollary 4, it follows that:

$$(a+b)^{\omega_{a+b}(\binom{2n}{n})} \text{ divides } M_B(2n,j,t;a,b), \tag{88}$$

for any j and for any natural number t such that $0 \leq j \leq n$.

Furthermore, by Corollary 4, it follows that:

$$(a+b)^{\omega_{a+b}(\binom{2n}{n})} \text{ divides } B(2n,m,a,b), \tag{89}$$

for any natural number m such that $m \geq 2$.

The Eq. (89) leads to Eq. (16).

6.4 A Proof of Eq. (17)

By setting $n := 2n+1$ in Eq. (75), it follows that:

$$M_B(2n+1,j,1;a,b) = \binom{2n+1}{j} \sum_{u=0}^{n-j} \binom{2n+1-j}{u} \times \binom{2n+1-j-u}{j+u} (ab)^{j+u} \times (a+b)^{2n+1-2j-2u}. \tag{90}$$

where $0 \leq j \leq n$.

By using the substitution $u = n-j-s$, Eq. (90) becomes, as follows:

$$M_B(2n+1,j,1;a,b) = (a+b) \sum_{s=0}^{n-j} \binom{2n+1}{j} \binom{2n+1-j}{n+s+1} \times \binom{n+s+1}{2s+1} (ab)^{n-s}(a+b)^{2s}. \tag{91}$$

It is readily verified that:

$$\binom{2n+1}{j}\binom{2n+1-j}{n+s+1} = \binom{2n+1}{n+s+1} \times \binom{n-s}{j}. \tag{92}$$

By Eq. (92), Eq. (91) becomes, as follows:

$$M_B(2n+1,j,1;a,b) = (a+b) \sum_{s=0}^{n-j} \binom{2n+1}{n+s+1} \times \binom{n+s+1}{2s+1}\binom{n-s}{j} \times (ab)^{n-s}(a+b)^{2s}. \tag{93}$$

By using Eq. (62) in Eq. (93), it follows that $M_B(2n+1,j,1;a,b)$ is equal to the following sum:

$$(a+b)\sum_{s=0}^{n-j}\left[\frac{(2n+1)\binom{2n}{n}\binom{n}{s}^2}{(2s+1)\binom{2s}{s}}\right] \times \binom{n-s}{j}(ab)^{n-s}(a+b)^{2s}. \tag{94}$$

Let $P(2n+1,a,b)$ denote the sum $\sum_{s=0}^{n-j}[\frac{(2n+1)\binom{2n}{n}\binom{n}{s}^2}{(2s+1)\binom{2s}{s}}]\binom{n-s}{j}(ab)^{n-s}(a+b)^{2s}$. By Eq. (94), it follows that:

$$\omega_{a+b}\big(M_B(2n+1,j,1;a,b)\big) = 1+\omega_{a+b}\big(P(2n+1,a,b)\big). \tag{95}$$

Let p be a prime number such that p divides $a+b$. Obviously,

$$\begin{aligned} &v_p\left(\left[\frac{(2n+1)\binom{2n}{n}\binom{n}{s}^2}{(2s+1)\binom{2s}{s}}\right]\binom{n-s}{j} \times (ab)^{n-s}(a+b)^{2s}\right) \\ &\geq v_p\left(\left[\frac{(2n+1)\binom{2n}{n}\binom{n}{s}^2}{(2s+1)\binom{2s}{s}}\right] \times (ab)^{n-s}(a+b)^{2s}\right). \end{aligned} \tag{96}$$

By using Eqns. (66) and (70), we know that:

$$\begin{aligned} &v_p\left(\left[\frac{(2n+1)\binom{2n}{n}\binom{n}{s}^2}{(2s+1)\binom{2s}{s}}\right] \times (a+b)^{2s}(ab)^{n-s}\right) \\ &\geq v_p\left((2n+1)\binom{2n}{n}\right). \end{aligned} \tag{97}$$

where $0\leq s\leq n$. By Eqns. (96) and (97), it follows that:

$$\begin{aligned} &v_p\left(\left[\frac{(2n+1)\binom{2n}{n}\binom{n}{s}^2}{(2s+1)\binom{2s}{s}}\right]\binom{n-s}{j} \times (ab)^{n-s}(a+b)^{2s}\right) \\ &\geq v_p\left((2n+1)\binom{2n}{n}\right). \end{aligned} \tag{98}$$

where $0\leq s\leq n$. Then, it follows that:

$$v_p(P(2n+1,a,b)) \geq v_p((2n+1)\binom{2n}{n}). \tag{99}$$

Now, by using Relation (13) and Eq. (99), it follows that:

$$\begin{aligned} &\omega_{a+b}\big(P(2n+1,a,b)\big) \\ &= \min_{\substack{p \text{ prime} \\ p|a+b}}\left\lfloor\frac{v_p\big(P(2n+1,a,b)\big)}{v_p(a+b)}\right\rfloor \\ &\geq \min_{\substack{p \text{ prime} \\ p|a+b}}\left\lfloor\frac{v_p\left((2n+1)\binom{2n}{n}\right)}{v_p(a+b)}\right\rfloor. \end{aligned} \tag{100}$$

By using Relation (13) and Eq. (100), it follows that:

$$\omega_{a+b}(P(2n+1,a,b)) \geq \omega_{a+b}((2n+1)\binom{2n}{n}). \tag{101}$$

By Eqns. (95) and (101), it follows that:

$$\omega_{a+b}\big(M_B(2n+1,j,1;a,b)\big) \geq 1+\omega_{a+b}\left((2n+1)\binom{2n}{n}\right). \tag{102}$$

By Eq. (102), it follows that

$$(a+b)^{1+\omega_{a+b}((2n+1)\binom{2n}{n})} \text{ divides } M_B(2n+1,j,1;a,b), \tag{103}$$

for any j such that $0\leq j\leq n$.

Note that $q(2n+1,a,b)=(a+b)^{1+\omega_{a+b}((2n+1)\binom{2n}{n})}$. By Corollary 4, it follows that:

$$(a+b)^{1+\omega_{a+b}((2n+1)\binom{2n}{n})} \text{ divides } M_B(2n+1,j,t;a,b); \tag{104}$$

for any j and for any natural number t such that $0\leq j\leq n$.

Furthermore, by Corollary 4, it follows that:

$$(a+b)^{1+\omega_{a+b}((2n+1)\binom{2n}{n})} \text{ divides } B(2n+1,m,a,b), \tag{105}$$

for any natural number m such that $m\geq 2$.

The Eq. (105) implies Eq. (17). This completes the proof of Theorem 2.

7 A Proof of Corolary 1

It is well-known that:

$$\sum_{k=0}^{n}\binom{n}{k}^2 = \binom{2n}{n}. \tag{106}$$

By Eq. (106), it follows that:

$$B(n,2,1,1) = \binom{2n}{n}. \tag{107}$$

By using Eq. (14) of Theorem 1, it follows that:

$$v_2(B(2n,2,1,1)) = v_2(\binom{2n}{n}). \tag{108}$$

Note that we use notation v_2 instead of ω_2 because 2 is a prime number.

By Eq. (107), it follows that:

$$B(2n,2,1,1) = \binom{4n}{2n}. \tag{109}$$

By using Eq. (109), Eq. (108) becomes

$$v_2(\binom{4n}{2n}) = v_2(\binom{2n}{n}). \tag{110}$$

The last equation above proves the Eq. (19).

Furthermore, by using Eq. (15) of Theorem 1, it follows that:

$$v_2(B(2n+1,2,1,1)) = 1 + v_2((2n+1)\binom{2n}{n}). \tag{111}$$

By Eq. (107), it follows that:

$$B(2n+1,2,1,1) = \binom{4n+2}{2n+1}. \tag{112}$$

By using Eq. (112), Eq. (111) becomes, as follows:

$$\begin{aligned} v_2(\binom{4n+2}{2n+1}) &= 1 + v_2((2n+1)\binom{2n}{n}), \\ &= 1 + v_2(2n+1) + v_2(\binom{2n}{n}), \\ &= 1 + v_2(\binom{2n}{n}). \end{aligned} \tag{113}$$

The last equation above proves the Eq. (18). This completes the proof of Corollary 1.

8 A Proof of Corollary 2

For the proof of Corollary 2, we use Theorem 2. It is known that:

$f_n = B(n,3,1,1)$;

where f_n is the n-th Franel number.

By using Eq. (16), it follows that

$$v_2(f_{2n}) \geq v_2(\binom{2n}{n}).$$

This proves the Eq. (21).

By using Eq. (17), it follows that

$$v_2(f_{2n+1}) \geq 1 + v_2((2n+1)\binom{2n}{n}).$$

Obviously,

$$v_2((2n+1)\binom{2n}{n}) = v_2(\binom{2n}{n}).$$

By using two last equations above, it follows that:

$$v_2(f_{2n+1}) \geq 1 + v_2(\binom{2n}{n}).$$

This proves the Eq. (20).

9 A Proof of Theorem 3

It is known that:

$D_n = B(n,2,1,2)$;

where D_n is n-th central Dellanoy number.

By using Eq. (15), it follows that:

$$\begin{aligned} v_3(D_{2n+1}) &= 1 + v_3((2n+1)\binom{2n}{n}), \\ &= 1 + v_3((2n+1)) + v_3(\binom{2n}{n}). \end{aligned} \tag{114}$$

The last equation above proves Eq. (22).

Note that we use notation v_3 instead of ω_3 because 3 is a prime number.

By using Eq. (14), it follows that:

$$v_3(D_{2n}) = v_3(\binom{2n}{n}).$$

The last equation above proves Eq. (23).

This completes the proof of Theorem 3.

10 Proofs of Theorem 5 and Corollary 3

Proof of Theorem 5 directly follows from the our proof of Theorem 1. Just replace ab with a, and $a+b$ with b in the proof of Theorem 1.

Also Corollary 3 follows from Theorem 1 and Eq. (3).

11 A Proof of Theorem 6

Proof of Theorem 6 is similar to the proof of Theorem 1.

Note that:

$$M_n(a,b) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{k}\binom{n-k}{k} \cdot \frac{1}{k+1} \cdot a^k \cdot b^{n-2k}. \tag{115}$$

11.1 A Proof of Eq. (30)

By setting $n := 2n$ in Eq. (115), it follows that:

$$M_{2n}(a,b) = \sum_{k=0}^{n} \binom{2n}{k}\binom{2n-k}{k} \cdot \frac{1}{k+1} \cdot a^k \cdot b^{2n-2k}. \tag{116}$$

By using the substitution $k = n-s$, Eq. (116) becomes gradually:

$$\begin{aligned} & M_{2n}(a,b) \\ &= \sum_{s=0}^{n} \binom{2n}{n-s}\binom{n+s}{n-s} \cdot \frac{1}{n+1-s} \cdot b^{2s} \cdot a^{n-s}, \\ &= \sum_{s=0}^{n} \binom{2n}{n+s}\binom{n+s}{2s} \cdot \frac{1}{n+1-s} \cdot b^{2s} \cdot a^{n-s}. \end{aligned} \tag{117}$$

By using Eq. (50), Eq. (117) becomes, as follows:

$$M_{2n}(a,b)=\sum_{s=0}^{n}\Big(\frac{\binom{2n}{n}\binom{n}{s}^2}{\binom{2s}{s}}\Big)\cdot\frac{1}{n+1-s}\cdot b^{2s}\cdot a^{n-s}. \tag{118}$$

It is readily verified that:

$$\binom{n}{s}\cdot\frac{1}{n+1-s}=\frac{1}{n+1}\cdot\binom{n+1}{s}. \tag{119}$$

By using Eq. (119), Eq. (118) becomes, as follows:

$$M_{2n}(a,b)=\sum_{s=0}^{n}\Big(\frac{C_n\binom{n}{s}\binom{n+1}{s}}{\binom{2s}{s}}\Big)\cdot b^{2s}\cdot a^{n-s}. \tag{120}$$

Let p be a prime number such that divides b. For $s=0$, the first summand in the right-side of the Eq. (120) is $C_n\cdot a^n$. Since a and b are relatively prime, it follows that:

$$v_p(C_n\cdot a^n)=v_p(C_n). \tag{121}$$

Let $1\leq s\leq n$. Then, for the $(s+1)$-th summand in the right-side of Eq. (120), we have:

$$v_p\left(\frac{C_n\binom{n}{s}\binom{n+1}{s}}{\binom{2s}{s}}b^{2s}a^{n-s}\right)\geq v_p(C_n)-v_p\left(\binom{2s}{s}\right)+2s\,v_p(b). \tag{122}$$

By using Ineq. (53), from Eq. (122), we obtain that:

$$v_p\left(\frac{C_n\binom{n}{s}\binom{n+1}{s}}{\binom{2s}{s}}b^{2s}a^{n-s}\right)\geq v_p(C_n)+s\big(2v_p(b)-1\big). \tag{123}$$

Obviously, for $1\leq s\leq n$,

$$v_p(C_n)+s\cdot(2v_p(b)-1)\geq v_p(C_n)+s\geq v_p(C_n)+1. \tag{124}$$

By Ineqs. (123) and (124), for $1\leq s\leq n$, it follows that:

$$v_p\Big(\Big(\frac{C_n\binom{n}{s}\binom{n+1}{s}}{\binom{2s}{s}}\Big)\cdot b^{2s}\cdot a^{n-s}\Big)\geq v_p(C_n)+1. \tag{125}$$

By using Eqns. (121) and (125), we obtain that:

$$v_p(M_{2n}(a,b))=v_p(C_n), \tag{126}$$

where p is a prime number that divides b. Furthermore, by using Relation (13), it follows that:

$$\begin{aligned}\omega_b\big(M_{2n}(a,b)\big)&=\min_{\substack{p\text{ prime}\\ p|b}}\left\lfloor\frac{v_p\big(M_{2n}(a,b)\big)}{v_p(b)}\right\rfloor\\&=\min_{\substack{p\text{ prime}\\ p|b}}\left\lfloor\frac{v_p(C_n)}{v_p(b)}\right\rfloor\\&=\omega_b(C_n).\end{aligned} \tag{127}$$

The Eq. (127) completes the proof of Eq. (30).

11.2 A Proof of Eq. (31)

By setting $n:=2n+1$ in Eq. (115), we obtain that:

$$\begin{aligned}M_{2n+1}(a,b)&=\sum_{k=0}^{n}\binom{2n+1}{k}\binom{2n+1-k}{k}\frac{1}{k+1}\\&\quad\times a^k b^{2n+1-2k}\\&=b\sum_{k=0}^{n}\binom{2n+1}{k}\binom{2n+1-k}{k}\frac{1}{k+1}\\&\quad\times a^k b^{2n-2k}.\end{aligned} \tag{128}$$

By using the substitution $k=n-s$, Eq. (128) becomes gradually:

$$\begin{aligned}&M_{2n+1}(a,b)\\&\quad=b\sum_{s=0}^{n}\binom{2n+1}{n-s}\binom{n+s+1}{n-s}\frac{1}{n-s+1}\\&\qquad\times b^{2s}a^{n-s}\\&\quad=b\sum_{s=0}^{n}\binom{2n+1}{n+s+1}\binom{n+s+1}{2s+1}\frac{1}{n+1-s}\\&\qquad\times b^{2s}a^{n-s}.\end{aligned} \tag{129}$$

By using Eq. (62), Eq. (129) becomes, as follows:

$$\begin{aligned}M_{2n+1}(a,b)&=b\sum_{s=0}^{n}\left[\frac{(2n+1)\binom{2n}{n}\binom{n}{s}^2}{(2s+1)\binom{2s}{s}}\right]\\&\quad\times\frac{1}{n+1-s}b^{2s}a^{n-s}.\end{aligned} \tag{130}$$

By using Eq. (119), it follows that

$$\begin{aligned}M_{2n+1}(a,b)&=b\sum_{s=0}^{n}\left[\frac{(2n+1)C_n\binom{n}{s}\binom{n+1}{s}}{(2s+1)\binom{2s}{s}}\right]\\&\quad\times b^{2s}a^{n-s}.\end{aligned} \tag{131}$$

The rest of the proof is similar to the proof of Eq. (15) of Theorem 1

Let p be a prime that divides b. Then, by using Lemma 1, it can be shown that:

$$v_p\left(\sum_{s=0}^{n}\left[\frac{(2n+1)C_n\binom{n}{s}\binom{n+1}{s}}{(2s+1)\binom{2s}{s}}\right]\quad\times b^{2s}a^{n-s}\right)=v_p\big((2n+1)C_n\big). \tag{132}$$

This completes the proof of Theorem 6.

12 A Proof of Theorem 4

We use Theorem 6 and Eq. (12). By setting $a:=2$ and $b:=3$ in the Eq. (30), it follows that:

$$v_3(M_{2n}(2,3))=v_3(C_n). \tag{133}$$

By Eq. (12), it follows that

$$s_{2n+1} = M_{2n}(2,3). \tag{134}$$

By using Eqns. (133) and (134), it follows that:

$$v_3(s_{2n+1}) = v_3(C_n), \tag{135}$$

where n is a non-negative integer. The Eq. (24) follows from Eq. (135) and the fact that $v_3(S_n) = v_3(s_n)$.

Similarly, by setting $a := 2$ and $b := 3$ in Eq. (31), it follows that:

$$v_3(M_{2n+1}(2,3)) = 1 + v_3((2n+1)\cdot C_n). \tag{136}$$

By Eq. (12), it follows that

$$s_{2n+2} = M_{2n+1}(2,3). \tag{137}$$

By using Eqns. (136) and (137), it follows that:

$$v_3(s_{2n+2}) = 1 + v_3(2n+1) + v_3(C_n), \tag{138}$$

where n is a non-negative integer.

Now, Eq. (25) follows from Eq. (138) and the fact that $v_3(S_n) = v_3(s_n)$.

This completes the proof of Theorem 4.

Remark 3 Theorem 4 can be almost proved by using Theorem 3, Eqns. (5), (6), and Kummer's theorem. See [18].

13 Concluding Remarks

Remark 4 Let us consider the following sum $B(n,m,1,-1) = \sum_{k=0}^{n} \binom{n}{k}^m (-1)^k$, where n and m are natural numbers such that $m \geq 2$. Obviously,

$$B(2n-1,m,1,-1) = 0. \tag{139}$$

By using Eq. (7) and applying Kummer's formula, it follows that

$$B(2n,2,1,-1) = (-1)^n \binom{2n}{n}. \tag{140}$$

By using Eq. (8) and applying Dixon's formula, it follows that

$$B(2n,3,1,-1) = (-1)^n \binom{2n}{n}\binom{3n}{2n}. \tag{141}$$

In 1998, Calkin proved [2, Thm. 1, p. 1] that the alternating binomial sum $B(2n,m,1,-1)$ is divisible by $\binom{2n}{n}$ for all non-negative integers n and all positive integers m. In 2007, Guo, Jouhet, and Zeng proved, among other things, two generalizations of Calkin's result [6, Thm. 1.2, Thm. 1.3, p. 2]. In 2010, Cao and Pan [3] proved an extension of Calkin's result [2, Thm. 1]. In 2020, Calkin's result [2, Thm. 1] was proved by using new class of binomial sums [20, Section 5]. See also [17, Section 8, p. 14].

Let a and b be relatively prime integers such that $a+b \neq 0$. Note that:

$$B(n,m,a,b) \equiv a^n \cdot B(n,m,1,-1) \mod (a+b). \tag{142}$$

By Eqns. (139) and (142), it follows that $B(2n-1,m,a,b)$ is always divisible by $a+b$. By Eqns. (140) and (142), it follows that $B(2n,2,a,b)$ is divisible by $a+b$ if and only if $\binom{2n}{n}$ is divisible by $a+b$. Similarly, by Eqns. (141) and (142), it follows that $B(2n,3,a,b)$ is divisible by $a+b$ if and only if $\binom{2n}{n}\binom{3n}{n}$ is divisible by $a+b$.

Furthermore, by Calkin's result [2, Thm. 1, p. 1], if $\binom{2n}{n}$ is divisible by $a+b$ then $B(2n,m,a,b)$ is divisible by $a+b$ for all natural numbers n and m.

Remark 5 It is known [33, page 2] and [34, Eq. (1.5)] that

$$H_n = M_n(1,3), \tag{143}$$

where H_n are restricted hexagonals numbers [7]. By using Theorem 6, it can be shown that:

$$\begin{aligned} v_3(H_{2n}) &= v_3(C_n), \\ v_3(H_{2n+1}) &= 1 + v_3(2n+1) + v_3(C_n). \end{aligned} \tag{144}$$

Remark 6 It is known [33, page 2] that

$$C_{n+1} = M_n(1,2), \tag{145}$$

where C_n denotes the nth Catalan number. By using Theorem 6, it can be shown that:

$$\begin{aligned} v_2(C_{2n+1}) &= v_2(C_n), \\ v_2(C_{2n+2}) &= 1 + v_2(C_n). \end{aligned} \tag{146}$$

Remark 7 By using Theorem 1 and Kummer's theorem [16, Theorem 1. 2, p. 2], we can efficiently calculate $\omega_{a+b}(B(n,2,a,b))$.

Let us consider the following sum:

$$B(4046,2,37,62) = \sum_{k=0}^{4046} \binom{4046}{k}^2 \times 37^{4046-k} 62^k. \tag{147}$$

Here $a := 37$, $b := 62$, $2n := 4046$, and $a+b := 99$. Obviously, $\gcd(37,62) = 1$.

By Eq. (14) of Theorem 1, it follows that:

$$\omega_{99}\left(B(4046,2,37,62)\right) = \omega_{99}\left(\binom{4046}{2023}\right). \tag{148}$$

Since $99 = 3^2 \cdot 11$, it follows $v_3(99) = 2$ and $v_{11}(99) = 1$. We need to find $v_3\left(\binom{4046}{2023}\right)$ and $v_{11}\left(\binom{4046}{2023}\right)$.

The expansion of 2023 in the base 3 is $(2202221)_3$. If we add the number $(2202221)_3$ to itself, there are five 'carry-overs' in total. By Kummer's theorem,

$$v_3\left(\binom{4046}{2023}\right) = 5. \tag{149}$$

The expansion of 2023 in the base 11 is $(157A)_{11}$. If we add the number $(157A)_{11}$. to itself, there are three 'carry-overs' in total. By Kummer's theorem,

$$v_{11}\left(\binom{4046}{2023}\right) = 3. \tag{150}$$

By Eq. (13), it follows that

$$\begin{aligned}\omega_{99}\left(\binom{4046}{2023}\right) &= \min\left\{\left\lfloor\frac{v_{11}\left(\binom{4046}{2023}\right)}{v_{11}(99)}\right\rfloor, \right. \\ &\qquad \left.\left\lfloor\frac{v_3\left(\binom{4046}{2023}\right)}{v_3(99)}\right\rfloor\right\} \\ &= \min\left\{\left\lfloor\frac{3}{1}\right\rfloor, \left\lfloor\frac{5}{2}\right\rfloor\right\} \\ &= \min\{3,2\} \\ &= 2.\end{aligned} \tag{151}$$

By Theorem 1, it follows that

$$\omega_{99}\big(B(4046,2,37,62)\big) = 2. \tag{152}$$

Hence, the sum $B(4046,2,37,62)$ is divisible by 99^2 and not divisible by 99^3. Moreover, by Theorem 2, it follows that the sum $B(4046,m,37,62)$ is always divisible by 99^2 for all natural numbers m such that $m \geq 2$.

Also, let us consider the following sum

$$\begin{aligned}B(4047,2,37,62) = &\sum_{k=0}^{4047}\binom{4047}{k}^2 \\ &\times 37^{4047-k}62^k.\end{aligned} \tag{153}$$

By Eq. (15) of Theorem 1, it follows that:

$$\begin{aligned}&\omega_{99}\big(B(4047,2,37,62)\big) \\ &\qquad = 1+\omega_{99}\left(4047\binom{4046}{2023}\right).\end{aligned} \tag{154}$$

We have

$$\begin{aligned}v_3\left(4047\binom{4046}{2023}\right) &= v_3(4047)+v_3\left(\binom{4046}{2023}\right) \\ &= 1+5=6.\end{aligned} \tag{155}$$

Furthermore,

$$\begin{aligned}v_{11}\left(4047\binom{4046}{2023}\right) &= v_{11}(4047)+v_{11}\left(\binom{4046}{2023}\right) \\ &= 0+3=3.\end{aligned} \tag{156}$$

By Eq. (13), it follows that:

$$\begin{aligned}\omega_{99}\left(4047\binom{4046}{2023}\right) &= \min\left\{\left\lfloor\frac{v_{11}\left(4047\binom{4046}{2023}\right)}{v_{11}(99)}\right\rfloor, \right. \\ &\qquad \left.\left\lfloor\frac{v_3\left(4047\binom{4046}{2023}\right)}{v_3(99)}\right\rfloor\right\} \\ &= \min\left\{\left\lfloor\frac{3}{1}\right\rfloor, \left\lfloor\frac{6}{2}\right\rfloor\right\} \\ &= \min\{3,3\} = 3.\end{aligned} \tag{157}$$

By Eq. (15) of Theorem 1, it follows that:

$$\begin{aligned}\omega_{99}\big(B(4047,2,37,62)\big) &= 1+\omega_{99}\left(4047\binom{4046}{2023}\right) \\ &= 1+3=4.\end{aligned} \tag{158}$$

It follows that:

$$\omega_{99}\big(B(4047,2,37,62)\big) = 4. \tag{159}$$

Hence, the sum $B(4047,2,37,62)$ is divisible by 99^4 and not divisible by 99^5.

Moreover, by Theorem 2, it follows that the sum $B(4047,m,37,62)$ is always divisible by 99^4 for all natural numbers m such that $m \geq 2$.

Remark 8 By using Eqns. (35) and (37), we can derive a slight generalization of MacMahon's formula (8). It can be shown that:

$$\begin{aligned}M_B(n,j,2;a,b) = &\binom{n}{j}\sum_{l=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n}{2l}\binom{2l}{l} \\ &\times\binom{n+l-j}{n}(ab)^l(a+b)^{n-2l}.\end{aligned} \tag{160}$$

By setting $j := 0$ in Eq. (160), Eq. (8) follows.

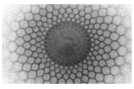
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TTMP vol 3 (3), 2026
doi: 10.30511/TTMP.2026.2095466.1086

Transactions in Theoretical and Mathematical Physics



Quantitative Non-Unique Čirić Fixed-Point Theory under Residual-Sensitive C-Class Controls

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Received: 24 July 2026 / Accepted: 21 August 2026 / Published: 24 August 2026

Abstract Building on the non-unique fixed-point theory of Čirić and its known extensions by c-comparison functions, we introduce a residual-sensitive three-variable C-class condition designed according to an orbit-extraction principle. The global condition involves the mutual distance and the two pointwise residuals, but on a Picard pair (x, Tx) it collapses to the scalar recurrence $d(Tx, T^2x) \leq \phi\big(d(x, Tx)\big)$, which governs summable decay of successive Picard displacements. This structural hypothesis and its diagonal reduction, rather than the use of a nonlinear comparison function by itself, constitute the principal contribution. For an orbitally continuous self-map on a T-orbitally complete metric space, the recurrence makes the map weakly Picard and yields both an a priori tail estimate and a genuine a posteriori estimate in terms of the current residual $d(T^n x, T^{n+1}x)$. These bounds induce an explicit retraction-displacement modulus. Consequently, we obtain set-wise generalized Ulam stability, well-posedness with respect to the fixed-point set, and a Hausdorff data-dependence estimate for the entire fixed-point set under perturbations of the operator. A nonlinear example with two fixed points exhibits non-geometric residual decay and cannot satisfy a corresponding constant-coefficient inequality with $q < 1$. A second example shows that the linear Hausdorff perturbation bound is sharp. The results place residual-sensitive C-class conditions within the quantitative theory of weakly Picard operators without imposing uniqueness.

Keywords: non-unique fixed point, Čirić mapping, c-comparison function, weakly Picard operator, C-class control, a posteriori error estimate, Ulam stability, data dependence.

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1 Introduction

Banach's contraction principle remains the basic model for constructive fixed-point theory [3], but its uniform coefficient forces uniqueness and excludes many nonuniform iterative mechanisms. Classical alternatives due to Kannan, Chatterjea, Rakotch and Čirić substantially broadened the admissible contractive structure. Of particular relevance here is Čirić's non-unique framework, in which a minimum-distance inequality is compatible with several fixed points and nevertheless forces Picard convergence [6]. Non-unique Čirić theory has since been extended through rational and hybrid inequalities, interpolation, generalized metric spaces and nonlinear comparison functions; see, for example, [1, 7–11]. An important point for the present paper is that this literature already provides a replacement of a constant factor $q < 1$ with a nonlinear comparison function. In particular, the b-metric result of Alsulami, Karapınar and Rakočević [1] contains, in the metric case, the estimate

$$m_T(x, y) - c_T(x, y) \leq \phi\big(d(x, y)\big). \tag{1}$$

with a c-comparison function ϕ. Thus, nonlinear control alone is not the contribution sought here.

A second relevant line of development is the theory of weakly Picard operators. A map is weakly Picard when every Picard orbit converges to some fixed point, not necessarily the same one. This point of view naturally accommodates non-unique fixed points and leads to retraction-displacement estimates, well-posedness, Ulam-type stability and data dependence [4, 12–14]. These quantitative consequences are particularly useful when the fixed-point set is not a singleton.

C-class functions provide another flexible language for writing generalized contractive inequalities. They originated as two-variable controls and continue to appear in recent fixed-point results in various settings [2, 15]. In a non-unique

Čirić argument, however, an unrestricted multivariable C-class function does not by itself give the summable recurrence needed to control the Picard tail. The decisive information is its behavior on the configuration generated by two consecutive iterates.

The motivation for the residual-sensitive formulation comes from a simple distinction between *global verification* and *orbital convergence*. As regards global, a generalized contractive condition may naturally depend not only on $d(x, y)$, but also on the residuals $d(x, Tx)$ and $d(y, Ty)$, which measure the local displacement of each point under the operator. These quantities can reflect nonuniform behavior that a single constant coefficient cannot represent. Along a Picard orbit, however, convergence is governed by the successive residuals

$$r_n = d\big(T^n x, T^{n+1}x\big). \tag{2}$$

The useful design principle is therefore to allow residual dependence in the global inequality while requiring that, on the Picard pair (x, Tx), the auxiliary three-variable control reduces to a one-dimensional comparison rule for r_{n+1} in terms of r_n. This principle explains the diagonal requirement introduced below. When $y = Tx$, the first two inputs of the auxiliary function coincide, because $d(x, y) = d(x, Tx)$, and the crossing term in the Čirić inequality vanishes. The condition $F(t, t, s) \le \phi(t)$ then extracts the recurrence

$$d\big(Tx, T^2x\big) \le \phi\big(d(x, Tx)\big). \tag{3}$$

Thus, the three-variable hypothesis is not introduced merely as another generalized contraction. It is a globally verifiable structural condition deliberately shaped so that its restriction to consecutive iterates yields the orbital recurrence required for quantitative convergence. The paper's logical structure follows this motivation. We first isolate the orbital summability principle and derive a priori and a posteriori Picard estimates from the scalar residual recurrence. We then prove that the residual-sensitive C-class condition reduces to that recurrence. The principal novelty therefore lies in the structural hypothesis together with this reduction mechanism. The retraction-displacement estimate, generalized Ulam stability, set-wise well-posedness, and Hausdorff data dependence follow from the resulting quantitative weakly Picard framework. The linear specialization is shown to be sharp. We also include a nonlinear two-fixed-point example for which the effective residual ratio tends to one, so no uniform coefficient $q < 1$ is possible. We do not claim that this example lies outside all previously known nonlinear comparison-function conditions; it illustrates why a residual-sensitive global formulation and a current-residual error estimate are natural in a non-geometric regime.

2 Preliminaries and residual-sensitive controls

Throughout, (X, d) is a metric space and $T : X \to X$ is a self-map. We write

$$\begin{aligned} &\mathrm{Fix}(T) = \{x \in X : Tx = x\}, \\ &O(x, T) = \{T^n x : n \in \mathbb{N} \cup \{0\}\}. \end{aligned} \tag{4}$$

For an initial point $x_0 \in X$, let $x_n = T^n x_0$ and define the *Picard residual sequence*

$$r_n = d(x_n, x_{n+1}) = d\big(T^n x_0, T^{n+1}x_0\big), \qquad n \ge 0. \tag{5}$$

The term *residual-sensitive* refers to a contractive condition that records pointwise displacements such as $d(x, Tx)$ and $d(y, Ty)$ and is designed so that, when $y = Tx$, it yields a recurrence controlling r_{n+1} by r_n.

Definition 1 The metric space (X, d) is called T-orbitally complete if every Cauchy sequence contained in an orbit $O(x, T)$ converges to a point of X.

Definition 2 The map T is called orbitally continuous if, whenever $T^{n_j}x \to z$ for some $x \in X$ and subsequence $\{n_j\}$, one has $T^{n_j+1}x \to Tz$.

Ordinary completeness implies orbital completeness, and ordinary continuity implies orbital continuity.

Definition 3 A map $\phi : [0, \infty) \to [0, \infty)$ is called a c-comparison function if ϕ is nondecreasing, $\phi(0) = 0$, and

$$\Theta_\phi(t) := \sum_{k=0}^{\infty} \phi^k(t) < \infty \qquad (t > 0), \tag{6}$$

where ϕ^k denotes the k-fold iterate.

This formulation is equivalent to the summability property commonly associated with c-comparison functions in metric fixed-point theory; see [1, 5, 12]. In particular,

$$\phi(t) < t \qquad (t > 0), \tag{7}$$

and Θ_ϕ is nondecreasing and continuous at zero. The linear choice $\phi(t) = qt$, $0 \le q < 1$, gives

$$\Theta_\phi(t) = \frac{t}{1 - q}. \tag{8}$$

Definition 4 A continuous function $F : [0, \infty)^3 \to [0, \infty)$ is called a three-variable modified C-class function if

$$F(a, b, c) \le a \qquad (a, b, c \ge 0), \tag{9}$$

and

$$F(a, b, c) = a \quad \Longrightarrow \quad a = 0 \text{ or } b = 0 \text{ or } c = 0. \tag{10}$$

Given a c-comparison function ϕ, the function F is called ϕ-diagonally admissible if

$$F(t, t, s) \le \phi(t) \qquad (t, s \ge 0). \tag{11}$$

The terminology in (9)–(10) is motivated by the C-class principle of Ansari [2]. The additional condition (11) is the residual-sensitive requirement: it prescribes exactly what the auxiliary function must do on the diagonal configuration generated by a Picard pair.

Proposition 1 *Let ϕ be a c-comparison function. Then*

$$F_\phi(a,b,c) = \phi(a) + \big(a - \phi(a)\big)\frac{|a-b|}{1+|a-b|}\frac{c}{1+c} \tag{12}$$

is a ϕ-diagonally admissible three-variable modified C-class function.

Proof Continuity and nonnegativity are immediate. Since both fractions in (12) belong to $[0,1)$,

$$\phi(a) \le F_\phi(a,b,c) \le a. \tag{13}$$

If $a > 0$, then $\phi(a) < a$ and the product of the two fractions is strictly smaller than one, so $F_\phi(a,b,c) < a$. Hence equality with a can occur only when $a = 0$, which is stronger than (10). Finally, $b = a$ gives $F_\phi(a,a,c) = \phi(a)$.

For $x, y \in X$, define the Čirić minimum terms

$$m_T(x,y) = \min\{d(Tx,Ty), d(x,Tx), d(y,Ty)\}, \tag{14}$$

$$c_T(x,y) = \min\{d(x,Ty), d(y,Tx)\}. \tag{15}$$

Definition 5 Let ϕ be a c-comparison function and let F be ϕ-diagonally admissible. We call T an (F,ϕ) residual-sensitive non-unique Čirić map if

$$m_T(x,y) \le F\big(d(x,y), d(x,Tx), d(y,Ty)\big) + c_T(x,y) \tag{16}$$

for all $x, y \in X$.

Condition (16) embodies the orbit-extraction principle described in the introduction. Its three inputs retain information about the separation of x and y and their individual residuals, so the inequality can be checked globally without requiring a uniform contraction ratio. At the same time, the diagonal admissibility of F is tuned to the single configuration needed in the Picard argument. If $u, v \in \mathrm{Fix}(T)$, then $m_T(u,v) = 0$, so (16) does not constrain the distance between distinct fixed points. This is why uniqueness is not built into the condition.

3 Orbital summability and Picard convergence

We begin with the mechanism that actually drives the iterative argument.

Theorem 1 (Orbital summability principle) *Let $T : X \to X$ be orbitally continuous and let (X,d) be T-orbitally complete. Assume that there exists a c-comparison function ϕ such that*

$$d(Tx,T^2x) \le \phi\big(d(x,Tx)\big) \qquad (x \in X). \tag{17}$$

Then T is a weakly Picard operator: for every $x_0 \in X$, the Picard sequence $x_{n+1} = Tx_n$ converges to a point $P_T x_0 \in \mathrm{Fix}(T)$.

If $r_n = d(x_n, x_{n+1})$, then

$$r_n \le \phi^n(r_0), \qquad n \ge 0, \tag{18}$$

and the following estimates hold:

$$d(x_n, P_T x_0) \le \sum_{j=n}^{\infty} \phi^j(r_0) = \Theta_\phi\big(\phi^n(r_0)\big), \tag{19}$$

$$d(x_n, P_T x_0) \le \Theta_\phi(r_n). \tag{20}$$

Equation (19) is an a priori estimate, while (20) is an a posteriori estimate based on the current Picard residual.

Proof If $r_n = 0$ for some n, then $x_n \in \mathrm{Fix}(T)$ and the orbit is constant thereafter. Otherwise, (17) applied at x_n gives

$$r_{n+1} \le \phi(r_n). \tag{21}$$

Monotonicity of ϕ yields (18) by induction. For $m > n$, the triangle inequality gives

$$d(x_n, x_m) \le \sum_{j=n}^{m-1} r_j \le \sum_{j=n}^{m-1} \phi^j(r_0). \tag{22}$$

The series in (6) has a convergent tail, hence $\{x_n\}$ is Cauchy. Orbital completeness gives a limit $p \in X$. Orbital continuity implies

$$x_{n+1} = Tx_n \longrightarrow Tp. \tag{23}$$

while $x_{n+1} \to p$, so $Tp = p$. Set $P_T x_0 = p$.

Letting $m \to \infty$ in the preceding estimate gives (19). Starting instead at the observed residual r_n, repeated use of (17) yields

$$r_{n+k} \le \phi^k(r_n), \qquad k \ge 0. \tag{24}$$

so

$$d(x_n, P_T x_0) \le \sum_{k=0}^{\infty} r_{n+k} \le \sum_{k=0}^{\infty} \phi^k(r_n) = \Theta_\phi(r_n). \tag{25}$$

which is (20).

The next proposition is the structural bridge between the global three-variable inequality and the scalar recurrence that controls the orbit.

Proposition 2 (Diagonal reduction to the residual recurrence) *Let T satisfy (16). Then*

$$d(Tx,T^2x) \le \phi\big(d(x,Tx)\big) \qquad (x \in X). \tag{26}$$

Proof Fix $x \in X$ and put

$$r = d(x, Tx), \qquad s = d\big(Tx, T^2x\big). \tag{27}$$

If $r = 0$, then $Tx = x$ and $s = 0$. Suppose $r > 0$. Apply (16) to the pair (x, Tx). Since

$$c_T(x, Tx) = \min\big\{d(x, T^2x), d(Tx, Tx)\big\} = 0. \tag{28}$$

and

$$m_T(x, Tx) = \min\{s, r, s\} = \min\{r, s\}. \tag{29}$$

we obtain

$$\min\{r, s\} \le F(r, r, s) \le \phi(r). \tag{30}$$

If $s \ge r$, then $r \le \phi(r) < r$, a contradiction. Hence $s < r$, and therefore $s \le \phi(r)$.

Proposition 2 identifies the mathematical core of the framework. The global inequality (16) is allowed to contain three variables and residual information, but the Picard substitution $y = Tx$ forces the crossing term to vanish and the first two arguments of F to coincide. Diagonal admissibility then removes the auxiliary structure and leaves exactly the scalar recurrence (17). The subsequent quantitative conclusions depend on this recurrence through Theorem 1.

Theorem 2 (Quantitative non-unique Čirić theorem) *Let $T : X \to X$ be orbitally continuous on a T-orbitally complete metric space. If T satisfies (16) for a c-comparison function ϕ and a ϕ-diagonally admissible modified C-class function F, then all conclusions of Theorem 1 hold.*

Proof Proposition 2 gives (17), so Theorem 1 applies.

Corollary 1 (Classical scalar comparison form) *Under the orbital assumptions above, suppose*

$$m_T(x, y) - c_T(x, y) \le \phi\big(d(x, y)\big) \tag{31}$$

for all $x, y \in X$, where ϕ is a c-comparison function. Then T is weakly Picard and satisfies (18)–(20).

Proof Take $F(a, b, c) = \phi(a)$. Then (16) is exactly (31), and F is ϕ-diagonally admissible.

Remark 1 Corollary 1 is an explicitly quantitative metric-space form of the nonlinear non-unique Čirić mechanism already present in [1]. The role of Theorem 2 is not merely to replace a constant coefficient by a comparison function. Its new element is the residual-sensitive structural condition and the proof that this globally stated condition reduces, on Picard pairs, to the recurrence that activates the quantitative weakly Picard theory developed below.

Corollary 2 (Constant coefficient) *If $\phi(t) = qt$, $0 \le q < 1$, then*

$$d(T^n x, P_T x) \le \frac{q^n}{1 - q} d(x, Tx) \tag{32}$$

and

$$d(T^n x, P_T x) \le \frac{d(T^n x, T^{n+1} x)}{1 - q}. \tag{33}$$

Corollary 3 (Absence of nontrivial periodic points) *Under the assumptions of Theorem 2, every periodic point of T is a fixed point.*

Proof If x has prime period $k > 1$, then all residuals along the cycle are positive and satisfy $r_{n+1} < r_n$. After one full period this gives

$$r_0 > r_1 > \cdots > r_k = r_0. \tag{34}$$

which is impossible.

4 Retraction-displacement, stability and data dependence

The results in this section are unified by one estimate. The a posteriori bound in Theorem 2 first gives

$$d(x, P_T x) \le \Theta_\phi\big(d(x, Tx)\big). \tag{35}$$

where $P_T x$ is the Picard limit. Since $P_T x \in \mathrm{Fix}(T)$, the same inequality bounds the distance from any approximate fixed point to the entire fixed-point set. Generalized Ulam stability and well-posedness follow by applying this estimate to points with small residual, while Hausdorff data dependence follows by viewing a fixed point of one operator as an approximate fixed point of the perturbed operator. Thus, Proposition 3 and Theorems 3–4 are different consequences of a single residual-to-solution-set modulus.

Proposition 3 (Picard limit map and retraction-displacement) *Under the assumptions of Theorem 2, define*

$$P_T : X \to \mathrm{Fix}(T), \qquad P_T x = \lim_{n \to \infty} T^n x. \tag{36}$$

Then

$$P_T \circ T = P_T, \qquad P_T|_{\mathrm{Fix}(T)} = I_{\mathrm{Fix}(T)}, \qquad P_T^2 = P_T, \tag{37}$$

and

$$d(x, P_T x) \le \Theta_\phi\big(d(x, Tx)\big) \qquad (x \in X). \tag{38}$$

Thus, P_T is a set retraction onto $\mathrm{Fix}(T)$ *and satisfies an explicit retraction-displacement estimate.*

Proof The identities in (37) follow from the fact that the orbits of x and Tx differ only in their first term and that fixed points generate constant orbits. Estimate (38) is (20) with $n = 0$.

The wording "set retraction" is used because continuity of P_T is not assumed; this agrees with the weakly Picard framework in [13, 14].

Definition 6 For a nonempty subset $A \subseteq X$, let

$$\operatorname{dist}(x, A) = \inf_{a \in A} d(x, a). \tag{39}$$

We say that the fixed-point problem for T is set-wise well posed if every sequence $\{z_n\}$ satisfying $d(z_n, Tz_n) \to 0$ also satisfies

$$\operatorname{dist}\big(z_n, \operatorname{Fix}(T)\big) \longrightarrow 0. \tag{40}$$

We say that it is generalized Ulam stable with modulus Ω if

$$d(x, Tx) \le \varepsilon \quad \Longrightarrow \quad \operatorname{dist}\big(x, \operatorname{Fix}(T)\big) \le \Omega(\varepsilon). \tag{41}$$

Theorem 3 (Generalized Ulam stability and well-posedness) *Under the assumptions of Theorem 2, the fixed-point problem is generalized Ulam stable with modulus* Θ_ϕ*:*

$$d(x, Tx) \le \varepsilon \quad \Longrightarrow \quad \operatorname{dist}(x, \operatorname{Fix}(T)) \le \Theta_\phi(\varepsilon). \tag{42}$$

It is also set-wise well posed.

Proof Since $P_T x \in \operatorname{Fix}(T)$, Proposition 3 gives

$$\operatorname{dist}\big(x, \operatorname{Fix}(T)\big) \le d(x, P_T x) \le \Theta_\phi\big(d(x, Tx)\big). \tag{43}$$

Monotonicity yields (42). If $d(z_n, Tz_n) \to 0$, then

$$\operatorname{dist}\big(z_n, \operatorname{Fix}(T)\big) \le \Theta_\phi\big(d(z_n, Tz_n)\big) \longrightarrow 0. \tag{44}$$

because Θ_ϕ is continuous at zero.

For perturbations, define the excess and Hausdorff distance between nonempty sets $A, B \subseteq X$ by

$$\begin{aligned} \operatorname{exc}(A, B) &= \sup_{a \in A} \operatorname{dist}(a, B), \\ H(A, B) &= \max\big\{\operatorname{exc}(A, B), \operatorname{exc}(B, A)\big\}. \end{aligned} \tag{45}$$

Theorem 4 (Hausdorff data dependence of the fixed-point set) *Let $T, S : X \to X$ each satisfy the assumptions of Theorem 2, with c-comparison functions ϕ_T and ϕ_S, respectively. Suppose*

$$\delta := \sup_{x \in X} d(Tx, Sx) < \infty. \tag{46}$$

Then

$$\operatorname{exc}(\operatorname{Fix}(T), \operatorname{Fix}(S)) \le \Theta_{\phi_S}(\delta), \tag{47}$$

$$\operatorname{exc}(\operatorname{Fix}(S), \operatorname{Fix}(T)) \le \Theta_{\phi_T}(\delta), \tag{48}$$

and consequently

$$H(\operatorname{Fix}(T), \operatorname{Fix}(S)) \le \max\{\Theta_{\phi_T}(\delta), \Theta_{\phi_S}(\delta)\}. \tag{49}$$

If both maps have the same control ϕ, then

$$H(\operatorname{Fix}(T), \operatorname{Fix}(S)) \le \Theta_\phi(\delta). \tag{50}$$

Proof Let $u \in \operatorname{Fix}(T)$. Then

$$d(u, Su) = d(Tu, Su) \le \delta. \tag{51}$$

Applying Theorem 3 to the map S gives

$$\operatorname{dist}\big(u, \operatorname{Fix}(S)\big) \le \Theta_{\phi_S}(\delta). \tag{52}$$

Taking the supremum over $u \in \operatorname{Fix}(T)$ proves (47). Interchanging T and S gives (48), and (49)–(50) follow from (45).

Corollary 4 (Linear perturbation estimate) *If both maps in Theorem 4 have the common control $\phi(t) = qt$, $0 \le q < 1$, then*

$$H(\operatorname{Fix}(T), \operatorname{Fix}(S)) \le \frac{\delta}{1 - q}. \tag{53}$$

The abstract retraction-displacement, Ulam and perturbation viewpoints are established parts of weakly Picard theory [13, 14]. Theorems 3 and 4 identify the explicit modulus produced when the residual-sensitive Čirić condition is reduced to the orbital recurrence.

5 Examples

The two examples have complementary purposes. Example 1 uses the genuine three-variable function F_ϕ and shows how residual sensitivity accommodates non-geometric local decay while still extracting a summable Picard recurrence. Example 2 uses the linear special case to isolate the fixed-set perturbation mechanism and demonstrate the sharpness of the resulting Hausdorff bound.

5.1 A nonlinear non-geometric example with two fixed points

Consider

$$\phi(t) = \frac{t}{1 + \sqrt{t}}, \qquad t \ge 0. \tag{54}$$

Lemma 1 *The function ϕ in (54) is a c-comparison function. It is increasing, concave and subadditive.*

Proof Direct differentiation shows that ϕ is increasing and concave on $(0,\infty)$, with $\phi(0)=0$ and $\phi(t)<t$ for $t>0$. A nonnegative concave function vanishing at zero is subadditive.

It remains to prove summability. Let $t_{n+1}=\phi(t_n)$, $t_0>0$, and put $s_n=t_n^{-1/2}$. Then

$$s_{n+1}=\sqrt{s_n(s_n+1)}. \tag{55}$$

so

$$s_{n+1}-s_n=\frac{1}{\sqrt{1+1/s_n}+1}>0. \tag{56}$$

The sequence $\{s_n\}$ cannot have a finite positive limit, because such a limit s would satisfy $s=\sqrt{s(s+1)}$. Hence $s_n\to\infty$. Choose N such that $s_N\geq 1$. For $n\geq N$,

$$s_{n+1}-s_n\geq\frac{1}{\sqrt{2}+1}=\sqrt{2}-1. \tag{57}$$

Therefore

$$t_n\leq\frac{1}{\left[s_N+(n-N)(\sqrt{2}-1)\right]^2},\qquad n\geq N. \tag{58}$$

and $\sum_{n=0}^{\infty}t_n<\infty$.

Example 1 Let

$$X=[0,1]\cup\{2\}. \tag{59}$$

with the Euclidean metric, and define

$$T(x)=\phi(x)=\frac{x}{1+\sqrt{x}}\quad(x\in[0,1]),\qquad T(2)=2. \tag{60}$$

Then X is complete, T is continuous, and

$$\mathrm{Fix}(T)=\{0,2\}. \tag{61}$$

Let F_ϕ be the genuine three-variable control in (12). For $x,y\in[0,1]$, subadditivity and monotonicity give

$$|\phi(x)-\phi(y)|\leq\phi\big(|x-y|\big). \tag{62}$$

Hence

$$\begin{aligned} m_T(x,y)&\leq d(Tx,Ty)\\ &\leq\phi\big(d(x,y)\big)\\ &\leq F_\phi\big(d(x,y),d(x,Tx),d(y,Ty)\big). \end{aligned} \tag{63}$$

If one of the points is 2, then $d(2,T2)=0$, so $m_T(x,y)=0$. Thus (16) holds on all of X.

The residual-sensitive formulation is particularly natural here. For arbitrary pairs, F_ϕ may depend on the mismatch between the mutual distance $d(x,y)$ and the residual $d(x,Tx)$, as well as on the second residual $d(y,Ty)$. This gives a flexible global sufficient condition. On a Picard pair (x,Tx), however, the first two arguments are equal, so the correction term in (12) vanishes and

$$F_\phi(r,r,s)=\phi(r). \tag{64}$$

The global three-variable inequality therefore retains off-orbit flexibility without weakening the exact recurrence needed to control successive residuals.

The example does not admit any corresponding constant coefficient $q<1$. For $x\in(0,1]$, put

$$r(x)=d(x,Tx)=x-\phi(x)=\frac{x^{3/2}}{1+\sqrt{x}}. \tag{65}$$

For the pair (x,Tx), first note that

$$\frac{r(Tx)}{r(x)}=\frac{1}{\sqrt{1+\sqrt{x}}+\sqrt{x}}<1, \tag{66}$$

so $m_T(x,Tx)=r(Tx)$ and $c_T(x,Tx)=0$. If a constant-coefficient inequality

$$m_T(x,y)-c_T(x,y)\leq qd(x,y). \tag{67}$$

held for some $q<1$, then (66) would give $r(Tx)/r(x)\leq q$ for every $x>0$. But

$$\lim_{x\downarrow 0}\frac{r(Tx)}{r(x)}=1. \tag{68}$$

a contradiction.

Thus the Picard decay is genuinely non-geometric. Nevertheless, Theorem 2 gives the computable residual estimate

$$d\big(T^nx,P_Tx\big)\leq\Theta_\phi\big(d(T^nx,T^{n+1}x)\big). \tag{69}$$

The current displacement is therefore not only part of the structural hypothesis; it is also an observable stopping quantity that bounds the remaining distance to the orbit's limiting fixed point. Every orbit starting in $[0,1]$ converges to 0, while the orbit starting at 2 remains at 2.

5.2 Sharpness of the linear data-dependence bound

Example 2 Let

$$X=[0,1]\cup[2,3]. \tag{70}$$

with the Euclidean metric and fix $\varepsilon\in(0,1)$. Define

$$T(x)=\begin{cases}x/2, & x\in[0,1],\\ 2+(x-2)/2, & x\in[2,3],\end{cases} \tag{71}$$

and

$$S_\varepsilon(x)=\begin{cases}\varepsilon+(x-\varepsilon)/2, & x\in[0,1],\\ 2+\varepsilon+\big(x-(2+\varepsilon)\big)/2, & x\in[2,3].\end{cases} \tag{72}$$

Both maps are continuous self-maps of X. Their fixed-point sets are

$$\mathrm{Fix}(T)=\{0,2\},\qquad \mathrm{Fix}(S_\varepsilon)=\{\varepsilon,2+\varepsilon\}. \tag{73}$$

For points in the same connected component,

$$d(Tx, Ty) = \frac{1}{2}d(x, y),$$
$$d(S_\varepsilon x, S_\varepsilon y) = \frac{1}{2}d(x, y). \tag{74}$$

For points in different components, the crossing term c_T (respectively c_{S_ε}) is at least 1, while m_T (respectively m_{S_ε}) is at most $1/2$. Hence both maps satisfy the classical non-unique Čirić inequality with $q = 1/2$.

Moreover,

$$\delta = \sup_{x\in X} d(Tx, S_\varepsilon x) = \frac{\varepsilon}{2}, \tag{75}$$

whereas

$$H\big(\mathrm{Fix}(T), \mathrm{Fix}(S_\varepsilon)\big) = \varepsilon. \tag{76}$$

Since $q = 1/2$, Corollary 4 gives

$$\frac{\delta}{1-q} = \frac{\varepsilon/2}{1/2} = \varepsilon. \tag{77}$$

which equals (76). Therefore the coefficient in the linear Hausdorff data-dependence bound is sharp.

This example is the residual-insensitive boundary case $F(a, b, c) = qa$ of the proposed framework. Its role is to show that, once the structural reduction has produced the residual-to-fixed-set modulus, the same modulus converts the measurable operator discrepancy δ into a best-possible bound for the displacement of the full fixed-point sets.

6 Relation to existing theory

Table 1 summarizes the positioning of the present result. The comparison is intentionally conservative: the existence part of the scalar c-comparison theory is acknowledged as known. The main novelty is the residual-sensitive three-variable structural hypothesis, its diagonal reduction to the Picard residual recurrence, and the explicit quantitative consequences derived from that recurrence.

Table 1 Position of the present framework relative to representative results.

Framework	Main mechanism	Relation to the present paper
Čirić [6]	Minimum-distance inequality with constant $q < 1$; non-unique fixed points allowed.	Recovered by $F(a, b, c) = qa$ and $\phi(t) = qt$. The present paper additionally derives current-residual error and fixed-set perturbation estimates.
Alsulami–Karapınar–Rakočević [1]	Non-unique Čirić inequality with a c/b-comparison function.	Establishes that nonlinear summable comparison control is already known. Corollary 1 is a quantitative metric specialization, not a novelty claim.
Karapınar–Agarwal [7] and Karapınar [8]	Non-unique Čirić extensions in generalized settings and through interpolation.	Complementary in setting and contractive form.
C-class methods [2, 15]	Functional auxiliary controls of C-class type.	The three-variable control is used as a residual-sensitive sufficient structural condition. Its diagonal restriction is what produces the Picard recurrence.
Weakly Picard theory [13, 14]	Retraction-displacement, well-posedness, Ulam stability and data dependence.	Provides the conceptual framework. Here the modulus Θ_ϕ is derived explicitly from the residual recurrence generated by the proposed structural condition.
Almost weakly Picard operators [4]	Recent weak forms of Picard convergence and comparison-function controls.	Shows the continuing relevance of non-unique iterative convergence; the present focus is the global-to-orbital reduction, quantitative residual control and fixed-set stability.

Three points deserve emphasis. First, the diagonal requirement (11) is not presented as a replacement for established c-comparison theory. It is a practical structural requirement ensuring that a flexible three-variable inequality collapses, on a Picard pair (x, Tx), to the summable residual recurrence (17). Second, the novelty is not the formal appearance of residual terms alone; it is the deliberate coupling of a globally verifiable residual-sensitive condition with a precise reduction mechanism and an explicit quantitative weakly Picard modulus. Third, the conclusions concern the whole fixed-point set. This is essential in the non-unique setting: one cannot in general estimate the displacement between two selected limits of perturbed maps, but Theorem 4 controls the Hausdorff distance between the fixed-point sets.

7 Conclusion

A non-unique Čirić theorem with nonlinear summable comparison control should not be justified merely by replacing a constant q with a nonlinear function, because such comparison-function extensions are already available. The contribution developed here is structural and quantitative. The essential orbital assumption was isolated as a recurrence for the Picard residual sequence. A residual-sensitive three-variable C-class inequality was then introduced as a globally stated sufficient condition whose diagonal restriction, on (x, Tx), produces precisely that recurrence.

This reduction bridges to the quantitative theory. It yields a priori information from the initial residual and a genuine a posteriori estimate from the current Picard displacement. The latter gives an explicit retraction-displacement modulus and, through the same estimate, set-wise generalized Ulam stability and well-posedness. For operator perturbations, the modulus controls the Hausdorff distance between the complete fixed-point sets; the constant-coefficient version of this estimate is sharp.

The framework also has clear limitations. Condition (16) is sufficient rather than necessary: some maps may satisfy the orbital recurrence directly without admitting a convenient global verification through a chosen three-variable function F. The conclusions require orbital completeness and orbital continuity, and the Picard limit retraction P_T need not be continuous. Because uniqueness is deliberately not imposed, the theory controls distance to the fixed-point set rather than selection of a particular fixed point. The Hausdorff perturbation result further assumes a finite uniform discrepancy $\sup_{x\in X} d(Tx, Sx)$, which may be restrictive on unbounded domains.

The present formulation is restricted to standard metric spaces to keep the global-to-orbital mechanism transparent. Extensions to b-metric spaces would require the weighted comparison series familiar from b-comparison functions. Multivalued weakly Picard mappings and ordered or graph-based settings are further natural directions, provided that an explicit summable residual estimate and an appropriate fixed-set distance can be preserved.

Acknowledgments

The first author gratefully acknowledges the Beijing Institute of Mathematical Sciences and Applications (BIMSA) for its hospitality and the stimulating research environment provided during a visit under the ICMRA programme.

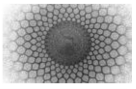
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TTMP vol 3 (3), 2026
doi: 10.30511/TTMP.2026.2096221.1090

Transactions in Theoretical and Mathematical Physics



The Photon Gas in Classical Mechanics: A Statistical-Mechanical Treatment of Classical Field Theory

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Received: 31 July 2026 / Accepted: 21 August 2026 / Published: 24 August 2026

Abstract The thermodynamics of cavity radiation is usually introduced through the quantum theory of a photon gas. In contrast, we investigate which features of thermal radiation follow from classical electrodynamics alone. Using two complementary approaches, we show that the proportionalities between energy density, radiation pressure, and energy flux arise without invoking photons or quantum statistics. The first approach uses explicit solutions of Maxwell's equations and spatio-temporal averaging of electromagnetic waves. The second approach develops the classical statistical mechanics of the electromagnetic field from first principles by constructing the canonical Hamiltonian, implementing the gauge constraints, formulating the partition function as a functional integral, and evaluating the corresponding field correlation functions. By combining wave analysis, statistical mechanics, constrained Hamiltonian dynamics, and functional methods within a single classical framework, this work provides a coherent classical formulation of electromagnetic radiation thermodynamics.

Keywords: Photon gas, Classical electrodynamics, Classical field theory, Statistical mechanics, Gauge invariance.

1 Introduction

The thermodynamics and statistical mechanics of electromagnetic radiation confined in a cavity is usually presented as the problem of a photon gas [1]. The photon picture – massless bosons obeying Bose-Einstein statistics – gives a clean and straightforward derivation of the Planck spectrum, the Stefan-Boltzmann law, and the relations between energy density, radiation pressure, and energy flux. This quantum approach is so successful that little else needs to be said. Historically, however, important results concerning thermal radiation were obtained before the introduction of the light quantum in 1905 [2, 3]. Stefan first discovered the Stefan-Boltzmann law experimentally in 1879, and Boltzmann derived it theoretically in 1884 [4]. Boltzmann's derivation used only thermodynamics and the relation $P = u/3$ between the pressure P and the energy density u – the latter following from Maxwell's electrodynamics. But there is a subtlety: Boltzmann's argument implicitly assumed that the internal energy U is a finite function of temperature and volume, an assumption that classical electrodynamics could not justify. Indeed, when one attempts a purely classical calculation of the energy density by counting modes and applying the equipartition theorem, one runs into the ultraviolet catastrophe and a divergent result. Thus, while the Stefan-Boltzmann law was *obtained* before quantum theory, it was not *derivable* from first principles within a consistent classical framework without additional (and ultimately unjustified) assumptions. Given this failure, one might ask: why revisit the classical treatment at all, let alone in a systematic field-theoretic manner? The answer is twofold. First, despite its inability to produce the correct spectrum or the correct absolute value of the Stefan-Boltzmann constant, classical electromagnetism can still predict certain universal *relations* between observable quantities correctly – in particular, the proportionality between energy flux, I and energy density, u, and similarly the proportionality between radiation pressure, P, and energy density. These proportionalities, when combined with standard thermodynamics, yield the Stefan-Boltzmann form $u \propto T^4$; only the numerical constant remains undetermined (and, in the classical calculation, divergent). Second, and more

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importantly for the present paper, electrodynamics is a classical gauge field theory. Constructing its statistical mechanics from scratch – starting from the Lagrangian, identifying the constraints, fixing the gauge, writing the partition function as a functional integral, and computing thermal averages — is a nontrivial exercise that rarely appears in textbooks on thermal physics. This exercise provides a valuable playground for understanding how field theory, gauge invariance, and statistical mechanics interact, and it serves as a natural warm-up for finite-temperature quantum field theory and path integral methods. The literature has also explored attempts to derive the blackbody spectrum within a purely classical framework, invoking assumptions such as zero-point radiation or stochastic electrodynamics; see, for example, the works of Boyer and Cole [5–8]. Numerous treatments of electromagnetic radiation discuss individual aspects of radiation pressure, blackbody thermodynamics, or the Hamiltonian formulation of Maxwell theory. The present work differs in that it presents these topics within a single self-contained classical framework that combines wave analysis, statistical mechanics, constrained Hamiltonian dynamics, and functional methods. This unified presentation is intended to provide advanced students with a coherent bridge between classical electrodynamics and finite-temperature field theory. In this paper, we focus on what classical field theory can do. We present two complementary classical derivations of the proportionalities $I \propto u$ and $P \propto u$. The first is more intuitive: it solves Maxwell's equations, decomposes the fields into Fourier modes, and uses spatial and temporal averaging to obtain the desired relations. The second is the paper's methodological core: we start from the Lagrangian, move to the Hamiltonian formalism, explicitly handle the gauge freedom, write the classical partition function as a functional integral, compute field correlation functions, and finally derive $P = u/3$ from the Maxwell stress tensor. Along the way, we also see why the average Poynting vector vanishes in equilibrium, a fact that distinguishes the equilibrium ensemble approach from the first wave-based derivation, where modes propagating in opposite directions can be treated separately. The paper is organized as follows. Section 2 briefly recalls the standard photon-gas argument for context and shows how the proportionalities lead to the Stefan-Boltzmann law via classical thermodynamics. Section 3 gives an intuitive, classical, wave-based derivation. Section 4 contains the systematic, constraint-based, field-theoretic treatment. We conclude in Section 5 with a discussion of what classical physics can and cannot achieve in this problem, and why the exercise remains useful for understanding the interface between field theory, statistical mechanics, and gauge invariance. Technical details and supporting derivations are provided in the Appendices.

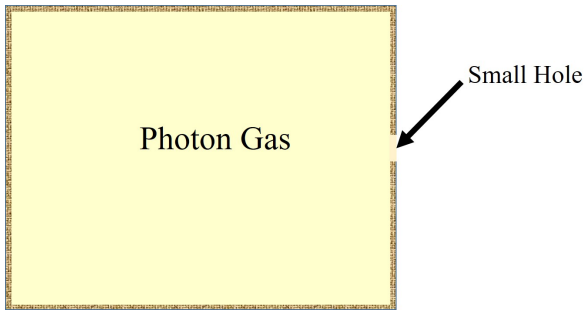


Fig. 1 A cavity filled with a photon gas and held at temperature T. The small hole acts as a blackbody aperture through which the equilibrium radiation can escape and be observed.

2 Photon gas and thermodynamic relations

Consider a photon gas consisting of photons confined within a cavity, as illustrated in Figure 1, and maintained in thermal equilibrium with the cavity walls at temperature T. In equilibrium, the energy radiated per unit surface area per unit time obeys the Stefan-Boltzmann law, i.e., it is proportional to the fourth power of the temperature. In addition, one can show that the energy density is proportional to both the rate of energy emission per unit area and the radiation pressure.

Remarkably, the Stefan-Boltzmann law can be obtained without any reference to the quantum nature of light, using only classical thermodynamics together with the relation between radiation pressure and energy density [9]. Suppose we know from electrodynamics (or from experiment) that for isotropic radiation

$$P = \frac{u}{3}, \tag{1}$$

where P is the radiation pressure and $u := U/V$ is the energy density. The first law of thermodynamics,

$$dU = TdS - PdV, \tag{2}$$

gives, at constant temperature,

$$\left(\frac{\partial U}{\partial V}\right)_T = T\left(\frac{\partial S}{\partial V}\right)_T - P. \tag{3}$$

Using the Maxwell relation that follows from the Helmholtz free energy [10],

$$\left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial P}{\partial T}\right)_V, \tag{4}$$

we obtain

$$\left(\frac{\partial U}{\partial V}\right)_T = T\left(\frac{\partial P}{\partial T}\right)_V - P. \tag{5}$$

Since $U = Vu(T)$ and $P = u(T)/3$, the left-hand side is simply $u(T)$, while the right-hand side becomes

$$\frac{T}{3}\frac{du}{dT} - \frac{u}{3}. \tag{6}$$

Equating the two sides yields a differential equation for $u(T)$, whose solutions is the Stefan-Boltzmann law,

$$u = AT^4, \tag{7}$$

where A is an undetermined constant. Kelly and Lee have discussed similar thermodynamic analyses of blackbody radiation [11, 12]. Thus, without invoking any quantum hypothesis, the temperature dependence $u \propto T^4$ follows solely from thermodynamics and the classical relation $P = u/3$. The constant A cannot be fixed by classical physics; deriving its correct value,

$$A = \frac{\pi^2 k_B^4}{15c^3\hbar^3}, \tag{8}$$

requires quantum theory and the Planck distribution, which agrees with the experiment [13].

Now we are left with a question: where do the proportionalities $P \propto u$ and $I \propto u$ come from? The standard textbook answer uses the photon picture and kinetic theory (see, for example, [14]). Consider a cavity as shown in Figure 1. We aim to determine the intensity of the radiation that escapes through a small hole in the wall. The radiation may contain photons of different frequencies ω. In the photon-gas picture used here, photons are assumed to be non-interacting,[1] so each frequency can be analyzed independently. Thus, we focus on a single frequency at a time.

To calculate the radiation rate, we first determine the number of photons passing through the small hole per unit time. Because the motion of photons is isotropic, the number of photons with velocity within the solid angle $d\Omega$ is independent of the orientation of that angle. In spherical coordinates, the solid angle element is $d\Omega = \sin\theta\, d\theta\, d\varphi$, where θ is the polar angle (measured from the z–axis) and φ is the azimuthal angle (measured in the xy–plane). If we denote by $f(\theta, \varphi)\, d\Omega$ the probability that photons move in a given direction, then

$$f(\theta, \varphi) = \frac{1}{4\pi}. \tag{9}$$

Let us now compute the number of photons that, moving in a given direction, strike a small surface area A of the wall during a short time interval dt. Figure 2 illustrates the geometry: all photons contained in the

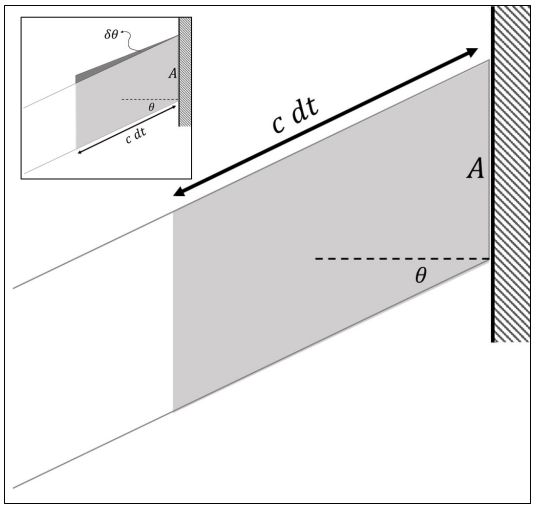


Fig. 2 Photons moving at angle θ reaching a surface element A during the time interval dt. The shaded volume corresponds to the photons traveling a distance $c\,dt$. The inset shows the infinitesimal angular spread $\delta\theta$, whose effect on the calculation is negligible to leading order.

parallelogram volume shown in grey, and moving in the specified direction, will reach the surface in time dt. One might object that we have drawn only a single angle in the figure, whereas we should consider an interval of angles. This is correct, but if we take an interval such as θ to $\theta + \delta\theta$, the difference is only that the parallelogram volume is slightly modified, producing a higher-order correction that does not affect the calculation. Thus, the number of photons colliding with the surface is equal to the volume of this parallelogram multiplied by the photon density and by the probability of having such a velocity:

$$N(\omega, \theta, \phi) = (c\,dt\, A\cos\theta)\, n_\omega\, \frac{d\Omega}{4\pi}, \tag{10}$$

where n_ω is the density of photons with frequency ω.

Two more steps are required to reach the desired result. First, we must integrate over all angles of incidence; second, we must include all frequencies together with their energies. Carrying out the angular integration gives

$$N(\omega) = \int (c\,dt\, A\cos\theta)\, n_\omega\, \frac{d\Omega}{4\pi} = \frac{cA dt\, n_\omega}{4}. \tag{11}$$

Note that the integration is taken only over half of the space, corresponding to the side of the wall where the photon gas is present. In this way, the collision flux of photons with frequency ω, i.e., the number of collisions per unit time per unit area, is $c\,n_\omega/4$.

All that remains is to account for the energy carried by each photon and sum over all frequencies. The energy of a photon is $\hbar\omega$, so the energy flux per unit time per

[1] In quantum electrodynamics (QED), photons can interact with each other via higher-order processes such as virtual electron-positron loops [15]. These effects are extremely weak at ordinary energies, but they have been experimentally observed [16].

unit area is

$$I = \int d\omega\, \hbar\omega\, \frac{c\, n_\omega}{4}. \tag{12}$$

The energy density has a similar expression: multiplying the number of photons per unit volume of frequency ω by their energy and integrating over all frequencies gives

$$u = \int d\omega\, \hbar\omega\, n_\omega. \tag{13}$$

Comparing the two relations, we immediately find

$$I = \frac{c}{4} u. \tag{14}$$

This is the result we sought. Similar straightforward arguments also establish the relation between radiation pressure and energy density,

$$P = \frac{u}{3}, \tag{15}$$

which likewise appears in most statistical physics textbooks. In fact, this relation has been obtained from classical reasoning in the seminal book by Landau and Lifshitz for ultra-relativistic particles [17]. This derivation, although it does not require quantum mechanics, is still based on the particle picture of a gas. However, the proportionality should be derivable within the wave picture of classical electromagnetism alone, without assuming the existence of particles or photons. In the following two sections, we present exactly such classical derivations. These derivations show that while the absolute normalization of the Stefan-Boltzmann law requires quantum physics, the functional relations among energy density, flux, and pressure are already contained in classical electromagnetism and classical statistical mechanics.

3 An Intuitive Approach Based on Classical Electromagnetic Waves

In thermodynamics, we deal with macroscopic quantities. Using statistical mechanics, we try to derive these macroscopic quantities from microscopic ones, usually by some averaging. In real measurements, we average mostly over time and sometimes over a region of space. For example, when we measure the pressure of a car tyre, we use a pressure gauge with a small but finite area, leave it in contact with the tyre for a time much longer than the microscopic time scales, and then read the pressure. In the theory part, if we wanted to perform time averaging directly, we would normally face the problem that the equations of motion cannot be solved analytically. Therefore, we usually turn to ensemble averaging. In the case of electromagnetic waves, however, we are dealing with a system that is exactly solvable analytically: Maxwell's equations in vacuum are linear, and their solutions are plane waves. Hence, we can follow the more intuitive path: start with the direct solution of Maxwell's equations, then perform spatial and temporal averages over scales larger than the microscopic ones (i.e., larger than the typical wavelength and oscillation period), and finally compute the macroscopic quantities of interest. As stated earlier, we focus on the relations between the energy flux, the energy density, and the radiation pressure.

3.1 Fourier representation of the fields

From a purely classical standpoint, the description of an empty cavity filled with radiation relies entirely on Maxwell's equations in vacuum, which govern the dynamics of the electric and magnetic fields $\vec{E}(\vec{x},t)$ and $\vec{B}(\vec{x},t)$. Moreover, Maxwell's equations ensure that once the electric field is specified, the magnetic field is determined accordingly [18]. It is therefore sufficient to focus our analysis on $\vec{E}$.

Maxwell's equations are linear, so their solutions, when expressed in Fourier space, decouple and can be analysed individually. In this representation, the dynamics of the system are described by the Fourier components $\vec{E}(\vec{k})$, where $\vec{k}$ is the wave vector [19]. The electric field can be written in terms of its Fourier modes as

$$\vec{E}(\vec{x},t) = \int \frac{d^3k}{(2\pi)^3} \left(\vec{E}(\vec{k}) e^{i\vec{k}\cdot\vec{x}} e^{-i\omega(\vec{k})t} + \text{c.c.} \right). \tag{16}$$

Here we have used the same symbol $\vec{E}$ for both the field and its Fourier components, a common but harmless abuse of notation. Note that the electric field $\vec{E}(\vec{x},t)$ is real, while its Fourier components $\vec{E}(\vec{k})$ are complex. Coulomb's law implies that

$$\vec{k} \cdot \vec{E}(\vec{k}) = 0. \tag{17}$$

From Maxwell's equations, it follows that the dispersion relation is $\omega(\vec{k}) = c\|\vec{k}\|$, where $\|\vec{k}\| := \sqrt{\vec{k}\cdot\vec{k}}$. By Faraday's law of induction, the magnetic field can also be expressed in terms of the Fourier components of the electric field:

$$\vec{B}(\vec{x},t) = \int \frac{d^3k}{(2\pi)^3} \left[\frac{\vec{k} \times \vec{E}(\vec{k})}{\omega(\vec{k})} \times e^{i\vec{k}\cdot\vec{x}} e^{-i\omega(\vec{k})t} + \text{c.c.} \right]. \tag{18}$$

3.2 Energy density and spatio-temporal averaging

The specific question we wish to address is the relationship between the energy density and the radiant flux in a chamber filled with electromagnetic fields. We begin by calculating the energy density, which is obtained directly from the fields:

$$u(\vec{x},t) = \frac{1}{2}\epsilon_0 \|\vec{E}(\vec{x},t)\|^2 + \frac{1}{2\mu_0}\|\vec{B}(\vec{x},t)\|^2, \tag{19}$$

where ϵ_0 and μ_0 denote the vacuum permittivity and permeability, respectively.

As noted above, in thermodynamics, energy density is not the instantaneous microscopic quantity introduced earlier, but rather an average over an appropriate time interval and spatial region. To illustrate this idea, let us recall the analogy with fluid mechanics: when we denote the fluid velocity field by $\vec{v}(\vec{x},t)$, we do not mean the instantaneous microscopic velocity of a single molecule located at position $\vec{x}$ at time t. Instead, we obtain the velocity field by averaging microscopic velocities over a spatial region much larger than molecular dimensions but much smaller than the system's macroscopic scale. In practice, one also averages over a short time interval, long compared to microscopic timescales but short relative to the macroscopic dynamics.

Returning to electromagnetism, we define the thermodynamic energy density as a local spatial average:

$$\underline{u}(\vec{x},t) = \frac{1}{V_1(\vec{x})}\int_{V_1(\vec{x})} d^3x'\, u(\vec{x}',t), \tag{20}$$

where $V_1(\vec{x})$ is a volume centered at $\vec{x}$, large compared to microscopic fluctuations but small compared to the system's size. For electromagnetic radiation, the relevant microscopic length scale is set by the wavelength. Since a cavity supports modes over a broad range of wavelengths, a natural choice is the wavelength at which the Planck spectrum reaches its maximum, as most of the radiative energy is concentrated near this wavelength. For solar radiation, the peak occurs at approximately $\lambda_{\text{peak}} \simeq 500\,\text{nm}$, which corresponds to a characteristic averaging volume of order $(\lambda_{\text{peak}})^3 \sim \mu\text{m}^3$. In addition to spatial averaging, one may also perform averaging over a short time interval, chosen to be long compared to microscopic oscillations but short relative to macroscopic dynamics.

For concreteness, let us consider the contribution of the electric field to the energy density (19). We have

$$\begin{aligned}\underline{u}_E(\vec{x},t) &:= \frac{1}{V_1(\vec{x})}\int_{V_1(\vec{x})} d^3x' \left(\frac{\epsilon_0}{2}\|\vec{E}(\vec{x}',t)\|^2\right) \\ &= \frac{\epsilon_0}{2V_1(\vec{x})}\int_{V_1(\vec{x})} d^3x' \int \frac{d^3k_1}{(2\pi)^3}\int \frac{d^3k_2}{(2\pi)^3} \\ &\quad\times\Big[\left(\vec{E}(\vec{k}_1)e^{i\vec{k}_1\cdot\vec{x}'}e^{-i\omega(\vec{k}_1)t} + \text{c.c.}\right) \\ &\qquad\cdot\left(\vec{E}(\vec{k}_2)e^{i\vec{k}_2\cdot\vec{x}'}e^{-i\omega(\vec{k}_2)t} + \text{c.c.}\right)\Big].\end{aligned} \tag{21}$$

The spatial average over $V_1(\vec{x})$ acts on the exponential factors $e^{\pm i(\vec{k}_1\pm\vec{k}_2)\cdot\vec{x}'}$. For a volume large compared to the typical wavelength, these integrals become sharply peaked. More precisely,

$$\frac{1}{V_1(\vec{x})}\int_{V_1(\vec{x})} d^3x'\, e^{i(\vec{k}_1\pm\vec{k}_2)\cdot\vec{x}'} \approx \delta_{\vec{k}_1,\mp\vec{k}_2} \qquad \text{(discrete notation).} \tag{22}$$

or in the continuum limit it tends to a Dirac delta function $(2\pi)^3\delta^3(\vec{k}_1 \pm \vec{k}_2)$ times a factor that depends on the shape of the volume, but crucially it vanishes when $\vec{k}_1 \neq \mp\vec{k}_2$. Therefore, only terms with $\vec{k}_2 = \pm\vec{k}_1$ survive the average. However, cross terms such as $\vec{E}(\vec{k})\cdot\vec{E}(-\vec{k})$ or its complex conjugate come with oscillatory factors $e^{\pm 2i\omega(\vec{k})t}$ which, when averaged over a time interval long compared to the oscillation period, vanish. Hence, after performing both spatial and temporal averaging, only the terms with $\vec{k}_2 = \vec{k}_1$ and without $e^{\pm 2i\omega t}$ survive. A detailed justification of why time averaging eliminates cross terms between different frequencies and why only the diagonal terms $\vec{k}_2 = \vec{k}_1$ survive is given in Appendix Appendix A. These terms come from the product $\vec{E}(\vec{k}_1)e^{i\vec{k}_1\cdot\vec{x}'}\cdot\vec{E}(\vec{k}_2)^* e^{-i\vec{k}_2\cdot\vec{x}'}$ with $\vec{k}_2 = \vec{k}_1$. The spatial average then gives a factor of unity, and the time average removes the oscillation. A similar analysis holds for the magnetic part (see Appendix Appendix B). Collecting the surviving contributions, one finds $\overline{\underline{u}}_B(\vec{x}) = \overline{\underline{u}}_E(\vec{x})$[2]

$$\begin{aligned}\overline{\underline{u}}(\vec{x}) &:= \overline{\underline{u}}_E(\vec{x}) + \overline{\underline{u}}_B(\vec{x}) \\ &= 2\epsilon_0\int \frac{d^3k}{(2\pi)^3}\,\vec{E}(\vec{k})^*\cdot\vec{E}(\vec{k}).\end{aligned} \tag{23}$$

3.3 Poynting vector and energy flux

Now we turn to the Poynting vector [20],

$$\vec{S} = \frac{1}{\mu_0}\,\vec{E}\times\vec{B}, \tag{24}$$

[2] Underline and overline denote average over volume and average over time, respectively.

which in turn, specifies the flux of energy. Using (16) and (22),

$$\vec{S}(\vec{x},t) = \frac{1}{\mu_0 V_1(\vec{x})}\int_{V_1(\vec{x})} d^3x' \int \frac{d^3k_1}{(2\pi)^3}\int \frac{d^3k_2}{(2\pi)^3} \times \left[\left(\vec{E}(\vec{k}_1)e^{i\vec{k}_1\cdot\vec{x}'}e^{-i\omega(\vec{k}_1)t} + \text{c.c.}\right) \times \left(\frac{\vec{k}_2}{\omega(\vec{k}_2)} \times \vec{E}(\vec{k}_2) \times e^{i\vec{k}_2\cdot\vec{x}'}e^{-i\omega(\vec{k}_2)t} + \text{c.c.}\right)\right]. \tag{25}$$

As before, we need the average over a small spatial volume and a short time interval. Applying the same averaging procedure (the spatial average enforces $\vec{k}_1 = \pm\vec{k}_2$ and the time average kills the oscillatory terms) together with the Coulomb law (17), and the identity $\mu_0\epsilon_0 = c^{-2}$ we obtain

$$\overline{\vec{S}}(\vec{x}) = c\int \frac{d^3k}{(2\pi)^3} u(\vec{k})\,\hat{k}, \tag{26}$$

where $u(\vec{k}) := 2\epsilon_0\vec{E}(\vec{k})^*\cdot\vec{E}(\vec{k})$, is the contribution of mode $\vec{k}$ to the average energy density (23).

We assume that the radiation spectrum is isotropic [21, 22], so that $u(\vec{k})$ depends only on $k = \|\vec{k}\|$. To find the energy escaping through a small hole, consider an infinitesimal surface element oriented perpendicular to the z-axis. Only modes with positive k_z contribute to the net outward flux. For each wave vector $\vec{k}$, the component of the Poynting vector normal to the surface is $c\,u(k)\cos\theta$. Integrating over the hemisphere $\theta \in [0, \pi/2]$ gives

$$I = c\int \frac{d^3k}{(2\pi)^3}\,u(k)\cos\theta = c\int \frac{dk}{(2\pi)^3}\,k^2u(k)\int_0^{2\pi} d\phi \times \int_0^{\pi/2}\cos\theta\sin\theta\,d\theta. \tag{27}$$

The angular integral equals π. The full energy density (integrated over all directions) is

$$u = \int \frac{d^3k}{(2\pi)^3}u(k) = 4\pi\int \frac{dk}{(2\pi)^3}k^2u(k). \tag{28}$$

Comparing (27) and (28) we recover (14).

Thus, by carefully separating modes according to their direction of propagation, we obtain the same relation $I = \frac{c}{4}u$ that, in the photon picture, follows from kinetic theory. The advantage of the present derivation is that it uses only classical wave physics and explicit averaging, without any appeal to light quanta.

3.4 Radiation pressure from the Maxwell stress tensor

The momentum flux is described by the Maxwell stress tensor

$$T_{ij} = \epsilon_0\left(E_iE_j - \frac{1}{2}\delta_{ij}\|\vec{E}\|^2\right) + \frac{1}{\mu_0}\left(B_iB_j - \frac{1}{2}\delta_{ij}\|\vec{B}\|^2\right). \tag{29}$$

For a surface perpendicular to the i-direction, the pressure is $P_i = -T_{ii}$. For example,

$$P_x = -\epsilon_0\left(E_x^2 - \tfrac{1}{2}\|\vec{E}\|^2\right) - \tfrac{1}{\mu_0}\left(B_x^2 - \tfrac{1}{2}\|\vec{B}\|^2\right). \tag{30}$$

In thermodynamic equilibrium, the radiation field is isotropic, so the pressure is the same in all directions. The diagonal components of the stress tensor therefore satisfy $P_x = P_y = P_z$. Averaging over these components and using the fact that the trace of the Maxwell stress tensor equals $-u$, we verify (15).

In the next section, we re-derive (14) and (15) by using a more formal ensemble method that does not rely on solving the equations of motion explicitly.

4 A Systematic Field Theoretic Treatment

In contrast to the intuitive, wave-based derivation of Section 3, we now develop a systematic statistical mechanical treatment of the classical electromagnetic field. Instead of solving Maxwell's equations and averaging over time and space, we work directly with the partition function and ensemble averages. This approach is more abstract, but it does not rely on the exact solvability of the equations of motion and naturally handles the constraints and the gauge invariance which characterise electrodynamics as a classical gauge field theory. The presentation is self-contained and follows the canonical Hamiltonian formulation, culminating in the derivation of $P = u/3$ from the Maxwell stress tensor.

In the previous section, we tried to advance the discussion by appealing to intuition and to what is actually being measured. Statistical mechanics, however, has developed a framework that strengthens these intuitive methods with rigorous mathematics. In this section, therefore, we encounter more mathematical manipulations. The appeal of the problem lies in the fact that we are dealing with a system that, in essence, is a field theory. We aim to construct a partition function for this field theory and thereby derive its thermodynamic relations. As we shall see, this field theory contains twists and subtleties that resemble the challenges encountered in quantum electrodynamics. It is worth confronting

these complexities here, where the physics of the story remains closer to intuition.

Before turning to electrodynamics, let us briefly review the general procedure of statistical mechanics. Instead of time averages, we employ ensemble averages; that is, we sum over different configurations of the system. Statistical mechanics relies on the ergodic hypothesis, which assumes that temporal and ensemble averages coincide. Thus, rather than following the detailed time evolution, we obtain the desired results by considering all possible configurations.

The first step is to identify the system's configurations. In classical mechanics, these are specified by the generalized coordinates and their conjugate momenta, i.e., by the system's location in phase space. For example, a particle moving in one dimension is described by the pair (q, p). The system's Hamiltonian H is then defined in terms of these variables.

The next step is to construct the appropriate ensemble, such as the canonical ensemble. In this case, the probability of a microstate (q, p) is proportional to the Boltzmann factor $e^{-\beta H(q,p)}$, where $\beta := \frac{1}{k_B T}$, with k_B being Boltzmann's constant (which relates energy to temperature), and T denoting the absolute temperature [23]. In this framework, the thermal average of any observable $F(q, p)$ is defined as

$$\langle F \rangle := \frac{1}{Z} \int dq\, dp\, F(q,p)\, e^{-\beta H(q,p)}, \tag{31}$$

where Z is the partition function,

$$Z := \int dq\, dp\, e^{-\beta H(q,p)}. \tag{32}$$

Quantities of interest include averages such as $\langle q^2 \rangle$, $\langle p^2 \rangle$, or $\langle qp \rangle$.

Returning now to electrodynamics, the steps we follow in a statistical approach are:

1. Define the canonical variables.
2. Write down the Hamiltonian of the system.
3. Construct the Boltzmann distribution.
4. Compute the desired quantities by averaging with respect to the Boltzmann weight.

Electromagnetism, however, introduces an additional layer of complexity that makes the problem both more challenging and more intriguing. The fundamental dynamical variables of electrodynamics are the electromagnetic potentials $(\phi, \vec{A})$, from which the fields are derived. Accordingly, the Lagrangian and Hamiltonian are formulated in terms of these potentials, and the equations of motion follow from their variation. The potentials also introduce the subtle issue of gauge freedom: distinct potential configurations may correspond to the same physical state. To avoid overcounting such gauge-equivalent configurations in the ensemble, the integration measure in (31) must be defined with care.

Presenting all of these arguments simultaneously would obscure their logic, so we have arranged them into a sequence of subsections. Each subsection isolates a distinct component of the calculation, making its role and conclusion transparent.

4.1 Maxwell's Canonical Equations

In this subsection, we identify the canonical field variables and their conjugate momenta, i.e., the phase space variables of Maxwell theory, and show how the Hamiltonian density is expressed in terms of them.

Maxwell's equations can be derived from the principle of least action, $S = \int dt\, L$, where the Lagrangian is

$$L = \frac{1}{2} \int d^3x \left(\epsilon_0 \vec{E}^2 - \frac{1}{\mu_0} \vec{B}^2 \right). \tag{33}$$

The electric and magnetic fields can be expressed in terms of the potential components $A^\mu = (\phi, \vec{A})$, which serve as the canonical variables of the theory [24]:

$$\vec{B} = \nabla \times \vec{A}, \qquad \vec{E} = -\nabla\phi - \partial_t \vec{A}. \tag{34}$$

Defining the Hamiltonian and the partition function requires the conjugate momenta. By definition,

$$\pi_\mu := \frac{\delta S}{\delta(\partial_t A^\mu)}. \tag{35}$$

This implies that the momentum conjugate to the vector potential is proportional to the electric field,

$$\pi_i = -\epsilon_0 E_i, \tag{36}$$

while that of the scalar potential vanishes,

$$\pi_0 = 0. \tag{37}$$

Although this may seem unusual, its significance becomes clear as we proceed.

We now construct the Hamiltonian:

$$\begin{aligned} H &:= \int d^3x\, \pi_\mu \partial_t A^\mu - L \\ &= \int d^3x \left[\frac{\vec{\pi}^{\,2}}{2\epsilon_0} + \frac{\vec{B}^{\,2}}{2\mu_0} - \vec{\pi} \cdot \nabla\phi \right] \\ &= \int d^3x \left[\frac{\vec{\pi}^{\,2}}{2\epsilon_0} + \frac{\vec{B}^{\,2}}{2\mu_0} + \phi\, \nabla \cdot \vec{\pi} \right]. \end{aligned} \tag{38}$$

In the last step, the boundary term from integration by parts has been discarded, since it does not affect the field dynamics.

After constructing the Hamiltonian, we verify that Hamilton's equations reproduce Maxwell's equations. For the scalar potential, Hamilton's equation gives

$$0 = \partial_t \pi_0 = -\frac{\delta H}{\delta \phi} = -\nabla \cdot \vec{\pi} = \epsilon_0 \nabla \cdot \vec{E}. \tag{39}$$

So Gauss's law in vacuum $\nabla \cdot \vec{E} = 0$ emerges as a secondary constraint enforced by the primary constraint $\pi_0 = 0$ [25]. For the vector potential,

$$\partial_t \vec{A} = \frac{\delta H}{\delta \vec{\pi}} = \frac{\vec{\pi}}{\epsilon_0} + \nabla \phi. \tag{40}$$

which reproduces $\vec{E} = -\nabla\phi - \partial_t \vec{A}$ via $\vec{\pi} = -\epsilon_0 \vec{E}$. Finally, for the conjugate momentum,

$$\partial_t \vec{\pi} = -\frac{\delta H}{\delta \vec{A}} = -\frac{1}{\mu_0} \nabla \times \vec{B}. \tag{41}$$

yielding Ampère's law in vacuum $\nabla \times \vec{B} = c^{-2} \partial_t \vec{E}$.

The Hamiltonian H is independent of π_0, so Hamilton's equation for the scalar potential formally yields

$$\partial_t \phi = \frac{\delta H}{\delta \pi_0} = 0. \tag{42}$$

At first sight this seems to constrain ϕ, but recall that the electromagnetic potentials admit gauge freedom:

$$A'^{\mu} := A^{\mu} + \partial^{\mu} \chi. \tag{43}$$

This gives rise to the same electric and magnetic fields as A^{μ}, regardless of the choice of χ. Thus, the condition $\partial_t \phi = 0$ is not a physical requirement but a reflection of the gauge redundancy. To make this explicit, we introduce the extended Hamiltonian

$$H' := H + \int d^3x \, \pi_0 \, \xi. \tag{44}$$

where ξ is an arbitrary Lagrange multiplier. Its equation of motion enforces the primary constraint $\pi_0 = 0$, while simultaneously setting

$$\partial_t \phi = \xi. \tag{45}$$

In this formulation, the time evolution of ϕ is governed by the undetermined function ξ, which reflects the gauge freedom of the theory. The physical dynamics remain unchanged: H' still reproduces Ampère's law and the other Maxwell equations.

To simplify the analysis, we fix the gauge by choosing $\phi = 0$. With this choice, the Hamiltonian can be expressed entirely in terms of $\vec{A}$ and $\vec{E}$:

$$H = \tfrac{1}{2} \int d^3x \left(\epsilon_0 \, \|\vec{E}(\vec{x})\|^2 + \tfrac{1}{\mu_0} \, \|\nabla \times \vec{A}(\vec{x})\|^2 \right). \tag{46}$$

This form is particularly useful for computing statistical averages. Because the Hamiltonian is quadratic in the canonical variables, the Boltzmann weight takes the form of a Gaussian.

While encouraging, this also highlights a complication: the number of degrees of freedom is infinite. This was anticipated from the outset, since we are dealing with a field theory. At each spatial point $\vec{x}$, there are three dynamical variables (A_1, A_2, A_3) together with their three conjugate momenta. Accordingly, the partition function takes the form of a functional integral:

$$Z = \int \mathcal{D}\vec{A}(\vec{x}) \, \mathcal{D}\vec{E}(\vec{x}) \times \exp\left[-\frac{\beta}{2} \int d^3x \left(\epsilon_0 \|\vec{E}(\vec{x})\|^2 + \frac{1}{\mu_0} \|\nabla \times \vec{A}(\vec{x})\|^2 \right) \right]. \tag{47}$$

Here, e.g., $\mathcal{D}\vec{A}(\vec{x})$ denotes integration over all possible configurations of the field $\vec{A}(\vec{x})$, a notation standard in the context of functional (path) integrals.

Although the partition function involves infinitely many degrees of freedom, its quadratic form makes it Gaussian. This observation allows us to evaluate it systematically by treating each Fourier mode as an independent harmonic oscillator.

To keep the progression clear, we proceed in stages:

1. **Fourier expansion:** Decompose $\vec{A}$ and $\vec{E}$ into spatial Fourier modes to isolate independent degrees of freedom mode by mode.
2. **Constraint handling:** Recognize that not all modes are physical since the gauge freedom and the constraints reduce the physical phase space, and rewrite the Hamiltonian and the partition function accordingly.
3. **Gaussian evaluation:** Once the Hamiltonian is cast into the appropriate quadratic form over the physical modes, perform the Gaussian integrals and extract the final expressions for the desired quantities.

We implement these steps in the following subsections.

4.2 Fourier Space and Constraints on Electric Fields

In this subsection, we express the fields in terms of their Fourier components and verify that this transformation is canonical, i.e., it preserves Hamilton's equations. In addition, to implement the constraint $\nabla \cdot \vec{E} = 0$, we see that some modifications to the Hamiltonian are appropriate.

We begin by writing the vector potential and electric field in Fourier space:

$$\vec{A}(\vec{x}) = \int \frac{d^3k}{(2\pi)^3} \vec{A}(\vec{k})\, e^{i\vec{k}\cdot\vec{x}}, \tag{48}$$

$$\vec{E}(\vec{x}) = \int \frac{d^3k}{(2\pi)^3} \vec{E}(\vec{k})\, e^{i\vec{k}\cdot\vec{x}}. \tag{49}$$

Because these fields are real, their Fourier components satisfy

$$\vec{E}(\vec{k})^* = \vec{E}(-\vec{k}), \qquad \vec{A}(\vec{k})^* = \vec{A}(-\vec{k}). \tag{50}$$

Although the Fourier expansion may look straightforward, it is crucial to verify that the transformation preserves the canonical structure of the theory. In other words, the Fourier transformation from the position-space variables to their momentum-space counterparts must be a canonical transformation, preserving the form of the equal-time Poisson brackets.

From (36), the fundamental brackets in position-space are

$$\{A_i(\vec{x}), E_j(\vec{y})\} = -\,\epsilon_0^{-1}\,\delta_{ij}\,\delta^{(3)}(\vec{x}-\vec{y}). \tag{51}$$

For the Fourier components

$$\vec{A}(\vec{k}) = \int d^3x\, \vec{A}(\vec{x})\, e^{-i\vec{k}\cdot\vec{x}}, \tag{52}$$

$$\vec{E}(\vec{k}) = \int d^3x\, \vec{E}(\vec{x})\, e^{-i\vec{k}\cdot\vec{x}}, \tag{53}$$

one obtains

$$\{A_i(\vec{k}), E_j(\vec{k}')^*\} = -\,(2\pi)^3\,\epsilon_0^{-1}\,\delta_{ij}\,\delta^{(3)}(\vec{k}-\vec{k}'). \tag{54}$$

Hence the canonical commutation relations are preserved under Fourier transformation, demonstrating that the map $(A_i(\vec{x}), E_j(\vec{x})) \mapsto (A_i(\vec{k}), E_j(\vec{k}))$ is indeed canonical.[3]

In this way, we can now express the Hamiltonian in terms of the Fourier components, which is straightforward:

$$\begin{aligned} H &= \frac{1}{2}\int d^3x \left(\epsilon_0 \|\vec{E}(\vec{x})\|^2 + \frac{1}{\mu_0}\|\vec{B}(\vec{x})\|^2\right) \\ &= \frac{1}{2}\int \frac{d^3k}{(2\pi)^3}\Big[\epsilon_0\, \vec{E}(\vec{k})^*\cdot\vec{E}(\vec{k}) \\ &\qquad + \frac{1}{\mu_0}\big(\vec{k}\times\vec{A}(\vec{k})^*\big)\cdot\big(\vec{k}\times\vec{A}(\vec{k})\big)\Big]. \end{aligned} \tag{55}$$

Accordingly, the functional integral can be written in terms of the Fourier variables. That is, we sum over $\vec{A}(\vec{k})$ and $\vec{E}(\vec{k})$ in the partition function instead of $\vec{A}(\vec{x})$ and $\vec{E}(\vec{x})$:

$$Z = \int \mathcal{D}\vec{A}(\vec{k})\,\mathcal{D}\vec{E}(\vec{k})\, \exp\Big\{-\beta\, H[\vec{A}(\vec{k}), \vec{E}(\vec{k})]\Big\}. \tag{56}$$

[3]This conclusion can be further substantiated by rewriting the action in Fourier space; see Appendix Appendix C.

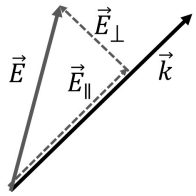


Fig. 3 Decomposition of the field $\vec{E}$ into its longitudinal and transverse components with respect to the wave vector $\vec{k}$, namely $\vec{E}_\parallel$ and $\vec{E}_\perp$, respectively.

The only subtlety in evaluating the partition function is that the electric field is not an unconstrained dynamical variable. A naive integration over all field configurations would overcount unphysical states. In vacuum, Gauss's law requires that $\nabla\cdot\vec{E}(\vec{x}) = 0$, so the functional integral must be restricted to divergence-free configurations. While implementing this constraint directly in the functional measure is possible, it is often more convenient to proceed indirectly. Specifically, we modify the Hamiltonian so that configurations violating Gauss's law are assigned zero statistical weight and therefore do not contribute to the partition function. In Fourier space, the constraint becomes $\vec{k}\cdot\vec{E}(\vec{k}) = 0$, which means that $\vec{E}(\vec{k})$ is perpendicular to $\vec{k}$. As shown in Figure 3, the field $\vec{E}(\vec{k})$ may be uniquely decomposed into components parallel and perpendicular to the wave vector $\vec{k}$:

$$\vec{E}(\vec{k}) = \vec{E}_\perp(\vec{k}) + \vec{E}_\parallel(\vec{k}). \tag{57}$$

If we can construct an operator that projects any $\vec{E}(\vec{k})$ onto its transverse component $\vec{E}_\perp(\vec{k})$, then only the allowed (divergence-free) part will contribute in the Hamiltonian, and the longitudinal part will be automatically discarded. We therefore look for an operator $\mathbf{P}(\vec{k})$ satisfying

$$\mathbf{P}(\vec{k})\cdot\vec{E}(\vec{k}) = \vec{E}_\perp(\vec{k}). \tag{58}$$

Since $\vec{E}$ is a three-component vector field, $\mathbf{P}$ is a 3×3 matrix that removes the component parallel to $\vec{k}$ and leaves the transverse components untouched. The projector is[4]

$$\mathbf{P}_{ij}(\vec{k}) = \delta_{ij} - \frac{k_i k_j}{k^2}. \tag{59}$$

Consider the electric-field contribution to the Hamiltonian (55). To enforce the transversality condition, we

[4]It is straightforward to verify that $\mathbf{P}$ is idempotent, $\mathbf{P}(\vec{k})^2 = \mathbf{P}(\vec{k})$. Moreover, since $\mathbf{P}_{ij}(\vec{k})k_j = 0$, the projector eliminates the longitudinal (unphysical) component of the field, leaving only the two transverse polarizations associated with physical electromagnetic waves.

replace that term by

$$H_E = \frac{\epsilon_0}{2}\int \frac{d^3k}{(2\pi)^3}\,\vec{E}(\vec{k})^*\cdot\mathbf{P}(\vec{k})\cdot\vec{E}(\vec{k}), \tag{60}$$

where

$$\vec{E}^*\cdot\mathbf{P}\cdot\vec{E} := \sum_{i,j} E_i^*\,P_{ij}\,E_j. \tag{61}$$

We may therefore integrate over all electric-field configurations in the partition function. In this formulation, the longitudinal components parallel to $\vec{k}$ carry no energy and hence enter the functional integral with unit weight. In contrast, the transverse components contribute exactly as in the original constrained theory. It is important to note that the longitudinal modes, although energetically inert, generate an overall divergent multiplicative factor in the partition function Z, reflecting the infinite volume associated with the gauge redundancy. This divergence is, however, physically irrelevant. Observable quantities are computed as ensemble averages–ratios of functional integrals, as in (31)–and the same divergent normalization factor appears in both the numerator and denominator. As a result, it cancels identically, and the unphysical degrees of freedom decouple from all physical observables. The remaining contributions come solely from the transverse modes, yielding finite, well-defined results.

It is also straightforward to verify that

$$\Big(\vec{k}\times\vec{A}(\vec{k})^*\Big)\cdot\Big(\vec{k}\times\vec{A}(\vec{k})\Big) = k^2\,\vec{A}(\vec{k})^*\cdot\mathbf{P}(\vec{k})\cdot\vec{A}(\vec{k}), \tag{62}$$

so the projector $\mathbf{P}(\vec{k})$ arises naturally in the magnetic term as well, ensuring that $\vec{A}(\vec{k})$ has no longitudinal component. Thus the full Hamiltonian can be written compactly as

$$H = \frac{1}{2}\int \frac{d^3k}{(2\pi)^3}\Big[\epsilon_0\,\vec{E}(\vec{k})^*\cdot\mathbf{P}(\vec{k})\cdot\vec{E}(\vec{k}) + \frac{k^2}{\mu_0}\,\vec{A}(\vec{k})^*\cdot\mathbf{P}(\vec{k})\cdot\vec{A}(\vec{k})\Big]. \tag{63}$$

4.3 Calculation of field correlations using Gaussian integrals

The derivation that follows is a standard exercise in the evaluation of quadratic (Gaussian) functional integrals. Readers interested only in the resulting correlation functions may safely skip ahead to Eqs. (74), (75), and (81).

The partition function used to compute field correlations is based on the Hamiltonian (63), augmented by source terms linear in the fields. We write

$$H_{\rm tot} = H + H_J, \tag{64}$$

with

$$H = \frac{\epsilon_0}{2}\int \frac{d^3k}{(2\pi)^3}\Big[\vec{E}_\perp(\vec{k})^*\cdot\vec{E}_\perp(\vec{k}) + c^2k^2\,\vec{A}_\perp(\vec{k})^*\cdot\vec{A}_\perp(\vec{k})\Big]. \tag{65}$$

and

$$H_J = \int \frac{d^3k}{(2\pi)^3}\Big(E_\perp(\vec{k})^*\cdot J_E(\vec{k}) + A_\perp(\vec{k})^*\cdot J_A(\vec{k})\Big). \tag{66}$$

The source terms are introduced so that field correlations can be obtained by functional differentiation of the partition function. Without loss of generality, they are supposed to be transverse, i.e.,

$$\mathbf{P}(\vec{k})\cdot J_E(\vec{k}) = J_E(\vec{k}), \qquad \mathbf{P}(\vec{k})\cdot J_A(\vec{k}) = J_A(\vec{k}), \tag{67}$$

and satisfy the reality conditions

$$J_E(\vec{k})^* = J_E(-\vec{k}), \qquad J_A(\vec{k}) = J_A(-\vec{k})^*. \tag{68}$$

Henceforth, we omit the subscript $\perp$ for convenience, with the understanding that $E(\vec{k})$ and $A(\vec{k})$ always denote transverse components. Following (31) we define the correlator[5]

$$\big\langle E_i(\vec{k})E_j^*(\vec{k}')\big\rangle := \frac{\int \mathcal{D}E\,\mathcal{D}A\,e^{-\beta H_{\rm tot}}E_i(\vec{k})E_j^*(\vec{k}')}{\int \mathcal{D}E\,\mathcal{D}A\,e^{-\beta H_{\rm tot}}} = \frac{(2\pi)^6}{\beta^2 Z}\frac{\delta^2 Z}{\delta[J_E(\vec{k})^*]_i\,\delta[J_E(\vec{k}')]_j}\bigg|_{\substack{J_E=J_A=0\\ J_E^*=J_A^*=0}}. \tag{69}$$

where we have adopted the convention

$$\frac{\delta[J_E(\vec{k})]_i}{\delta[J_E(\vec{k}')]_j} = \delta_{ij}\,\delta^3(\vec{k}-\vec{k}'), \tag{70}$$

used (50) and (68), and noted that

$$H_J = \int \frac{d^3k}{(2\pi)^3}\Big(E(\vec{k})\cdot J_E(-\vec{k}) + A(\vec{k})\cdot J_A(-\vec{k})\Big). \tag{71}$$

Similarly,

$$\big\langle A_i(\vec{k})A_j^*(\vec{k}')\big\rangle = \frac{\int \mathcal{D}E\,\mathcal{D}A\,e^{-\beta H_{\rm tot}}A_i(\vec{k})A_j^*(\vec{k}')}{\int \mathcal{D}E\,\mathcal{D}A\,e^{-\beta H_{\rm tot}}} = \frac{(2\pi)^6}{\beta^2 Z}\frac{\delta^2 Z}{\delta[J_A(\vec{k})^*]_i\,\delta[J_A(\vec{k}')]_j}\bigg|_{\substack{J_E=J_A=0\\ J_E^*=J_A^*=0}}. \tag{72}$$

[5]Here and below, the functional measures $\mathcal{D}E$ and $\mathcal{D}A$ denote integrations over the transverse field configurations only, satisfying $\vec{k}\cdot\vec{E}(\vec{k}) = \vec{k}\cdot\vec{A}(\vec{k}) = 0$. The longitudinal components are not part of the functional integration.

To evaluate the Gaussian integrals, it suffices to complete the square in the exponential.[6] Working in Fourier space is advantageous because different wave numbers decouple and can be treated independently. The result is

$$Z \propto \exp\left\{\frac{\beta}{2\epsilon_0}\int\frac{d^3k}{(2\pi)^3}\left[J_E^*(\vec{k})\cdot\mathbf{P}(\vec{k})\cdot J_E(\vec{k}) + \frac{1}{c^2k^2}\,J_A^*(\vec{k})\cdot\mathbf{P}(\vec{k})\cdot J_A(\vec{k})\right]\right\}. \tag{73}$$

where, following (67), the projection operator $\mathbf{P}(\vec{k})$ is explicitly displayed for clarity. Since the proportionality constant contains no current-dependent terms, it is omitted. Using (73) in (69) and (72), we obtain

$$\langle E_i(\vec{k})E_j^*(\vec{k}')\rangle = \frac{(2\pi)^3 k_BT}{\epsilon_0}\,\delta^3(\vec{k}-\vec{k}')\,P_{ij}(\vec{k}), \tag{74}$$

$$\langle A_i(\vec{k})A_j^*(\vec{k}')\rangle = \frac{(2\pi)^3 \mu_0 k_BT}{k^2}\,\delta^3(\vec{k}-\vec{k}')\,P_{ij}(\vec{k}), \tag{75}$$

where the relation $\beta^{-1} = k_BT$ has been applied. Mixed correlations such as $\langle E_iA_j\rangle$ vanish identically, since the block-diagonal structure of the Hamiltonian forces the corresponding source terms to decouple.

Using the correlation functions (74) and (75), we can evaluate the thermal average of the Hamiltonian (65). The result is

$$\langle H\rangle = V\mathcal{N}\,(2k_BT), \tag{76}$$

where V denotes the cavity volume,

$$V := \int d^3x = (2\pi)^3\,\delta^3(\vec{k}-\vec{k}\,')\Big|_{\vec{k}'=\vec{k}}, \tag{77}$$

and $\mathcal{N}$ is the total number of allowed wave–vectors

$$\mathcal{N} := \int\frac{d^3k}{(2\pi)^3}. \tag{78}$$

Equation (76) shows that the thermal average of the Hamiltonian coincides with the familiar equipartition result: each wave–vector contributes the energy of two simple harmonic oscillators per unit volume. In the continuum description, however, both V and $\mathcal{N}$ are apparently infinite. The volume V arises from the normalization of the Dirac delta function,

$$(2\pi)^3\delta^3(\vec{0}). \tag{79}$$

corresponding to the infinite-volume limit, whereas $\mathcal{N}$ diverges because the continuum theory contains infinitely many electromagnetic modes with arbitrarily large wave vectors. Accordingly, we should understand both quantities as regulated by finite infrared and ultraviolet cutoffs, respectively. We retain the compact notation above for simplicity and return to the role of these regulators when discussing the energy density and radiation pressure later in this section.

The correlation of the magnetic fields follows directly from the correlator (75). Since

$$\vec{B} = \nabla\times\vec{A}, \qquad \vec{B}(\vec{k}) = i\,\vec{k}\times\vec{A}(\vec{k}). \tag{80}$$

we have

$$\begin{aligned}\left\langle B_i(\vec{k})B_j^*(\vec{k}')\right\rangle &= \frac{(2\pi)^3\mu_0 k_BT}{k^2}\,\epsilon_{imr}\epsilon_{jns}\,k_mk_n \\ &\quad\times P_{rs}(\vec{k})\,\delta^3(\vec{k}-\vec{k}') \\ &= (2\pi)^3\mu_0 k_BT\,P_{ij}(\vec{k})\,\delta^3(\vec{k}-\vec{k}').\end{aligned} \tag{81}$$

where ϵ_{imr} is the Levi-Civita symbol. In reaching the final form, we have used

$$\begin{aligned}&\epsilon_{imr}\,k_mk_r = 0,\\ &\epsilon_{imr}\epsilon_{jns}\delta_{rs} = \delta_{ij}\delta_{mn} - \delta_{in}\delta_{mj}.\end{aligned} \tag{82}$$

This result is manifestly consistent with $\nabla\cdot\vec{B}(\vec{x}) = 0$, or equivalently $\vec{k}\cdot\vec{B}(\vec{k}) = 0$.

Remark. The transverse projector $P_{ij}(\vec{k})$ appearing in all correlators reflects the fact that only physical, divergence-free modes contribute. This is the statistical mechanics analogue of gauge invariance in the canonical formalism.

The correlations obtained in this section are very similar to the relations (A.4) and (A.5), although they have been derived differently. To obtain (A.4) and (A.5), we used the equations of motion and averaged over time or within a finite volume. Here, by contrast, we carried out ensemble averaging, and the two approaches coincide. According to the ergodic theorem, time (or spatial) averages and ensemble averages are equivalent. Furthermore, the transverse projector $\mathbf{P}_{ij}$ appears explicitly in (74) and (81), whereas in (A.4) and (A.5) it does not. The distinction stems from the fact that the constraints are already built into the solutions of the equations of motion. Consequently, the resulting fields are automatically transverse, and the projector need not be written explicitly. In the statistical approach, however, one integrates over arbitrary field configurations, so the constraints must be imposed explicitly.

Another distinction concerns the constants multiplying the Dirac delta functions. In the statistical derivation presented here, these coefficients appear naturally. In the former approach, one would obtain them by invoking the equipartition theorem, which assigns an average energy k_BT to each oscillating mode. This yields coefficients consistent with those derived in the present framework.

[6] See Appendix Appendix D for details.

4.4 Energy density and radiant power

We now return to our central question: the relationship between energy density and radiant power. Within the present statistical framework, we cannot establish this connection directly. While the electromagnetic energy density can be computed in this way (albeit with a divergent result), the radiated power requires analysis of the Poynting vector (24), and, in particular, its average value. Because the correlations between electric and magnetic fields vanish,

$$\langle E_i(\vec{k}) B_j(\vec{k}) \rangle = 0, \tag{83}$$

the average Poynting vector in thermal equilibrium is identically zero. This vanishing of the average Poynting vector is the correct equilibrium result (no net energy flow). It does not contradict the relation $I = \frac{c}{4}u$ derived in Section 3, because there we specifically isolated modes moving toward a hole, i.e., integrated only over a hemisphere in k-space. This outcome is consistent with physical intuition: a nonzero average flux would imply a net energy transfer, contradicting the very notion of equilibrium.

In section 3, by solving the equations of motion, we were able to identify the propagation direction of individual waves. At a given point, left-moving and right-moving modes could be distinguished, and the flux associated with each category computed. Such a decomposition, however, is not available in the ensemble approach, which yields only the equilibrium result.

Nevertheless, these calculations remain highly instructive. First, they sharpen our intuition about electromagnetic systems in thermal equilibrium and, more broadly, about the principles of statistical mechanics. Second, they provide a natural entry point into field-theoretic methods and path integrals, which play a central role in quantum electrodynamics. Finally, they enable estimation of radiation pressure – a quantity of fundamental importance that, as discussed earlier, leads directly to the Stefan-Boltzmann law.

To compute the thermal average of the radiant pressure, we note that, as already explained in subsection 3.4, the pressure can be obtained from the stress tensor. While this calculation can be carried out in Fourier space, here we present the derivation directly in position space. Using (49) together with (74), we find

$$\langle E_i(\vec{x}) E_j(\vec{x}\,') \rangle = \frac{k_B T}{\epsilon_0} \int \frac{d^3k}{(2\pi)^3} P_{ij}(\vec{k})\, e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}. \tag{84}$$

For the correlation at coincident points, $\vec{x} = \vec{x}\,'$, this reduces to[7]

$$\begin{aligned} \langle E_i(\vec{x}) E_j(\vec{x}) \rangle &= \frac{k_B T}{\epsilon_0} \int \frac{d^3k}{(2\pi)^3} P_{ij}(\vec{k}) \\ &= \frac{2}{3} \frac{k_B T}{\epsilon_0} \mathcal{N}\, \delta_{ij}. \end{aligned} \tag{85}$$

where $\mathcal{N}$ is defined in (78). This result, in particular, shows that

$$\langle E_i(\vec{x})^2 \rangle = \frac{1}{3} \langle \|\vec{E}(\vec{x})\|^2 \rangle. \tag{86}$$

An analogous relation holds for the magnetic field. Substituting into the pressure expression (30) and performing the ensemble average, we arrive at

$$\langle P \rangle = \epsilon_0 \frac{\langle \|\vec{E}(\vec{x})\|^2 \rangle}{6} + \frac{\langle \|\vec{B}(\vec{x})\|^2 \rangle}{6\mu_0} = \frac{\langle u \rangle}{3}, \tag{87}$$

in agreement with (15).

Before closing this section, let us emphasize that (85) is fully consistent with the equipartition theorem. In particular, it yields

$$\frac{\epsilon_0}{2} \langle \|\vec{E}(\vec{x})\|^2 \rangle = \mathcal{N} k_B T, \tag{88}$$

which expresses the expected thermal contribution of each mode to the electric field energy.

At this point, it is appropriate to comment on the formal divergences encountered above. In the continuum limit, the mode count $\mathcal{N}$ diverges because of the unlimited number of electromagnetic modes with arbitrarily large wave vectors. Introducing a finite ultraviolet cutoff makes the energy density and the coincident-point correlation functions finite, while the usual infrared regularisation handles the infinite-volume normalisation. Since both the pressure and the energy density are proportional to the same regulated quantity $\mathcal{N}$, the regulator dependence cancels in their ratio. Consequently, the equation of state $P = \frac{u}{3}$ is independent of the cutoff, even though the individual quantities diverge as the cutoff is removed.

To summarise, classical statistical mechanics correctly reproduces the universal proportionalities

$$P = \frac{u}{3}, \qquad I = \frac{c}{4}u. \tag{89}$$

the latter having been derived in Section 3. Together with thermodynamics, these relations imply the Stefan–Boltzmann law $u \propto T^4$. Classical theory fails to determine the finite value of the proportionality constant: because of the ultraviolet catastrophe, the energy density diverges as the ultraviolet cutoff is removed. Nevertheless, the present analysis remains valuable both as a classical derivation of these universal relations and as a natural bridge to finite-temperature quantum field theory.

[7] Details are given in Appendix Appendix E.

5 Conclusion

In this paper, we have revisited the classical foundations of the statistical mechanics of electromagnetic radiation. Starting from Maxwell's equations and classical thermodynamics, we have shown that two fundamental proportionalities – the energy flux $I = \frac{c}{4}u$ and the radiation pressure $P = \frac{u}{3}$ – follow without any appeal to photons or quantum statistics. The former follows from an intuitive wave-based derivation that uses spatial and temporal averaging and isolates modes propagating toward a hole; the latter is rederived in a systematic, constraint-based field-theoretic approach that explicitly handles gauge invariance via a transverse projector and evaluates thermal averages from the classical partition function.

These derivations make it clear that while the classical theory is unable to produce the correct blackbody spectrum (it suffers from the ultraviolet catastrophe and gives a divergent energy density) and therefore cannot fix the absolute constant in the Stefan-Boltzmann law $u = AT^4$, the functional relations among energy density, flux, and pressure are already contained in classical electromagnetism and classical statistical mechanics. In this sense, the classical theory does not fail: it correctly predicts the structure of the macroscopic laws, even though it fails at the level of the spectral distribution.

The formulation presented here addresses a gap often overlooked in the standard literature. It demonstrates how to systematically construct the partition function for a classical gauge field theory, enforce gauge constraints, and evaluate Gaussian functional integrals and thermal averages within a concrete physical system. This framework establishes a clear classical foundation for more advanced topics, such as finite-temperature quantum field theory and path-integral methods. The two complementary approaches—the heuristic averaging method and the systematic ensemble method—clarify the interplay between classical fields, gauge invariance, and thermodynamics.

Appendices

Appendix A: Time Averaging of Rapid Oscillations

Physical measurements are always performed over a finite time interval, which naturally introduces a time average. This averaging plays a role analogous to spatial averaging and further justifies the approximations used in Section 3. To illustrate the effect, consider a simple one-dimensional example: the time-averaged energy density of a superposition of two monochromatic waves. The (complex) electric field is

$$E(x,t) = E_1 e^{i(k_1 x - \omega_1 t)} + E_2 e^{i(k_2 x - \omega_2 t)}. \tag{A.1}$$

Computing the energy density requires squaring the field, which produces two types of contributions: (i) terms from the square of each individual wave, and (ii) cross terms arising from the product of the two waves. We focus on the cross term, denoted u_{12}. Its time average over an interval of duration T is

$$\begin{aligned} \bar{u}_{12} &:= \frac{1}{T}\int_{-T/2}^{T/2} dt\, u_{12}(t) \\ &= \frac{1}{T} E_1 E_2^* e^{i(k_1-k_2)x} \int_{-T/2}^{T/2} dt\, e^{i(\omega_1-\omega_2)t} + \text{c.c.} \end{aligned} \tag{A.2}$$

Evaluating the integral gives

$$\bar{u}_{12} \propto \frac{2}{(\omega_1-\omega_2)T} \sin\big[(\omega_1-\omega_2)T\big]. \tag{A.3}$$

The sine factor is bounded between -1 and 1, but the prefactor can be extremely small. For example, even the frequency difference between yellow and green light is of order 10^{15} rad/s. With a measurement time of the order of seconds, the resulting contribution is suppressed to about 10^{-15} – effectively zero. Only if the frequency difference were as small as 2π radians per second would the integral yield a contribution of order unity. For optical frequencies, this would require a measurement apparatus capable of resolving frequencies to fifteen significant digits.

Thus, time averaging eliminates cross terms between waves of different frequencies, leaving only the incoherent sum of their individual energy densities. The net effect is that only the energies of the individual modes survive, and the total energy density is obtained by summing over them. The magnetic-field contribution follows by the same reasoning.

More formally, the averaging procedure can be expressed through correlation functions,

$$\begin{aligned} \Big\langle \vec{E}(\vec{k},t)\cdot\vec{E}(\vec{k}',t)\Big\rangle_{t,V} &= 0, \\ \Big\langle \vec{E}(\vec{k},t)\cdot\vec{E}(\vec{k}',t)^*\Big\rangle_{t,V} &= C_E\,\delta(\vec{k}-\vec{k}'). \end{aligned} \tag{A.4}$$

$$\begin{aligned} \Big\langle \vec{B}(\vec{k},t)\cdot\vec{B}(\vec{k}',t)\Big\rangle_{t,V} &= 0, \\ \Big\langle \vec{B}(\vec{k},t)\cdot\vec{B}(\vec{k}',t)^*\Big\rangle_{t,V} &= C_B\,\delta(\vec{k}-\vec{k}'). \end{aligned} \tag{A.5}$$

where C_E and C_B are constants.These relations emerge after solving Maxwell's equations and subsequently performing the time and spatial averages. This justifies the step in Section 3 where we dropped cross terms between different Fourier modes. In Section 4, however,

we demonstrate how such expressions can be derived directly from statistical mechanics, without recourse to the explicit solution of the field equations.

Appendix B: Magnetic Contribution to the Energy Density

Substituting the Fourier representation of the magnetic field (18) into the definition of the energy density (19), we obtain

$$\begin{aligned}
&\underline{u}_B(\vec{x},t) \\
&= \frac{1}{2\mu_0 V_1(\vec{x})} \int_{V_1(\vec{x})} d^3x' \int \frac{d^3k_1}{(2\pi)^3} \int \frac{d^3k_2}{(2\pi)^3} \\
&\times \left[\left(\frac{\vec{k}_1}{\omega(\vec{k}_1)} \times \vec{E}(\vec{k}_1) \times e^{i\vec{k}_1\cdot\vec{x}'} e^{-i\omega(\vec{k}_1)t} + \text{c.c.} \right) \right. \\
&\left. \cdot \left(\frac{\vec{k}_2}{\omega(\vec{k}_2)} \times \vec{E}(\vec{k}_2) \times e^{i\vec{k}_2\cdot\vec{x}'} e^{-i\omega(\vec{k}_2)t} \right) \right] + \text{c.c.}
\end{aligned} \tag{B.6}$$

Applying the approximation (22), this expression simplifies to

$$\begin{aligned}
&\underline{u}_B(\vec{x},t) \\
&= \frac{1}{2\mu_0} \int \frac{d^3k}{(2\pi)^3} \left[-\frac{\big(\vec{k}\times\vec{E}(\vec{k})\big)\cdot\big(\vec{k}\times\vec{E}(-\vec{k})\big)}{\omega(\vec{k})^2} \right. \\
&\qquad \left. \times e^{-2i\omega(\vec{k})t} \right. \\
&\left. + \frac{\big(\vec{k}\times\vec{E}(\vec{k})^*\big)\cdot\big(\vec{k}\times\vec{E}(\vec{k})\big)}{\omega(\vec{k})^2} \right] + \text{c.c.}
\end{aligned} \tag{B.7}$$

and averaging over a suitable time period gives

$$\overline{\underline{u}}_B(\vec{x}) = \frac{1}{\mu_0} \int \frac{d^3k}{(2\pi)^3} \frac{\vec{k}\times\vec{E}(\vec{k})^*\cdot\vec{k}\times\vec{E}(\vec{k})}{\omega(\vec{k})^2}. \tag{B.8}$$

Using (17), the dispersion relation $\omega(\vec{k})^2 = c^2k^2$, and the identity $c^{-2} = \epsilon_0\mu_0$ we obtain

$$\overline{\underline{u}}_B(\vec{x}) = \epsilon_0 \int \frac{d^3k}{(2\pi)^3} \vec{E}(\vec{k})\cdot\vec{E}(\vec{k})^*. \tag{B.9}$$

Appendix C: Canonical Structure of the Fourier Transformation

In this appendix we verify that the Fourier transformation relating the position-space and momentum-space canonical variables is a *canonical transformation*.

Using (48) and (49), the quadratic terms of the Lagrangian (33) can be written in Fourier space as

$$\begin{aligned}
\int d^3x\, \|\vec{E}(\vec{x})\|^2 &= \int \frac{d^3k}{(2\pi)^3}\, \vec{E}(\vec{k})\cdot\vec{E}^*(\vec{k}), \\
\int d^3x\, \|\nabla\times\vec{A}(\vec{x})\|^2 &= \int \frac{d^3k}{(2\pi)^3} \big(\vec{k}\times\vec{A}(\vec{k})\big) \\
&\quad \cdot \big(\vec{k}\times\vec{A}^*(\vec{k})\big).
\end{aligned} \tag{C.10}$$

Since the relation between the electric field and the potentials is linear, their Fourier transforms are likewise linearly related:

$$\vec{E}(\vec{k}) = -\partial_t\vec{A}(\vec{k}) - i\vec{k}\,\phi(\vec{k}). \tag{C.11}$$

Substituting this into the Lagrangian (33) yields

$$\begin{aligned}
L = \frac{1}{2}\int \frac{d^3k}{(2\pi)^3} \Big[&\epsilon_0 \big(\partial_t\vec{A}(\vec{k}) + i\vec{k}\,\phi(\vec{k})\big) \\
&\cdot \big(\partial_t\vec{A}^*(\vec{k}) - i\vec{k}\,\phi^*(\vec{k})\big) \\
&- \big(\vec{k}\times\vec{A}(\vec{k})\big)\cdot\big(\vec{k}\times\vec{A}^*(\vec{k})\big)\Big].
\end{aligned} \tag{C.12}$$

From this expression the canonical momentum conjugate to $\vec{A}(\vec{k})$ follows directly:

$$\begin{aligned}
\vec{\pi}_{\vec{A}(\vec{k})} &:= \frac{\delta L}{\delta\partial_t\vec{A}(\vec{k})} \\
&= \frac{\epsilon_0}{(2\pi)^3}\big(\partial_t\vec{A}^*(\vec{k}) - i\vec{k}\,\phi^*(\vec{k})\big) \\
&= -\frac{\epsilon_0}{(2\pi)^3}\vec{E}^*(\vec{k}).
\end{aligned} \tag{C.13}$$

Thus the conjugate momentum to $\vec{A}(\vec{k})$ is precisely $-\frac{\epsilon_0}{(2\pi)^3}\vec{E}^*(\vec{k})$, in agreement with the Poisson bracket (54). This establishes that the Fourier transformation is not merely a convenient change of variables, but a genuine *canonical transformation*.

Appendix D: Completing the square in the exponential

The contribution of the electric field to the total Hamiltonian H_{tot} (64) can be written as

$$\begin{aligned}
\mathcal{H}_E &:= \int \frac{d^3k}{(2\pi)^3} \Big[\frac{\epsilon_0}{2}\vec{E}(\vec{k})\cdot\vec{E}(-\vec{k}) \\
&\qquad + \vec{E}(\vec{k})\cdot\vec{J}_E(-\vec{k}) \Big] \\
&= \frac{\epsilon_0}{2}\int \frac{d^3k}{(2\pi)^3} \left[\left(\vec{E}(-\vec{k}) + \frac{1}{\epsilon_0}\vec{J}_E(-\vec{k}) \right) \right. \\
&\qquad \left. \cdot \left(\vec{E}(\vec{k}) + \frac{1}{\epsilon_0}\vec{J}_E(\vec{k}) \right) \right] \\
&\quad - \frac{1}{2\epsilon_0}\int \frac{d^3k}{(2\pi)^3} \vec{J}_E(-\vec{k})\cdot\vec{J}_E(\vec{k}).
\end{aligned} \tag{D.14}$$

Appendix E: Evaluating the integral (85)

We expand in spherical coordinates:

$$\int \frac{d^3k}{(2\pi)^3} P_{ij}(\vec{k}) = \left(\int \frac{dk}{(2\pi)^3} k^2\right) \times \int d\Omega \left(\delta_{ij} - \hat{k}_i \hat{k}_j\right). \tag{E.15}$$

The angular and radial parts separate, yielding

$$\int d\Omega \left(\delta_{ij} - \hat{k}_i \hat{k}_j\right) = \frac{8\pi}{3}\delta_{ij}, \tag{E.16}$$

and

$$\int \frac{dk}{(2\pi)^3} k^2 = \frac{1}{4\pi} \int \frac{d^3k}{(2\pi)^3} = \frac{1}{4\pi}\mathcal{N}. \tag{E.17}$$

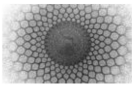
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TTMP vol 3 (3), 2026
doi: 10.30511/TTMP.2026.2096322.1092

Transactions in Theoretical and Mathematical Physics



A Dynamical Transition from Confinement to Blow-Up in an Oscillatory Differential Equation

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Received: 01 August 2026 / Accepted: 25 August 2026 / Published: 25 August 2026

Abstract In this paper, we investigate the nonlinear ordinary differential equation

$$y' = y^p(\sin y + \varepsilon), \qquad p > 1,$$

with positive initial data. We show that the qualitative behavior of solutions strongly depends on the perturbation parameter ε. For $0 \leq \varepsilon < 1$, the nonlinearity possesses infinitely many zeros, which generate invariant intervals trapping the trajectories. Consequently, all solutions are global and converge to equilibrium points. In contrast, when $\varepsilon > 1$, all zeros disappear, and the equation becomes strictly positive, leading to finite-time blow-up for every positive solution via an Osgood-type argument. The results provide a simple example illustrating how a small structural perturbation may completely change the global dynamics of an oscillatory differential equation.

Keywords Finite-time blow-up, Global existence, Oscillatory nonlinearity, Invariant intervals, Ordinary differential equations, Osgood criterion.

1 Introduction

The qualitative study of nonlinear ordinary differential equations has attracted considerable attention because of its importance in nonlinear analysis, mathematical physics, population dynamics, and reaction–diffusion theory. In particular, researchers have extensively investigated questions of global existence, asymptotic behaviour, and finite-time blow-up over the past decades; see, for instance, the classical monographs by Hartman [1], Hale [2], and Friedman [3].

A fundamental model in this context is the scalar autonomous equation

$$y'(t) = f(y(t)), \tag{1}$$

where the long-time behavior of solutions strongly depends on the growth and sign properties of the nonlinearity f. One of the classical criteria for finite-time blow-up is the Osgood criterion introduced by Osgood [4], which states that blow-up occurs whenever

$$\int^{+\infty} \frac{ds}{f(s)} < +\infty, \tag{2}$$

under suitable positivity assumptions on f. This criterion has played a major role in the qualitative theory of differential equations and has been extended in several directions; see, e.g., [5, 6].

For superlinear nonlinearities such as

$$f(y) = y^p, \qquad p > 1, \tag{3}$$

all positive solutions blow up in finite time. However, the situation changes drastically when the nonlinearity possesses zeros or oscillatory behavior. In such a case, equilibrium points may generate invariant regions trapping the trajectories and preventing blow-up phenomena.

Oscillatory nonlinearities appear naturally in several mathematical models and may lead to unexpected qualitative behaviors. In particular, the equation

$$y'(t) = y(t)^p \sin(y(t)) \tag{4}$$

provides a simple example in which infinitely many equilibria divide the phase space into invariant intervals. Consequently, trajectories remain confined between two consecutive zeros and converge asymptotically toward equilibrium points.

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The purpose of the present paper is to investigate how a constant perturbation modifies this confinement mechanism. More precisely, we study the equation

$$y'(t) = y(t)^p\big(\sin(y(t)) + \varepsilon\big), \qquad p > 1, \tag{5}$$

where $\varepsilon \geq 0$ is a real parameter, we show that the qualitative behavior of solutions undergoes a sharp transition depending on the value of ε. When $0 \leq \varepsilon < 1$, the perturbed nonlinearity still possesses infinitely many zeros. These zeros continue to generate invariant intervals, and all solutions remain global. On the other hand, when $\varepsilon > 1$, all equilibria disappear because $\sin y + \varepsilon > 0$ for every $y \in \mathbb{R}$. The equation then behaves like a purely superlinear differential equation, and all positive solutions blow up in finite time by an Osgood-type argument.

Therefore, this work illustrates how a simple perturbation can destroy the invariant-interval structure generated by oscillatory nonlinearities, producing a transition from global existence to finite-time blow-up.

The paper is organized as follows. Section 2 introduces the initial-value problem. Section 3 analyzes the equilibrium structure and the associated invariant intervals. Section 3.1 establishes the global-existence and blow-up alternatives. Section 4 gives an estimate of the blow-up time. Concluding remarks are presented in Section 5.

2 Setting and assumptions

We consider the initial-value problem

$$\begin{cases} y'(t) = y(t)^p\big(\sin(y(t)) + \varepsilon\big), \\ y(0) = y_0 > 0, \end{cases} \tag{6}$$

where $p > 1$ and $\varepsilon \geq 0$. We assume that $y(t)$ is a classical solution defined on its maximal interval of existence.

3 Structure of equilibria

The positive equilibria satisfy

$$\sin y + \varepsilon = 0. \tag{7}$$

Hence:

1. if $0 \leq \varepsilon < 1$, the equation admits infinitely many equilibria;
2. if $\varepsilon = 1$, the equation is in the critical regime and possesses degenerate equilibria;
3. if $\varepsilon > 1$, there is no equilibrium.

When equilibria exist, we denote consecutive equilibrium points by $a_k < b_k$ and the corresponding open intervals by (a_k, b_k).

3.1 Main theorem

Theorem 1 (Global existence versus blow-up transition) *Consider the initial-value problem (6).*

(i) Subcritical case: $0 \leq \varepsilon < 1$. *Every positive solution is global and remains trapped in the interval determined by its initial data. Moreover,*

$$\lim_{t\to+\infty} y(t) = y_*, \tag{8}$$

where y_* is an equilibrium satisfying

$$y_*^p(\sin y_* + \varepsilon) = 0. \tag{9}$$

Since $y_* > 0$, this is equivalent to $\sin y_* + \varepsilon = 0$.

(ii) Critical case: $\varepsilon = 1$. *Every positive solution is global and is either stationary or satisfies*

$$\lim_{t\to+\infty} y(t) = \frac{3\pi}{2} + 2k\pi \tag{10}$$

for an integer k determined by the initial condition.

(iii) Supercritical case: $\varepsilon > 1$. *Every positive solution blows up in finite time.*

3.2 Proof of the theorem

Proof (i) Let $0 \leq \varepsilon < 1$. Fix two consecutive positive equilibrium points $a < b$ such that $y_0 \in (a, b)$. By construction, $\sin y + \varepsilon$ does not change sign on (a, b). Therefore, the right-hand side of (5) has constant sign on this interval, and the solution $y(t)$ is monotone.

We claim that (a, b) is invariant. Indeed, assume by contradiction that the solution reaches one of the endpoints, say a, at some finite time $t_1 > 0$. Since

$$a^p(\sin a + \varepsilon) = 0, \tag{11}$$

the constant function $y(t) \equiv a$ is also a solution. By uniqueness of solutions for ordinary differential equations, a trajectory cannot cross the equilibrium point a, contradicting the assumption. The same argument applies to b. Hence

$$a < y(t) < b \qquad \text{for all } t \geq 0. \tag{12}$$

Since $y(t)$ is monotone and bounded, the finite limit

$$y_* := \lim_{t\to+\infty} y(t) \tag{13}$$

exists. Passing to the limit in the differential equation shows that

$$y_*^p(\sin y_* + \varepsilon) = 0. \tag{14}$$

Because $y_* > 0$, we conclude that $\sin y_* + \varepsilon = 0$. Thus y_* is an equilibrium point, and the solution is global.

Equivalently, the real line is partitioned into invariant intervals by the zeros of $\sin y + \varepsilon$. Inside each interval the right-hand side has constant sign, the solution is monotone, and the trajectory remains bounded between two equilibria.

(ii) Let $\varepsilon = 1$. Then

$$1 + \sin y \geq 0 \tag{15}$$

for all $y \in \mathbb{R}$, with degenerate equilibria at

$$y_k = \frac{3\pi}{2} + 2k\pi, \qquad k \in \mathbb{Z}. \tag{16}$$

If $y_0 = y_k$ for some positive equilibrium y_k, the solution is stationary. Otherwise, the solution is strictly increasing and remains below the next equilibrium. It therefore exists globally and converges to that equilibrium.

(iii) Let $\varepsilon > 1$. For every $y \in \mathbb{R}$,

$$\sin y + \varepsilon \geq \varepsilon - 1 > 0. \tag{17}$$

Consequently,

$$y'(t) \geq (\varepsilon - 1) y(t)^p. \tag{18}$$

Comparison with the solution of

$$z'(t) = (\varepsilon - 1) z(t)^p, \qquad z(0) = y_0, \tag{19}$$

or direct application of the Osgood criterion (2), gives finite-time blow-up. □

4 Blow-up-time estimate

Proposition 1 *If* $\varepsilon > 1$, *the maximal existence time* T *of a positive solution satisfies*

$$T \leq \frac{y_0^{1-p}}{(p-1)(\varepsilon - 1)}. \tag{20}$$

Proof The explicit solution of the comparison equation (19) is

$$z(t) = \frac{y_0}{\left[1 - (p-1)(\varepsilon - 1) y_0^{p-1} t\right]^{1/(p-1)}}. \tag{21}$$

It blows up at

$$T_z = \frac{y_0^{1-p}}{(p-1)(\varepsilon - 1)}. \tag{22}$$

Since (18) implies $y(t) \geq z(t)$ while both solutions exist, the solution y must blow up no later than z. This proves (20). □

5 Concluding remarks

Remark 1 The preceding theorem shows that the disappearance of equilibrium points fundamentally changes the equation's qualitative behaviour. In the regime $0 \leq \varepsilon < 1$, the oscillatory structure generates invariant intervals preventing trajectories from escaping to infinity. In contrast, when $\varepsilon > 1$, all equilibria disappear, and the dynamics becomes purely expansive, leading to finite-time blow-up.

Remark 2 The critical value $\varepsilon = 1$ separates two fundamentally different dynamical regimes: a confinement regime characterized by invariant intervals and global existence, and a blow-up regime characterized by monotone growth and finite-time singularity. Hence, $\varepsilon = 1$ may be viewed as a bifurcation threshold for the global dynamics of the equation.

In this paper, we investigated the qualitative behavior of solutions to the oscillatory differential equation (5). Our analysis shows that the structure of the equilibrium points generated by the oscillatory nonlinearity strongly influences the global dynamics. For $0 \leq \varepsilon < 1$, the equation possesses infinitely many equilibria, which partition the phase space into invariant intervals. As a consequence, every solution remains confined inside the interval determined by its initial data and converges asymptotically toward an equilibrium point. In particular, all solutions are global. In contrast, when $\varepsilon > 1$, all equilibrium points disappear. The equation then becomes strictly positive and behaves like a superlinear autonomous equation. Using a comparison argument together with the Osgood criterion, we proved that every positive solution blows up in finite time. The critical value $\varepsilon = 1$ therefore represents a threshold separating two qualitatively different dynamical regimes: confinement with invariant intervals and global existence, and blow-up characterized by monotone growth and finite-time singularity. The present work shows how a simple perturbation can completely change the global behaviour of an oscillatory nonlinear differential equation by destroying its equilibrium structure. Future work may consider several extensions, including reaction–diffusion equations, coupled systems, and nonautonomous perturbations.

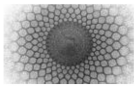
Acknowledgements The author would like to thank the anonymous referee for useful and detailed comments on a previous version of the manuscript.

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TTMP vol 3 (3), 2026
doi: 10.30511/TTMP.2026.2095670.1087

Transactions in Theoretical and Mathematical Physics



Generalized coherent states with shifted (displaced) arguments[1]

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Received: 26 July 2026 / Accepted: 26 August 2026 / Published: 31 August 2026

Abstract In the paper we developed a procedure for constructing generalized coherent states with shifted argument, as a result of the action of the generalized analogue of the displacement operator. This was based on the successive action of a pair of nonlinear ladder operators, which generate nonlinear coherent states. To examine the properties of coherent states with shifted argument, the rules of the diagonal operator ordering technique (DOOT) were used. The results obtained were verified for a series of particular cases, obtaining expressions consistent with those in the literature. The expressions obtained will be, among others, useful in quantum optics, e.g. for calculating the Wigner operator in the representation of coherent states.

keywords: coherent states, displacement operator, hypergeometric function.

1 Introduction

It is well known that, in many branches of physics, the so-called displacement operator is used. In essence, this operator translates the variable x, characteristic of a given physical quantity, by an amount Δx. It transforms a function of x into the same function evaluated at the shifted argument $x + \Delta x$.

In general, the unitary translation or displacement operator is the exponential operator

$$\hat{D}(\Delta x) = \exp\left(\Delta x \frac{\partial}{\partial x}\right),$$

which transfers an initial state or function $\Phi(x)$ into a state or function with shifted argument:

$$\exp\left(\Delta x \frac{\partial}{\partial x}\right) \Phi(x) = \Phi(x + \Delta x). \tag{1}$$

In quantum mechanics, displacement operators are used to change the position q and/or the momentum p in phase space. The associated generators $\hat{Q}$ and $\hat{P}$ satisfy

$$[\hat{Q}, \hat{P}] = i, \qquad \hbar = 1.$$

The displacement operator is then defined by

$$\hat{D}(q, p) = \exp\left[i\left(p\hat{Q} - q\hat{P}\right)\right]. \tag{2}$$

The next step toward quantum optics is the introduction of the complex variable z, together with the canonical creation operator $\hat{a}^\dagger$ and annihilation operator $\hat{a}$:

$$\begin{aligned} \begin{pmatrix} z \\ z^* \end{pmatrix} &= \frac{1}{\sqrt{2}} \begin{pmatrix} q + ip \\ q - ip \end{pmatrix}, \\ \begin{pmatrix} \hat{a} \\ \hat{a}^\dagger \end{pmatrix} &= \frac{1}{\sqrt{2}} \begin{pmatrix} \hat{Q} + i\hat{P} \\ \hat{Q} - i\hat{P} \end{pmatrix}. \end{aligned} \tag{3}$$

The inverse relations are

$$\begin{aligned} q &= \frac{1}{\sqrt{2}}(z^* + z), & p &= \frac{i}{\sqrt{2}}(z^* - z), \\ \hat{Q} &= \frac{1}{\sqrt{2}}(\hat{a}^\dagger + \hat{a}), & \hat{P} &= \frac{i}{\sqrt{2}}(\hat{a}^\dagger - \hat{a}). \end{aligned} \tag{4}$$

Since the complex variable z enters the formalism of coherent states (CSs), the preceding relations give the following expression for the displacement operator in quantum optics:

$$\hat{D}(z) = \exp\left(z\hat{a}^\dagger - z^*\hat{a}\right). \tag{5}$$

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[1] The article is intended to be a tribute to Emeritus Professor Hong-Yi Fan, from the University of Science and Technology of China (USTC), China, for his contribution to the creation and promotion of the IWOP calculation technique in Quantum Mechanics.

In the physical world, however, some phenomena require operators that are more complicated than the canonical operators associated with the linear one-dimensional harmonic oscillator. Whereas canonical operators generate linear coherent states (LCSs), these more complicated operators correspond to nonlinear coherent states (NCSs). It is therefore natural to assume that the formalism of nonlinear or generalized coherent states involves a generalized analogue of the displacement operator.

The formalism of LCSs underwent considerable development following the introduction of the technique of integration within an ordered product (IWOP). This significant computational advance, which makes full use of Dirac's symbolic representation, was developed by Hong-Yi Fan and his collaborators beginning in 1982; see Refs. [1–4] and the references therein. It made calculations within the formalism of coherent states more transparent and easier to interpret. Nevertheless, applications of the IWOP technique have largely remained restricted to canonical coherent states, without being extended to nonlinear coherent states.

To generalize the IWOP technique to nonlinear systems beyond the one-dimensional harmonic oscillator, we developed a related method called the *diagonal ordering operation technique* (DOOT) [5]. We subsequently applied this technique to several nonlinear oscillators characterized by different classes of coherent states [6–10].

Consequently, the construction of the generalized analogue of the displacement operator $\hat{D}(z)$ is based on the *DOOT–IWOP correspondence principle*. This principle belongs to a broader framework relating the characteristic quantities of nonlinear systems to those of linear systems. It states that every characteristic quantity X_{DOOT} in the DOOT formalism corresponds to a quantity X_{IWOP} in the IWOP formalism through the following limit:

$$\begin{aligned} &\lim_{\substack{p=q\\ \widetilde{a}=\widetilde{b}}} X_{\mathrm{DOOT}} = X_{\mathrm{IWOP}}, \\ &\lim_{\substack{p=q\\ \widetilde{a}=\widetilde{b}}} \hat{D}_{\mathrm{DOOT}}(z) = \hat{D}_{\mathrm{IWOP}}(z). \end{aligned} \tag{6}$$

Here, as will be shown below, the integers p and q, together with the sets of real parameters $\widetilde{a}$ and $\widetilde{b}$, characterize the DOOT formalism.

In this article, we apply the rules of the DOOT technique to examine the expressions and properties of coherent states with shifted or displaced arguments. All the intermediate results are verified, by taking the preceding limit, in the case of the canonical coherent states associated with the one-dimensional harmonic oscillator (HO-1D).

In this sense, the article may be regarded as a tribute to Professor Hong-Yi Fan, one of the initiators and leading proponents of the IWOP calculation technique for canonical coherent states. The application of this technique has not only clarified several questions involving quantum operators and coherent states, but has also produced new formulas useful in the theory of complex functions.

2 The case of canonical (linear) coherent states

In this section, we recall several properties of canonical coherent states and displacement operators that will be useful below. For the one-dimensional harmonic oscillator, the normalized canonical coherent states have the same expression in the Barut–Girardello, Klauder–Perelomov, and Gazeau–Klauder definitions [11].

These states can be expanded in the orthonormal Fock basis $\{|n\rangle\}_{n=0}^{\infty}$, whose elements are eigenstates of the number operator:

$$\hat{N}|n\rangle = n|n\rangle, \qquad \langle n|m\rangle = \delta_{nm}. \tag{7}$$

The canonical coherent states are then given by

$$|z\rangle = e^{-\frac{1}{2}|z|^2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} |n\rangle. \tag{8}$$

They are labeled by the complex variable

$$z = |z|e^{i\phi}, \qquad 0 \le |z| < \infty, \qquad 0 \le \phi < 2\pi. \tag{9}$$

The same expression can be obtained using the displacement operator:

$$\begin{aligned} \hat{D}(z) &= \exp\left(z\hat{a}^{\dagger} - z^{*}\hat{a}\right) \\ &= e^{-\frac{1}{2}|z|^2} e^{z\hat{a}^{\dagger}} e^{-z^{*}\hat{a}}. \end{aligned} \tag{10}$$

This normally ordered form follows from the Baker–Campbell–Hausdorff formula

$$\begin{aligned} \exp\left(\hat{X} + \hat{Y}\right) = \exp(\hat{X}) \exp(\hat{Y}) \\ \times \exp\left(-\frac{1}{2}[\hat{X}, \hat{Y}]\right). \end{aligned} \tag{11}$$

provided that $[\hat{X}, \hat{Y}]$ commutes with both $\hat{X}$ and $\hat{Y}$.

The canonical creation and annihilation operators, $\hat{a}^{\dagger}$ and $\hat{a}$, satisfy

$$[\hat{a}, \hat{a}^{\dagger}] = 1. \tag{12}$$

Their actions on the Fock states and the corresponding bra states are

$$\begin{aligned}
&\hat{a}|n\rangle = \sqrt{n}\,|n-1\rangle,\\
&\langle n|\hat{a}^\dagger = \sqrt{n}\,\langle n-1|,\\
&\hat{a}^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle,\\
&\langle n|\hat{a} = \sqrt{n+1}\,\langle n+1|,\\
&\left(\hat{a}^\dagger\right)^n|0\rangle = \sqrt{n!}\,|n\rangle,\\
&\langle 0|\hat{a}^n = \sqrt{n!}\,\langle n|,\\
&\langle n|\hat{a}^\dagger\hat{a}|n\rangle = n,\\
&\langle n|\hat{a}\hat{a}^\dagger|n\rangle = n+1.
\end{aligned} \tag{13}$$

To these operators, we may add the particle-number operator, usually called simply the number operator, $\hat{N} = \hat{a}^\dagger\hat{a}$. This operator will be useful below. It is self-adjoint:

$$\hat{N}^\dagger = \hat{N} = \hat{a}^\dagger\hat{a}. \tag{14}$$

The coherent states can be obtained by applying the displacement operator $\hat{D}(z)$ to the vacuum state $|0\rangle$. Using

$$\hat{a}|0\rangle = 0, \qquad e^{-z^*\hat{a}}|0\rangle = |0\rangle. \tag{15}$$

we obtain

$$\begin{aligned}
|z\rangle &= \hat{D}(z)|0\rangle\\
&= e^{-\frac{1}{2}|z|^2}e^{z\hat{a}^\dagger}e^{-z^*\hat{a}}|0\rangle\\
&= e^{-\frac{1}{2}|z|^2}e^{z\hat{a}^\dagger}|0\rangle.
\end{aligned} \tag{16}$$

The Hermitian-conjugate relation is

$$\begin{aligned}
\langle z| &= \langle 0|\hat{D}^\dagger(z) = \langle 0|\hat{D}(-z)\\
&= \langle 0|e^{z^*\hat{a}-z\hat{a}^\dagger}\\
&= \langle 0|e^{-\frac{1}{2}|z|^2}e^{z^*\hat{a}}e^{-z\hat{a}^\dagger}.
\end{aligned} \tag{17}$$

Since coherent states are normalized to unity,

$$\begin{aligned}
\langle z|z\rangle &= \langle 0|\hat{D}^\dagger(z)\hat{D}(z)|0\rangle\\
&= 1.
\end{aligned} \tag{18}$$

It follows that the displacement operator is unitary:

$$\hat{D}^\dagger(z)\hat{D}(z) = \hat{D}(z)\hat{D}^\dagger(z) = \hat{I}. \tag{19}$$

We now apply a second displacement operator, depending on the complex variable σ, to the previously obtained coherent state. Since $\hat{a}|z\rangle = z|z\rangle$, we successively obtain

$$\begin{aligned}
\hat{D}(\sigma)|z\rangle &= e^{-\frac{1}{2}|\sigma|^2}e^{\sigma\hat{a}^\dagger}e^{-\sigma^*\hat{a}}|z\rangle\\
&= e^{-\frac{1}{2}|\sigma|^2-\sigma^*z}e^{\sigma\hat{a}^\dagger}|z\rangle\\
&= e^{-\frac{1}{2}|\sigma|^2-\frac{1}{2}|z|^2-\sigma^*z}\sum_{l=0}^{\infty}\frac{z^l}{\sqrt{l!}}\,e^{\sigma\hat{a}^\dagger}|l\rangle.
\end{aligned} \tag{20}$$

The action of the exponential operator on a Fock state is

$$e^{\sigma\hat{a}^\dagger}|l\rangle = \sum_{m=0}^{\infty}\frac{\sigma^m}{m!}\sqrt{\frac{(l+m)!}{l!}}\,|l+m\rangle. \tag{21}$$

Substituting this expression gives

$$\begin{aligned}
\hat{D}(\sigma)|z\rangle = e^{-\frac{1}{2}|\sigma|^2-\frac{1}{2}|z|^2-\sigma^*z}&\\
\times\sum_{m=0}^{\infty}\sum_{l=0}^{\infty}\frac{(l+m)!}{\sqrt{(l+m)!}\,l!\,m!}&\\
\times z^l\sigma^m\,|l+m\rangle.&
\end{aligned} \tag{22}$$

Introducing the new summation index

$$n = l+m, \qquad l = n-m \geq 0, \qquad 0 \leq m \leq n. \tag{23}$$

we find

$$\begin{aligned}
\hat{D}(\sigma)|z\rangle &= e^{-\frac{1}{2}|\sigma|^2-\frac{1}{2}|z|^2-\sigma^*z}\\
&\quad\times\sum_{n=0}^{\infty}\frac{1}{\sqrt{n!}}\sum_{m=0}^{n}\frac{n!}{(n-m)!\,m!}\\
&\quad\times z^{n-m}\sigma^m\,|n\rangle\\
&= e^{-\frac{1}{2}|\sigma|^2-\frac{1}{2}|z|^2-\sigma^*z}\\
&\quad\times\sum_{n=0}^{\infty}\frac{1}{\sqrt{n!}}\left[\sum_{m=0}^{n}\binom{n}{m}z^{n-m}\sigma^m\right]|n\rangle\\
&= e^{-\frac{1}{2}|\sigma|^2-\frac{1}{2}|z|^2-\sigma^*z}\\
&\quad\times\sum_{n=0}^{\infty}\frac{(z+\sigma)^n}{\sqrt{n!}}|n\rangle\\
&= e^{\frac{1}{2}(\sigma z^*-\sigma^*z)}|z+\sigma\rangle.
\end{aligned} \tag{24}$$

Consequently, the coherent state with shifted argument satisfies

$$\begin{aligned}
|z+\sigma\rangle &= e^{-\frac{1}{2}(\sigma z^*-\sigma^*z)}\hat{D}(\sigma)\hat{D}(z)|0\rangle\\
&= \hat{D}(z+\sigma)|0\rangle.
\end{aligned} \tag{25}$$

which means that the successive displacements are equivalent to a single displacement up to an phase factor. The above relation can be written in an equivalent manner,

$$|z+\sigma\rangle e^{-\ \frac{1}{2}\ |z+\sigma|^2}\sum_{n=0}^{\infty}\frac{(z+\sigma)^n}{\sqrt{n\ !}}|n> \tag{2.14}$$

This expression of CSs with shifted argument could also be deduced directly, if we consider that the sum of two complex numbers z and σ is also a complex number$z+\sigma$. It is observed that the expression on the right-hand side is symmetric in the variables z andσ, which thus have "equal rights".

3 Generalized coherent states

For several decades since the introduction, in 1926, by Schrödinger, of the notion of coherent state [12], the scientific world was only interested in the canonical CSs, connected with the one-dimensional harmonic oscillator. Over time, however, other types of CSs began to appear and were named nonlinear CSs (NCSs) or, more recently, generalized CSs (GCSs). As a counterpart, the canonical CSs of HO-1D were named linear CSs.

Mainly, any nonlinear or generalized CSs have the following structure:

$$|z\rangle = \frac{1}{\sqrt{N(|z|^2)}} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{\rho(n)}} |n\rangle, \tag{26}$$

where $\{\rho(n)\}_{n=0}^{\infty}$ is the sequence of strictly positive numbers, called *structure constants*, because they determine the structure and character of the normalization function. This must be an analytical function:

$$N(|z|^2) = \sum_{n=0}^{\infty} \frac{1}{\rho(n)} (|z|^2)^n. \tag{27}$$

This function must be strictly positive and convergent within a disc, a circle or domain of the complex z-plane. This condition assures that the probabilities are well-defined. Since the numbers are strictly positive, it follows automatically that the terms of the series are positive.

In essence, generalized CSs are generated by a conjugate (adjoint) pair of ladder operators: the creation $\hat{A}_+$ and the annihilation $\hat{A}_-$, where $(\hat{A}_+)^\dagger = \hat{A}_-$ and $(\hat{A}_-)^\dagger = \hat{A}_+$. These operators are connected with the canonical operators $\hat{a}^\dagger$ and $\hat{a}$ through a function depending on the particle number operator $\hat{N} = \hat{a}^\dagger \hat{a}$. This real function $f(\hat{N})$ is called the nonlinearity function.

Nonlinear operators have the following structure:

$$\hat{A}_- = \hat{a}\, f(\hat{N}), \qquad \hat{A}_+ = f(\hat{N})\, \hat{a}^\dagger. \tag{28}$$

The nonlinearity function $f(\hat{N})$ can be, up to a point, arbitrary, as long as it ensures that the normalization function is an analytic function (non-analytic ones having no physical meaning). Moreover, the nonlinearity function $f(\hat{N})$ satisfies the following general relation:

$$f(\hat{N}+m)\,|n+p\rangle = f(n+p+m)\,|n+p\rangle. \tag{29}$$

At the limit the pair of newly introduced operators reduces to the canonical operators:

$$\lim_{\substack{p=q \\ a=b}} \begin{pmatrix} \hat{A}_- \\ \hat{A}_+ \end{pmatrix} = \lim_{f(\hat{N})\to 1} \begin{pmatrix} \hat{A}_- \\ \hat{A}_+ \end{pmatrix} = \begin{pmatrix} \hat{a} \\ \hat{a}^\dagger \end{pmatrix}. \tag{30}$$

We will choose the nonlinearity function in the following manner:

$$f(\hat{N}) = \sqrt{\frac{\prod_{j=1}^{q}(b_j - 1 + \hat{N})}{\prod_{i=1}^{p}(a_i - 1 + \hat{N})}}. \tag{31}$$

Its eigenvalue equation is

$$f(\hat{N})\,|n\rangle = \sqrt{\frac{\prod_{j=1}^{q}(b_j - 1 + n)}{\prod_{i=1}^{p}(a_i - 1 + n)}}\,|n\rangle. \tag{32}$$

The actions of these operators and their Hermitian conjugates on the Fock vectors are

$$\begin{aligned} &\hat{A}_-|n\rangle = \sqrt{e(n)}\,|n-1\rangle, \\ &\langle n|\,\hat{A}_+ = \sqrt{e(n)}\,\langle n-1|, \\ &\hat{A}_+|n\rangle = \sqrt{e(n+1)}\,|n+1\rangle, \\ &\langle n|\,\hat{A}_- = \sqrt{e(n+1)}\,\langle n+1|, \\ &\langle n|\,\hat{A}_+\hat{A}_-|n\rangle = e(n). \end{aligned} \tag{33}$$

Here we used the notation:

$$\begin{aligned} &e(n) \equiv n\ \ (f(n))^2 = n\ \ g(n) \quad , \\ &g(n) \equiv (f(n))^2 = \frac{\prod_{j=1}^{q}(b_j - 1 + n)}{\prod_{i=1}^{p}(a_i - 1 + n)} \end{aligned} \tag{3.9}$$

The repeated action of the creation operator on the vacuum ket vector $|0>$ (and similarly for their hermitian conjugate $0|$) is summarized as follows:

$$\left(\hat{A}_+\right)^n |0\rangle = \sqrt{\prod_{s=1}^{n} e(s)}\,|n\rangle = \sqrt{\rho(n)}\,|n\rangle, \tag{34}$$

and

$$\langle 0|\left(\hat{A}_-\right)^n = \langle n|\sqrt{\prod_{s=1}^{n} e(s)} = \langle n|\sqrt{\rho(n)}. \tag{35}$$

where we introduced a new notation

$$\rho(n) = [e(n)]! = \prod_{s=1}^{n} e(s) = n!\,[g(n)]!, \tag{36}$$

with

$$\lim_{f(n)\to 1} \rho(n) = n!, \qquad \rho(0) = 1. \tag{37}$$

Here we used the *functional factorial* defined as

$$[x(n)]! \equiv x(1)x(2)\cdots x(n), \qquad [x(0)]! = 1, \tag{38}$$

as well as the Pochhammer symbol

$$(x)_n = \prod_{s=1}^{n}(x+s-1) = \frac{\Gamma(x+n)}{\Gamma(x)}. \tag{39}$$

Consequently, the structure function can be written as

$$\rho(n) = n!\,\frac{\prod_{j=1}^{q}(b_j)_n}{\prod_{i=1}^{p}(a_i)_n} \equiv n!\,[g(n)]!, \tag{40}$$

where

$$[g(n)]! \equiv \frac{\prod_{j=1}^{q}(b_j)_n}{\prod_{i=1}^{p}(a_i)_n}, \qquad [g(0)]! = 1. \tag{41}$$

This entity, in which a function that depends on a number appears instead of a number, can be called *functional factorial*.

In the following we will also use the short notations:

$$\binom{\rho(l)}{\rho(n)} = \frac{\rho(l)}{\rho(n)\,\rho(l-n)}, \tag{42}$$

and

$$\binom{g(l)}{g(n)} \equiv \frac{[g(l)]!}{[g(n)]!\,[g(l-n)]!}, \tag{43}$$

so that

$$\binom{\rho(l)}{\rho(n)} = \binom{l}{n}\binom{g(l)}{g(n)}. \tag{44}$$

The entity $\binom{\rho(l)}{\rho(n)}$ can be interpreted as the *functional binomial coefficients*, because at the limit, this turns into the usual binomial coefficient:

$$\lim_{\substack{p=q\\ \widetilde{a}=\widetilde{b}=1}} \binom{\rho(l)}{\rho(n)} = \binom{l}{n}. \tag{45}$$

Generally, the ladder operators $\hat{A}_+$ and $\hat{A}_-$ are not commutable, but, as we will see below, they obey the rules of the *Diagonal Operator Ordering Technique* (DOOT), which we introduced in a previous paper [5]. The main rules of the DOOT are:

a) Inside the symbol $\#\cdots\#$ the order of operators $\hat{A}_+$ and $\hat{A}_-$ can be permuted, $\hat{A}_-\hat{A}_+ = \hat{A}_+\hat{A}_-$, so that finally we obtain a function or an expression that is normally ordered (operator $\hat{A}_+$ is located on the left side and operator $\hat{A}_-$ is on its right side).

b) Inside the symbol $:\cdots:$ we can perform all algebraic operations (derivations, integrations), according to the usual rules.

Consequently, all operator functions involved in the DOOT technique must have a formal series expansion of a specific form in order to yield non-zero states:

$$f(\hat{A}_+, \hat{A}_-) = \sum_{n=0}^{\infty} C_n\,(\hat{A}_+\hat{A}_-)^n. \tag{46}$$

c) The ladder operators $\hat{A}_+$ and $\hat{A}_-$ can be treated as simple c-numbers;

d) the vacuum state projector $|0><0|$, in the frame of DOOT, has the following normal ordered form:

$$\frac{1}{{}_pF_q(\hat{A}_+\hat{A}_-)} = \sum_{n=0}^{\infty} \frac{(\hat{A}_+\hat{A}_-)^n}{\rho(n)},$$

where the abbreviated notation for the coefficients are: $\widetilde{a} \equiv \{a_1, a_2, \ldots, a_p\} \equiv \{a_i\}_1^p$ and ${}_pF_q(\widetilde{a};\widetilde{b};x)$ for the generalized hypergeometric function.

It follows from the above rules of the DOOT technique that creation and annihilation operators commute only if they are located within the ## sign. Outside this sign, these operators do not commute:

$$\hat{A}_-\hat{A}_+ \neq \hat{A}_+\hat{A}_-, \tag{47}$$

but within the ## sign they commute:

$$\#\hat{A}_-\hat{A}_+\# = \#\hat{A}_+\hat{A}_-\#. \tag{48}$$

Furthermore, to obtain a non-zero result, the creation and annihilation operators within the ordered product must have the same power.

$$(\hat{A}_-)^m(\hat{A}_+)^n = (\hat{A}_+)^n(\hat{A}_-)^m = \delta_{nm}(\hat{A}_+\hat{A}_-)^n. \tag{49}$$

To simplify writing formulas, everywhere possible, without causing confusion, in the following we will write the hypergeometric functions without mentioning the set of numbers $\widetilde{a}$ and $\widetilde{b}$ and indicating only their argument, e.g. ${}_pF_q(\widetilde{a};\widetilde{b};x) \equiv {}_pF_q(x)$. Instead, we will use the same $|z\rangle$ notation for both generalized CSs and canonical CSs. We will explicitly specify the sets of numbers $\widetilde{a}$ and $\widetilde{b}$ only where strictly necessary, to avoid confusion.

With this notation, the generalized hypergeometric function will be written as

$$\begin{aligned} {}_pF_q(\widetilde{a};\widetilde{b};x) &\equiv {}_pF_q(x) \\ &= \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{p}(a_i)_n}{\prod_{j=1}^{q}(b_j)_n}\frac{x^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{x^n}{\rho(n)}. \end{aligned} \tag{50}$$

One of the important advantages of using the DOOT technique (as well as the IWOP technique) is that it allows us to express various entities in terms of creation/annihilation operators $\hat{A}_+$ and $\hat{A}_-$. Including the vacuum projector $|0\rangle\langle 0|$ which is expressed as a function that depends on the normal ordered product of creation and annihilation operators $\hat{A}_+\hat{A}_-$. Thus we end up working with functions that have operators as arguments, and these are viewed as simple *c*-numbers.

Let us emphasize here that the DOOT technique is a generalization of the technique of *Integration Within an Ordered Product* (IWOP), introduced by Hong-yi Fan

(see, e.g. [3] and references therein). But, the difference between the two techniques consists in the fact that the IWOP is applicable only for Bose operators, referring to the CSs of the HO-1D, while DOOT applies to any pair of creation/annihilation operators, adopting the above rules. In this sense, the IWOP appears as a particular case of DOOT.

The choice of ladder operators $\hat{A}_+$ and $\hat{A}_-$, having the above properties and which obey the rules of the DOOT technique was not accidental. This choice leads to the fact that the normalization function of CSs will be the generalized hypergeometric function itself, ${}_pF_q(x)$.

Consequently, let us define the CSs in the Barut-Girardello manner, i.e. as the eigenvalues of the annihilation operator

$$\hat{A}_-|z\rangle = z|z\rangle, \qquad \langle z|\hat{A}_+ = z^*\langle z|. \tag{51}$$

The second relation can be obtained either by complex conjugation of the first relation, or by using the property $\langle n|\hat{A}_+ = \sqrt{e(n)}\,\langle n-1|$, see Eq. (3.8).

Using this definition, as well as the rules of the DOOT technique, it is obtained that the expression of generalized CSs is

$$\begin{aligned} |z\rangle &= \frac{1}{\sqrt{{}_pF_q(|z|^2)}} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{\rho(n)}}|n\rangle \\ &= \frac{1}{\sqrt{{}_pF_q(|z|^2)}}\, {}_pF_q(z\hat{A}_+)|0\rangle, \end{aligned} \tag{52}$$

and their complex conjugate

$$\begin{aligned} \langle z| &= \frac{1}{\sqrt{{}_pF_q(|z|^2)}} \sum_{n=0}^{\infty} \frac{(z^*)^n}{\sqrt{\rho(n)}}\langle n| \\ &= \frac{1}{\sqrt{{}_pF_q(|z|^2)}}\, \langle 0|\, {}_pF_q(z^*\hat{A}_-). \end{aligned} \tag{53}$$

Using the DOOT rules, the projector, in the CSs representation can be written as

$$\begin{aligned} |z\rangle\langle z| &= \frac{{}_pF_q(z\hat{A}_+)|0\rangle\langle 0|\, {}_pF_q(z^*\hat{A}_-)}{{}_pF_q(|z|^2)} \\ &= \frac{1}{{}_pF_q(|z|^2)} \frac{{}_pF_q(z\hat{A}_+)\, {}_pF_q(z^*\hat{A}_-)}{{}_pF_q(\hat{A}_+\hat{A}_-)}. \end{aligned} \tag{54}$$

Because the normalization function must be a strictly positive convergent function, their convergence radius R_c of the series must fulfill the inequality:

$$\begin{aligned} 0 < |z| < R_c &= \sqrt{\lim_{n\to\infty} \frac{\rho(n+1)}{\rho(n)}} \\ &= \sqrt{\lim_{n\to\infty} (n+1) \frac{\prod_{j=1}^{q}(b_j+n)}{\prod_{i=1}^{p}(a_i+n)}}. \end{aligned} \tag{55}$$

Particularly, for the case of canonical coherent states, $\rho(n) = n!$, and we obtain the correct result, $R_c = \infty$.

From the theory of hypergeometric series, it follows that the function ${}_pF_q(\widetilde{a};\widetilde{b};x) \equiv {}_pF_q(\{a_i\}_1^p;\{b_j\}_1^q;x)$ exhibits the following behavior:

- if $p \le q$ and $|z| < \infty$, the series converges over the entire complex plane.
- if $p = q+1$ and $|z| < 1$, is convergent, and for $|z| > 1$ is divergent.
- if $p > q+1$ and $|z| \neq 0$, is divergent, if the series does not terminate and becomes a polynomial.

In addition to these, a series of conditions are imposed on the parameters which appear in the expression of generalized hypergeometric functions [13]. For example, for the case $p = q+1$ convergence will be achieved if:

- for $|z| = 1$ if $\eta = 1$;
- for $|z| = 1$ and $|z| \neq 1$, if $0 \le \eta \le 1$, where

$$\eta = \mathrm{Re}\left(\sum_{i=1}^{p} a_i - \sum_{j=1}^{q} b_j\right). \tag{56}$$

From the perspective of the coherent state formalism, only the cases where the series converges are of interest.

In 1963 Klauder established that all generalized CSs (therefore, linear and non-linear) must satisfy the following minimum conditions, which were later called "the Klauder's prescriptions" [14]: (I). NCSs are normalized but non-orthogonal:

$$\langle z|z'\rangle = \frac{{}_pF_q(z^*z')}{\sqrt{{}_pF_q(|z|^2)}\sqrt{{}_pF_q(|z'|^2)}}, \tag{57}$$

and thus the generalized CSs form an overcomplete set.

(II). Continuity in the label variable z of the NCSs:

$$\lim_{z\to z'} \| \, |z\rangle - |z'\rangle \| = 0. \tag{58}$$

(III). NCSs must satisfy the resolution of the identity, or the closure relation which can be expressed by the resolution of the identity operator $\hat{I}$ in the vector space of quantum states, where $d\mu(z)$ is suitable integration measure.

$$\int d\mu(z)\, |z\rangle\langle z| = \hat{I} = \sum_{n=0}^{\infty} |n\rangle\langle n|. \tag{59}$$

As a consequence of this relation, the property of CSs to form an overcomplete set translates into the fact that any state can be decomposed on the set of CSs:

$$|z'\rangle = \int d\mu(z)\, |z\rangle\langle z|z'\rangle. \tag{60}$$

It can be verified that the integration measure is [5]:

$$d\mu(z) = \Gamma(\widetilde{a}/\widetilde{b})\,\frac{d\phi}{2\pi}\,d(|z|^2)\,{}_pF_q(|z|^2) \times G^{q+1,0}_{p,q+1}\left(|z|^2 \,\middle|\, \begin{matrix} \widetilde{a}-1 \\ 0,\ \widetilde{b}-1 \end{matrix}\right), \tag{61}$$

where we used the notation

$$\Gamma(\widetilde{a}/\widetilde{b}) \equiv \frac{\prod_{i=1}^{p}\Gamma(a_i)}{\prod_{j=1}^{q}\Gamma(b_j)}. \tag{62}$$

Here $G^{q+1,0}_{p,q+1}(|z|^2|\cdots)$ is the Meijer G-function which can be connected with the hypergeometric function ${}_pF_q(|z|^2)$ as follows [15]:

$${}_pF_q(|z|^2) = \Gamma(\widetilde{b}/\widetilde{a})\,G^{1,p}_{p,q+1}\left(-|z|^2 \,\middle|\, \begin{matrix} 1-\widetilde{a} \\ 0,\ 1-\widetilde{b} \end{matrix}\right), \tag{63}$$

where $1-\widetilde{a} \equiv \{1-a_1, 1-a_2, \ldots, 1-a_p\}$ and so on.

With the above considerations, the relation of decomposition of unity operator can be also written as

$$\int d\mu(z)\,|z\rangle\langle z| = \Gamma(\widetilde{a}/\widetilde{b}) \sum_{n,m=0}^{\infty} \frac{|n\rangle\langle m|}{\sqrt{\rho(n)\rho(m)}} \times \int d(|z|^2)\,G^{q+1,0}_{p,q+1}\left(|z|^2 \,\middle|\, \begin{matrix} \widetilde{a}-1 \\ 0,\ \widetilde{b}-1 \end{matrix}\right) \times \int_0^{2\pi} \frac{d\phi_z}{2\pi} z^n (z^*)^m. \tag{64}$$

Using the orthogonality of the angular integral, $\int_0^{2\pi} \frac{d\phi}{2\pi} z^n (z^*)^m = \delta_{nm}|z|^{2n}$, we get

$$\int d\mu(z)\,|z\rangle\langle z| = \Gamma(\widetilde{a}/\widetilde{b}) \times \sum_{n=0}^{\infty} \frac{|n\rangle\langle n|}{\rho(n)} \int_0^{\infty} d(|z|^2)\,(|z|^2)^n \times G^{q+1,0}_{p,q+1}\left(|z|^2 \,\middle|\, \begin{matrix} \widetilde{a}-1 \\ 0,\ \widetilde{b}-1 \end{matrix}\right). \tag{65}$$

Using the properties of the Meijer G-function, the integral equals $\rho(n)\Gamma(\widetilde{b}/\widetilde{a})$, so that the final result is

$$\int d\mu(z)\,|z\rangle\langle z| = \sum_{n=0}^{\infty} |n\rangle\langle n| = \hat{I}. \tag{66}$$

The proof is simple and can be found by consulting our preprint [16]. Let us also focus our attention on an integral that results from the decomposition relations of the unit operator and which will be useful to us further:

$$\int \frac{d^2z}{\pi}\,G^{q+1,0}_{p,q+1}\left(|z|^2 \,\middle|\, \begin{matrix} \widetilde{a}-1 \\ 0,\ \widetilde{b}-1 \end{matrix}\right)(|z|^2)^n = \Gamma(\widetilde{b}/\widetilde{a})\,\rho(n). \tag{67}$$

By analogy with the displacement operator defined by the canonical operators $\hat{a}^\dagger$ and $\hat{a}$, the generalized exponential displacement operator, corresponding to the ladder operators $\hat{A}_+$ and $\hat{A}_-$, can also be defined as:

$$\hat{D}(z) = \exp\left(z\hat{A}_+ - z^*\hat{A}_-\right). \tag{68}$$

At the end of this section, let us point out that, besides Barut-Girardello type CSs, other type of CSs can also be defined in the so called Klauder-Perelomov manner, as a result of the action of the generalized analogue of the displacement operator on the vacuum state:

$$|z\rangle_{\rm KP} = \frac{1}{\sqrt{N_{\rm KP}(|z|^2)}}\,\hat{D}(z)\,|0\rangle = \frac{1}{\sqrt{N_{\rm KP}(|z|^2)}}\,\exp\left(z\hat{A}_+\right)\,|0\rangle, \tag{69}$$

where we took into account that $\hat{A}_-|0\rangle = 0$. Expanding the exponential in a power series and taking into account the result of the action of the operator $\hat{A}_+$ on the vacuum state, we obtain that the set of CSs of Klauder-Perelomov type has the expression

$$|z\rangle_{\rm KP} = \frac{1}{\sqrt{{}_qF_p(\widetilde{b};\widetilde{a};|z|^2)}} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{\rho_{\rm KP}(n)}}|n\rangle = \frac{1}{\sqrt{{}_qF_p(\widetilde{b};\widetilde{a};|z|^2)}}\,{}_qF_p(\widetilde{b};\widetilde{a};z\hat{A}_+)|0\rangle. \tag{70}$$

The notation used here is

$$\rho_{\rm KP}(n) = \frac{(n!)^2}{\rho(n)}. \tag{71}$$

It can be seen that between the sets of CSs of the Barut-Girardello type, on the one hand, and the Klauder-Perelomov, on the other hand, there is a dualism. This dualism consists mainly of the interchange of the indices p and q, as well as of the sets of numbers $\{a_i\}$ and $\{b_j\}$, that is, instead of the hypergeometric function ${}_pF_q(\widetilde{a};\widetilde{b};x)$, the function ${}_qF_p(\widetilde{b};\widetilde{a};x)$ appears, as well as instead of the structure function $\rho(n)$, the function $\rho_{\rm KP}(n)$ appears [17].

4 Generalized analogue of the displacement operator

Previously we have seen that the generalized CSs can also be obtained by applying the generalized hypergeometric function ${}_pF_q(z\hat{A}_+)$ on the vacuum ket vector $|0\rangle$. This idea appeared for the first time, for a particular case of pseudoharmonic oscillator, in Mojaveri and Dehghani's paper [16].

On the other hand, any function that has as its argument the annihilation operator, therefore also ${}_pF_q(-z^*\hat{A}_-)$, applied to the vector $|0\rangle$ leaves it unchanged, i.e.

$$ {}_pF_q(-z^*\hat{A}_-)\,|0\rangle = |0\rangle. \tag{72}$$

It must be explicitly emphasized that a relation of the form $f(\hat{A}_-)|0\rangle = |0\rangle$, containing the annihilation operator $\hat{A}_-$ in its argument, is valid only if the function $f(\hat{A}_-)$ acts on the vacuum state vector $|0\rangle$. If this operator function acts upon a "non-vacuum" coherent state, then the result of this action is different from $|0\rangle$. This procedure will also be evident in the calculations that follow.

So, it is obvious that a generalized analogue of the displacement operator can be constructed, similar to the "usual" one $\hat{D}(z) = e^{-\frac{1}{2}|z|^2} e^{z\hat{a}^\dagger - z^*\hat{a}}$.

Consequently, by analogy, we can define the *generalized analogue of the displacement operator* as follows:

$$\hat{D}(z) \equiv \frac{1}{\sqrt{{}_pF_q(|z|^2)}}\, {}_pF_q(\ z\hat{A}_+)\ {}_pF_q(\ -z^*\hat{A}_-) \tag{4.2}$$

Correspondingly, its adjoint operator (Hermitian conjugate) is

$$\begin{aligned}\hat{D}^\dagger(z) &= \frac{1}{\sqrt{{}_pF_q(|z|^2)}}\, {}_pF_q(z^*\hat{A}_-)\, {}_pF_q(-z\hat{A}_+) \\ &= \frac{1}{\sqrt{{}_pF_q(|z|^2)}}\, {}_pF_q(-z\hat{A}_+)\, {}_pF_q(z^*\hat{A}_-) \\ &= \hat{D}(-z).\end{aligned} \tag{73}$$

However, the inequalities hold:

$$\hat{D}^{-1}(z) \neq \hat{D}(-z), \qquad \hat{D}^\dagger(z)\hat{D}(z) \neq 1, \tag{74}$$

therefore, the operator $\hat{D}(z)$ is not unitary.

In the complete definition of the displacement operator, the minus sign in ${}_pF_q(-z^*\hat{A}_-)$ is retained so that, in the harmonic limit, it coincides with the displacement operator of canonical coherent states.

As a consequence, the following relationship will have to be valid at the limit:

$$\begin{aligned}\lim_{\substack{p=q\\a=b}} \hat{D}(z) &= \lim_{\substack{p=q\\a=b}} \frac{1}{\sqrt{{}_pF_q(|z|^2)}} \\ &\quad \times {}_pF_q(z\hat{A}_+)\, {}_pF_q(-z^*\hat{A}_-) \\ &= e^{-\frac{1}{2}|z|^2} e^{z\hat{a}^\dagger - z^*\hat{a}} = \hat{D}(z).\end{aligned} \tag{75}$$

In the calculations, but only when the function ${}_pF_q(-z^*\hat{A}_-)$ acts on the vacuum state $|0\rangle$, below we can drop this function, so we will consider that the generalized analogue of the displacement operator has the expression (which agrees with [18]):

$$\hat{D}(z) \equiv \frac{1}{\sqrt{{}_pF_q(|z|^2)}}\, {}_pF_q(z\hat{A}_+). \tag{76}$$

Using the definition of the CSs (and their complex conjugate counterpart), let's highlight another property of the generalized analogue of the displacement operator, which can be considered as their eigenvalue equation

$$\begin{aligned}{}_pF_q(-z^*\hat{A}_-)|\sigma\rangle &= \sum_{n=0}^{\infty} \frac{(z^*)^n}{\rho(n)} (\hat{A}_-)^n |\sigma\rangle \\ &= \sum_{n=0}^{\infty} \frac{(z^*)^n}{\rho(n)} \sigma^n |\sigma\rangle \\ &= {}_pF_q(-z^*\sigma)|\sigma\rangle,\end{aligned} \tag{77}$$

and similarly

$$\langle\sigma|\, {}_pF_q(z\hat{A}_+) = \langle\sigma|\, {}_pF_q(z\sigma^*). \tag{78}$$

It means that the function ${}_pF_q(-z^*\sigma)$ is the eigenvalue of the operator ${}_pF_q(-z^*\hat{A}_-)$ attached to the eigenfunction $|\sigma\rangle$ and similar for their complex conjugate.

The diagonal elements of the generalized analogue of the displacement operator in CSs representation are

$$\begin{aligned}\langle\sigma|\hat{D}(z)|\sigma\rangle &= \frac{\langle\sigma|\, {}_pF_q(z\hat{A}_+)\, {}_pF_q(z^*\hat{A}_-)|\sigma\rangle}{\sqrt{{}_pF_q(|z|^2)}} \\ &= \frac{1}{\sqrt{{}_pF_q(|z|^2)}} \frac{{}_pF_q(z\sigma^*)\, {}_pF_q(z^*\sigma)}{{}_pF_q(|\sigma|^2)}.\end{aligned} \tag{79}$$

On the other hand, using the DOOT rules we have

$$\begin{aligned}|l\rangle\langle l| &= \frac{1}{\rho(l)} (\hat{A}_+)^l |0\rangle\langle 0| (\hat{A}_-)^l \\ &= \frac{1}{{}_pF_q(\hat{A}_+\hat{A}_-)} \frac{(\hat{A}_+\hat{A}_-)^l}{\rho(l)},\end{aligned} \tag{80}$$

and it can be easily verified that this expression fulfills the completeness relation

$$\sum_{l=0}^{\infty} |l\rangle\langle l| = \hat{I}. \tag{81}$$

Now, we will be able to write, which means that the displacement operator applied to the vacuum ket vector $|0\rangle$ transforms it into a vector corresponding to the generalized CSs $|z\rangle$.

$$|z\rangle = \hat{D}(z)\,|0\rangle. \tag{82}$$

For the moment, for reasons of clarity of exposition, we will return to the initial notation for generalized hypergeometric functions, ${}_pF_q(\widetilde{a};\widetilde{b};x)$.

Previously we saw that CSs satisfy the decomposition relation of the unit operator. Let us see how generalized states of displacement are involved in this relation, also taking into account the DOOT rules.

$$\begin{aligned}\hat{I} &= \int d\mu(z)\,|z\rangle\langle z| \\ &= \int d\mu(z)\,\hat{D}(z)|0\rangle\langle 0|\hat{D}^{\dagger}(z) \\ &= \frac{1}{{}_pF_q(\widetilde{a};\widetilde{b};\hat{A}_+\hat{A}_-)}\int d\mu(z)\,\hat{D}(z)\hat{D}^{\dagger}(z).\end{aligned} \tag{83}$$

From this follows a new property of the generalized analogue of the displacement operator:

$$\int d\mu(z)\,\hat{D}(z)\hat{D}^{\dagger}(z) = {}_pF_q(\widetilde{a};\widetilde{b};\hat{A}_+\hat{A}_-). \tag{84}$$

or, more explicitly, using the expression of integration measure

$$\begin{aligned}\int \frac{d^2z}{\pi}\,G^{q+1,0}_{p,q+1}\left(|z|^2\,\middle|\,\begin{matrix}\widetilde{a}-1\\ 0,\ \widetilde{b}-1\end{matrix}\right)\times {}_pF_q(\widetilde{a};\widetilde{b};|z|^2) \\ \times\,\hat{D}(z)\hat{D}^{\dagger}(z) \\ = \Gamma(\widetilde{b}/\widetilde{a})\,{}_pF_q(\widetilde{a};\widetilde{b};\hat{A}_+\hat{A}_-).\end{aligned} \tag{85}$$

Moreover, the CSs can also be defined as follows:

$$|z\rangle = \hat{D}(z)|0\rangle = \frac{1}{\sqrt{{}_pF_q(\widetilde{a};\widetilde{b};|z|^2)}}\,{}_pF_q(\widetilde{a};\widetilde{b};z\hat{A}_+)|0\rangle. \tag{86}$$

$$\langle z| = \langle 0|\hat{D}^{\dagger}(z) = \frac{\langle 0|\,{}_pF_q(\widetilde{a};\widetilde{b};z^*\hat{A}_-)}{\sqrt{{}_pF_q(\widetilde{a};\widetilde{b};|z|^2)}}. \tag{87}$$

Substituting into the decomposition relation the unit operator, as well as with the DOOT rules, we obtain

$$\begin{aligned}\hat{I} &= \int d\mu(z)\,|z\rangle\langle z| \\ &= \int d\mu(z)\,\frac{{}_pF_q(\widetilde{a};\widetilde{b};z\hat{A}_+)\,{}_pF_q(\widetilde{a};\widetilde{b};z^*\hat{A}_-)}{{}_pF_q(\widetilde{a};\widetilde{b};|z|^2)\,{}_pF_q(\widetilde{a};\widetilde{b};\hat{A}_+\hat{A}_-)}.\end{aligned} \tag{88}$$

After substituting the expression of the integration measure, we will obtain an important equality:

$$\begin{aligned}\int \frac{d^2z}{\pi}\,G^{q+1,0}_{p,q+1}\left(|z|^2\,\middle|\,\begin{matrix}\widetilde{a}-1\\ 0,\ \widetilde{b}-1\end{matrix}\right) \\ \times\,\#\,{}_pF_q(\widetilde{a};\widetilde{b};z\hat{A}_+)\,{}_pF_q(\widetilde{a};\widetilde{b};z^*\hat{A}_-)\,\# \\ = \Gamma(\widetilde{b}/\widetilde{a})\,{}_pF_q(\widetilde{a};\widetilde{b};\hat{A}_+\hat{A}_-).\end{aligned} \tag{89}$$

With the help of this integral, a series of similar integrals can be calculated, for example, an integral whose integrand also contains the product $z^n(z^*)^m$ (for calculation details, see [19]). The following will also be useful for this purpose:

– The differentiation rule for hypergeometric functions [15]:

$$\begin{aligned}\left(\frac{\partial}{\partial\hat{A}_+}\right)^n {}_pF_q(\widetilde{a};\widetilde{b};z\hat{A}_+) \\ = z^n\,\frac{\prod_{i=1}^{p}(a_i)_n}{\prod_{j=1}^{q}(b_j)_n}\,{}_pF_q(\widetilde{a}+n;\widetilde{b}+n;z\hat{A}_+).\end{aligned} \tag{90}$$

Consequently, we have

$$\begin{aligned}z^n\,{}_pF_q(\widetilde{a};\widetilde{b};z\hat{A}_+) \\ = \frac{\prod_{j=1}^{q}(1-b_j)_n}{\prod_{i=1}^{p}(1-a_i)_n} \\ \times\left(\frac{\partial}{\partial\hat{A}_+}\right)^n {}_pF_q(\widetilde{a}-n;\widetilde{b}-n;z\hat{A}_+).\end{aligned} \tag{91}$$

as well as their complex conjugate. Here we used the equality between the Pochhammer symbols [20]:

$$(a-n)_n = (-1)^n(1-a)_n. \tag{92}$$

– The angular integral has the following expression:

$$\begin{aligned}\int_0^{2\pi}\frac{d\phi}{2\pi}\,{}_pF_q(\widetilde{a};\widetilde{b};z\hat{A}_+)\,{}_pF_q(\widetilde{a};\widetilde{b};z^*\hat{A}_-) \\ = \sum_{n=0}^{\infty}\frac{(\hat{A}_+\hat{A}_-)^n}{[\rho(n)]^2}\,(|z|^2)^n.\end{aligned} \tag{93}$$

– The result of the integral after $|z|^2$ is

$$\begin{aligned}\int_0^{\infty} d(|z|^2)\,G^{q+1,0}_{p,q+1}\left(|z|^2\,\middle|\,\begin{matrix}\widetilde{a}-1\\ 0,\ \widetilde{b}-1\end{matrix}\right)(|z|^2)^n \\ = \Gamma(\widetilde{b}/\widetilde{a})\,\rho(n).\end{aligned} \tag{94}$$

Consequently, the generalized analogue of the displacement operators satisfy the relation

$$\int d\mu(z)\,\hat{D}(z)\hat{D}^{\dagger}(z) = {}_pF_q(\widetilde{a};\widetilde{b};\hat{A}_+\hat{A}_-). \tag{95}$$

Considering the DOOT rules, according to which the creation and annihilation operators commute, so that in the end we have an ordered product of the operators, we can easily verify the validity of the properties below.

The generalized analogue of the displacement operators are not unitary operators, in other words, $\hat{D}(z)\hat{D}^{-1}(z) \neq 1$, because it is evident that FF equals 1. This can be shown.

By considering the scalar product of self-consistent states, which proves that the coherent states are normalized to unity, we obtain:

$$\langle z|z\rangle = \langle 0|\hat{D}^{\dagger}(z)\hat{D}(z)|0\rangle = 1. \tag{96}$$

This result might lead to the false conclusion that generalized shift operators are unitary operators.

In order to demonstrate this assertion, we will make use of the relations that generalize the actions of ladder operators on Fock vectors:

$$
\begin{aligned}
(\hat{A}_+)^s|n\rangle &= \sqrt{\frac{[e(n+s)]!}{[e(n)]!}}\,|n+s\rangle,\\
(\hat{A}_-)^s|n\rangle &= \sqrt{\frac{[e(n)]!}{[e(n-s)]!}}\,|n-s\rangle,\\
\langle n|(\hat{A}_+)^s &= \sqrt{\frac{[e(n)]!}{[e(n-s)]!}}\,\langle n-s|,\\
\langle n|(\hat{A}_-)^s &= \sqrt{\frac{[e(n+s)]!}{[e(n)]!}}\,\langle n+s|.
\end{aligned}
\tag{97}
$$

Consequently, successively we have

$$
\begin{aligned}
\langle z|z\rangle &= \langle 0|\hat{D}^\dagger(z)\hat{D}(z)|0\rangle\\
&= \frac{1}{{}_pF_q(|z|^2)}\langle 0|{}_pF_q(z^*\hat{A}_-)\,{}_pF_q(z\hat{A}_+)|0\rangle\\
&= \frac{1}{{}_pF_q(|z|^2)}\sum_{n,m=0}^{\infty}\frac{(z^*)^m}{\rho(m)}\frac{z^n}{\rho(n)}\\
&\quad\times\langle 0|(\hat{A}_-)^m(\hat{A}_+)^n|0\rangle.
\end{aligned}
\tag{98}
$$

Now,

$$
\begin{aligned}
\langle 0|(\hat{A}_-)^m(\hat{A}_+)^n|0\rangle &= \sqrt{[e(n)]![e(m)]!}\,\langle n|m\rangle\\
&= [e(n)]!\,\delta_{nm}\\
&= \rho(n)\,\delta_{nm},
\end{aligned}
\tag{99}
$$

due to the fact that $\rho(n) = [e(n)]!$.

It is evident that this relation generally is not equal to unity, only in the case $n = m$.

Therefore, it is incorrect to state that the generalized analogue of the displacement operator is a unitary operator.

Regarding the name itself, however, we prefer to retain the term "generalized analogue of the displacement operator"—as it appears in the paper by Mojaveri and Dehghani [18]—which they introduced and used.

5 Construction of the displaced coherent states

Let's see what is the result of the successive application of two displacement operators, which depend on different arguments, respectively $\hat{D}(\varepsilon\ \sigma)$ and $\hat{D}(\lambda\ z)$ on the vacuum ket vector $|0>$. The goal is to obtain the CSs that have as argument a linear combination of complex variables $z = |z|\ \exp(i\ \phi_z)$ and $\sigma = |\sigma|\ \exp(i\ \phi_\sigma)$, that is $Z = \varepsilon\ z + \lambda\ \sigma$, where ε and λ are real numbers.

The complex number Z can be viewed from two perspectives:

Perspective 1: as a complex number resulting from the linear combination of two complex numbers, z and σ, with phases ϕ_z respectively ϕ_σ.

$$
\begin{aligned}
Z &= \varepsilon z + \lambda\sigma = \varepsilon|z|e^{i\phi_z} + \lambda|\sigma|e^{i\phi_\sigma},\\
|Z|^2 &= \varepsilon^2|z|^2 + \lambda^2|\sigma|^2 + 2\varepsilon\lambda|z|\,|\sigma|\cos(\phi_z - \phi_\sigma).
\end{aligned}
\tag{100}
$$

Perspective 2, as a single phasor (rotating vector) in the complex space $\mathbb{C}$, having magnitude $|Z|$ and phase ϕ_Z.

$$
\begin{aligned}
Z &= |Z|\exp(i\phi_Z) = \mathrm{Re}\,Z + i\,\mathrm{Im}\,Z,\\
\phi_Z &= \arctan\left(\frac{\varepsilon|z|\sin\phi_z + \lambda|\sigma|\sin\phi_\sigma}{\varepsilon|z|\cos\phi_z + \lambda|\sigma|\cos\phi_\sigma}\right).
\end{aligned}
\tag{101}
$$

The expressions obtained will be, among others, useful for calculating the Wigner operator in the representation of CSs [21]. Consequently, CSs having as argument the linear combination of complex numbers is defined as follows:

$$
|Z\rangle = |\varepsilon z + \lambda\sigma\rangle = \hat{D}(\lambda\sigma)\hat{D}(\varepsilon z)|0\rangle. \tag{102}
$$

Let us see what is the expression of the development of these CSs in terms of the Fock vectors $|n\rangle$.

We will start from the definition relation of CSs with the argument εz

$$
\hat{D}(\varepsilon z)|0\rangle = |\varepsilon z\rangle, \tag{103}
$$

and we will act on it with the displacement operator $\hat{D}(\lambda\sigma)$. We obtain, successively

$$
\begin{aligned}
\hat{D}(\lambda\sigma)|\varepsilon z\rangle &= \frac{{}_pF_q(\lambda\sigma\hat{A}_+)\,{}_pF_q(-\lambda\sigma^*\hat{A}_-)}{\sqrt{{}_pF_q(|\lambda\sigma|^2)}}\,|\varepsilon z\rangle\\
&= \frac{{}_pF_q(\lambda\sigma\hat{A}_+)\,{}_pF_q(-\lambda\sigma^*\hat{A}_-)}{\sqrt{{}_pF_q(|\lambda\sigma|^2)}}\\
&\quad\times\frac{{}_pF_q(\varepsilon z\hat{A}_+)}{\sqrt{{}_pF_q(|\varepsilon z|^2)}}|0\rangle.
\end{aligned}
\tag{104}
$$

Now,

$$
\begin{aligned}
{}_pF_q(\lambda\sigma\hat{A}_+)|\varepsilon z\rangle &= \frac{1}{\sqrt{{}_pF_q(|\varepsilon z|^2)}}\\
&\quad\times\sum_{m=0}^{\infty}\frac{(\lambda\sigma)^m}{\rho(m)}(\hat{A}_+)^m|\varepsilon z\rangle.
\end{aligned}
\tag{105}
$$

Using

$$
(\hat{A}_+)^m|n\rangle = \sqrt{\frac{\rho(n+m)}{\rho(n)}}|n+m\rangle, \tag{106}
$$

and substituting above, using a new summation index $l = n + m$ and proceeding similarly to the case of Eqs. (2.11) and (2.12), we obtain

$$
\begin{aligned}
{}_pF_q(\lambda\sigma\hat{A}_+)|\varepsilon z\rangle &= \frac{1}{\sqrt{{}_pF_q(|\varepsilon z|^2)}} \\
&\times \sum_{l=0}^{\infty} \frac{1}{\sqrt{\rho(l)}} \left[\sum_{m=0}^{l} \binom{\rho(l)}{\rho(m)} (\varepsilon z)^{l-m} (\lambda\sigma)^m \right] |l\rangle,
\end{aligned}
\tag{107}
$$

where we multiplied the numerator and denominator by $\sqrt{\rho(l)}$ and swapped the summations over m and l. Due to $\rho(l-m)$, we must have $m \leq l$, so the summation over m must stop at $m = l$.

In addition, since $m \leq l$ and the sum over the index l no longer depends on the index m, we can take $m = 0$.

The variable Z, defined previously, is ultimately also a complex number. Therefore, the shift operator corresponding to this variable is

$$
\hat{D}(Z) \equiv \frac{1}{\sqrt{{}_pF_q(|Z|^2)}} {}_pF_q(Z\hat{A}_+)\, {}_pF_q(-Z^*\hat{A}_-). \tag{108}
$$

Consequently, the coherent state in the variable Z is

$$
|Z\rangle = \hat{D}(Z)|0\rangle = \frac{1}{\sqrt{{}_pF_q(|Z|^2)}} {}_pF_q(Z\hat{A}_+)|0\rangle. \tag{109}
$$

Now,

$$
\begin{aligned}
{}_pF_q(Z\hat{A}_+)|0\rangle &= \sum_{n=0}^{\infty} \frac{Z^n}{\rho(n)} (\hat{A}_+)^n |0\rangle \\
&= \sum_{n=0}^{\infty} \frac{Z^n}{\sqrt{\rho(n)}} |n\rangle,
\end{aligned}
\tag{110}
$$

so that the final expression is

$$
|Z\rangle = \frac{1}{\sqrt{{}_pF_q(|Z|^2)}} \sum_{n=0}^{\infty} \frac{Z^n}{\sqrt{\rho(n)}} |n\rangle. \tag{111}
$$

The quantity between curly brackets $\{\}$ represents precisely the *generalized expression of Newton's binomial*. For this quantity we will use curly brackets:

$$
\{Z\}^l \equiv \{\varepsilon z + \lambda\sigma\}^l = \sum_{m=0}^{l} \binom{\rho(l)}{\rho(m)} (\varepsilon z)^{l-m} (\lambda\sigma)^m. \tag{112}
$$

So, a sum between curly brackets raised to a power represents the generalized expression of the Newton's binomial:

$$
\begin{aligned}
\{x + y\}^l &\equiv \sum_{m=0}^{l} \binom{\rho(l)}{\rho(m)} x^{l-m} y^m \\
&= \sum_{m=0}^{l} \binom{l}{m} \binom{g(l)}{g(m)} x^{l-m} y^m.
\end{aligned}
\tag{113}
$$

Note: We will use curly brackets $\{\}$ to avoid potential confusion with the generalized factorial $[]!$.

With this notation, we will obtain precisely the expression of the coherent state with a shifted argument.

$$
\begin{aligned}
{}_pF_q(\lambda\sigma\hat{A}_+)|\varepsilon z\rangle &= \frac{1}{\sqrt{{}_pF_q(|\varepsilon z|^2)}} \\
&\times \sum_{l=0}^{\infty} \frac{\{\varepsilon z + \lambda\sigma\}^l}{\sqrt{\rho(l)}} |l\rangle \\
&= |\varepsilon z + \lambda\sigma\rangle.
\end{aligned}
\tag{114}
$$

It should be noted that the generalized Newton binomial, defined above, appears in the context of calculations involving coherent states (where, incidentally, the structure constant also appears). In other cases, the generalized Newton binomial obviously differs from the standard binomial coefficient expression.

$$
\{x + y\}^l \neq (x + y)^l. \tag{115}
$$

Therefore, the two notations should not be confused—that is,

$$
\begin{aligned}
\{x + y\}^l &= \sum_{k=0}^{l} \binom{\rho(l)}{\rho(k)} x^{l-k} y^k \\
&= \sum_{k=0}^{l} \binom{l}{k} \binom{g(l)}{g(k)} x^{l-k} y^k, \\
(x + y)^l &= \sum_{k=0}^{l} \binom{l}{k} x^{l-k} y^k.
\end{aligned}
\tag{116}
$$

The connection between the two expressions is, see Eqs. (3.14)–(3.16)

$$
\lim_{\substack{g(l)\to 1 \\ g(l-k)\to 1}} \{x + y\}^l = (x + y)^l. \tag{117}
$$

Finally, after substitutions, we will get

$$
\hat{D}(\lambda\sigma)|\varepsilon z\rangle = \frac{\sqrt{{}_pF_q\left(|\{\varepsilon z + \lambda\sigma\}|^2\right)}}{\sqrt{{}_pF_q(|\lambda\sigma|^2)\, {}_pF_q(|\varepsilon z|^2)}} |Z\rangle, \tag{118}
$$

where the generalized hypergeometric function depending on the displaced variable is

$$
{}_pF_q\left(|\{Z\}|^2\right) \equiv {}_pF_q\left(|\{\varepsilon z + \lambda\sigma\}|^2\right). \tag{119}
$$

$$
\begin{aligned}
{}_pF_q(|Z|^2) &= \sum_{n=0}^{\infty} \frac{1}{\rho(n)} \left|\{\epsilon z + \lambda\sigma\}\right|^{2n} \\
&= \sum_{n=0}^{\infty} \frac{1}{\rho(n)} \sum_{s=0}^{n} \sum_{t=0}^{n} \binom{n}{s} \binom{n}{t} \\
&\quad \times (\epsilon z)^{n-s} (\lambda\sigma)^s (\epsilon z^*)^{n-t} (\lambda\sigma^*)^t.
\end{aligned}
\tag{120}
$$

The final expression for the shifted CSs, defined by the generalized analogue of the displacement operators is then

$$|\varepsilon z+\lambda\sigma\rangle = \sqrt{\frac{{}_pF_q(|\lambda\sigma|^2)\,{}_pF_q(|\varepsilon z|^2)}{{}_pF_q\left(|\{\varepsilon z+\lambda\sigma\}|^2\right)}} \times \hat{D}(\lambda\sigma)\hat{D}(\varepsilon z)|0\rangle, \tag{121}$$

or, in equivalent manner, as their expansion with respect to the Fock vectors

$$|\varepsilon z+\lambda\sigma\rangle = \frac{1}{\sqrt{{}_pF_q\left(|\{\varepsilon z+\lambda\sigma\}|^2\right)}} \times \sum_{n=0}^{\infty}\frac{\{\varepsilon z+\lambda\sigma\}^n}{\sqrt{\rho(n)}}|n\rangle. \tag{122}$$

Finally, this is the expression for *generalized CSs with displaced (shifted) argument.*

This can also be written in the following form:

$$|\varepsilon z+\lambda\sigma\rangle = \frac{{}_pF_q\left(\{\varepsilon z+\lambda\sigma\}\hat{A}_+\right)}{\sqrt{{}_pF_q\left(|\{\varepsilon z+\lambda\sigma\}|^2\right)}}|0\rangle, \tag{123}$$

and similar for their complex conjugate counterpart.

Let's find the expression of the integration measure $d\mu(\varepsilon z+\lambda\sigma) = d\mu(Z)$ from the decomposition relation of the unit operator.

In essence, the variable Z is a linear combination of two independent complex variables, z and σ; therefore, ultimately, it is also a complex variable. In the complex plane $\mathbb{C}$, generally the Lebesgue integration measure is

$$\begin{aligned} d\mu(Z) &= \frac{1}{\pi}d(\mathrm{Re}\,Z)\,d(\mathrm{Im}\,Z)\,h(|Z|^2) \\ &= d(|Z|^2)\,h(|Z|^2)\,\frac{d\phi_Z}{2\pi}. \end{aligned} \tag{124}$$

where $h\ \left(|Z|^2\right)$ is a weighting function that must be determined specifically at a later stage.

If we consider the linear combination of two independent complex variables with equal "rights", $z,\sigma\in\mathbb{C}$, then the integration measure in polar coordinates will be symmetric in both variables.

$$\begin{aligned} d\mu(z,\sigma) &= d\mu(z)\,d\mu(\sigma)\,{}_pF_q\left(|\{\varepsilon z+\lambda\sigma\}|^2\right) \\ &= d(|z|^2)\,h(|z|^2)\frac{d\phi_z}{2\pi}\,d(|\sigma|^2)\,h(|\sigma|^2)\frac{d\phi_\sigma}{2\pi} \\ &\quad\times {}_pF_q\left(|\{\varepsilon z+\lambda\sigma\}|^2\right). \end{aligned} \tag{125}$$

Consequently, the integral over the entire space $\mathbb{C}^2$, in polar coordinates, is a quadruple integral:

$$\begin{aligned} &\int_{\mathbb{C}^2} f(z,\sigma)\,d\mu(z,\sigma) \\ &\quad= \int_0^\infty d(|z|^2)\,h(|z|^2)\int_0^{2\pi}\frac{d\phi_z}{2\pi} \\ &\quad\times\int_0^\infty d(|\sigma|^2)\,h(|\sigma|^2)\int_0^{2\pi}\frac{d\phi_\sigma}{2\pi} \\ &\quad\times f\left(|z|e^{i\phi_z},|\sigma|e^{i\phi_\sigma}\right). \end{aligned} \tag{126}$$

The weight function of the one-variable integration measure is, see Eq. (3.31).

Given that the linear combination of the two complex variables, z and σ, is also a complex number $Z=\varepsilon z+\lambda\sigma$, expressing the formulas in terms of z and σ is equivalent to expressing them in terms of Z. Therefore, for the sake of simplicity, we will use the variable $\{Z\}$ where appropriate.

For simplification, we will use the variable $Z=\varepsilon z+\lambda\sigma$, so that the CSs will be

$$\begin{aligned} |Z\rangle &= \frac{1}{\sqrt{{}_pF_q\left(|\{Z\}|^2\right)}}\sum_{n=0}^{\infty}\frac{\{Z\}^n}{\sqrt{\rho(n)}}|n\rangle \\ &= \frac{1}{\sqrt{{}_pF_q\left(|\{Z\}|^2\right)}}{}_pF_q\Big(\{Z\}\hat{A}_+\Big)\,|0\rangle. \end{aligned} \tag{127}$$

$$\begin{aligned} \langle Z| &= \frac{1}{\sqrt{{}_pF_q\left(|\{Z\}|^2\right)}}\sum_{n=0}^{\infty}\frac{\{Z^*\}^n}{\sqrt{\rho(n)}}\langle n| \\ &= \frac{1}{\sqrt{{}_pF_q\left(|\{Z\}|^2\right)}}\langle 0|\,{}_pF_q\Big(\{Z^*\}\hat{A}_+\Big). \end{aligned} \tag{128}$$

and the decomposition relation of the unit operator will be written as

$$\int d\mu(Z)\,|Z\rangle\langle Z| = \hat{I}, \tag{129}$$

where the integration measure is

$$\begin{aligned} d\mu(Z) &= \frac{d^2Z}{\pi}\,h\big(|\{Z\}|^2\big)\,{}_pF_q\big(|\{Z\}|^2\big) \\ &= \frac{d\phi_Z}{2\pi}\,d\big(|\{Z\}|^2\big)\,h\big(|\{Z\}|^2\big)\,{}_pF_q\big(|\{Z\}|^2\big), \end{aligned} \tag{130}$$

and we need to find the weighting function $h\big(|\{Z\}|^2\big)$.

By substituting the corresponding expressions, we will use the expressions of the hypergeometric functions written as power series of the respective variables. Consequently, we will first have to solve the angular integral

$$\int_0^{2\pi}\frac{d\phi_Z}{2\pi}\{Z\}^l\,\{Z^*\}^s = \big(|\{Z\}|^2\big)^l\,\delta_{ls}. \tag{131}$$

The angular integral is non-zero only if we have equal exponents: $s = l$, which is in accordance with the condition of satisfying the closure relation of the Fock vectors.

Therefore, we will have:

$$\sum_{l=0}^{\infty} \frac{|l\rangle\langle l|}{\rho(l)} \times \int_0^{\infty} d(|Z|^2)\, h(|Z|^2)\, {}_pF_q(|Z|^2)\, (|Z|^2)^l = \hat{I}. \tag{132}$$

It is obvious that the above integral must have the value

$$\int_0^{\infty} d\big(|\{Z\}|^2\big)\, h\big(|\{Z\}|^2\big)\, {}_pF_q\big(|\{Z\}|^2\big)\, \big(|\{Z\}|^2\big)^l = \rho(l) = \Gamma(\widetilde{a}/\widetilde{b})\, \Gamma(l+1)\, \frac{\prod_{j=1}^{q} \Gamma(b_j + l)}{\prod_{i=1}^{p} \Gamma(a_i + l)}. \tag{133}$$

This is an integral of the type of Stieltjes moments problem and their solution imply the Meijer G-function:

$$h\big(|\{Z\}|^2\big) = \Gamma(\widetilde{a}/\widetilde{b})\, {}_pF_q\big(|\{Z\}|^2\big) \times G_{p,q+1}^{q+1,0}\left(|\{Z\}|^2 \,\middle|\, \begin{matrix} \{a_i - 1\}_1^p \\ 0,\ \{b_j - 1\}_1^q \end{matrix}\right). \tag{134}$$

Consequently, the integration measure will have the final expression

$$d\mu(Z) = \Gamma(\widetilde{a}/\widetilde{b})\, d\big(|\{Z\}|^2\big)\, \frac{d\phi_Z}{2\pi}\, {}_pF_q\big(|\{Z\}|^2\big) \times G_{p,q+1}^{q+1,0}\left(|\{Z\}|^2 \,\middle|\, \begin{matrix} \{a_i - 1\}_1^p \\ 0,\ \{b_j - 1\}_1^q \end{matrix}\right). \tag{135}$$

In the notation of old variables, we will have

$$|\varepsilon z + \lambda\sigma\rangle = \frac{1}{\sqrt{{}_pF_q(|\{\varepsilon z + \lambda\sigma\}|^2)}} \times \sum_{n=0}^{\infty} \frac{\{\varepsilon z + \lambda\sigma\}^n}{\sqrt{\rho(n)}} |n\rangle. \tag{136}$$

It is not difficult to verify that the coherent states with displaced (shifted) argument are normalized to unity.

$$\langle \varepsilon z + \lambda\sigma | \varepsilon z + \lambda\sigma\rangle = \frac{1}{{}_pF_q(|\{\varepsilon z + \lambda\sigma\}|^2)} \times \sum_{n=0}^{\infty} \frac{\big(|\{\varepsilon z + \lambda\sigma\}|^2\big)^n}{\rho(n)} = 1. \tag{137}$$

It is well known from quantum mechanics or quantum optics literature that the choice of the integration measure, implicitly of their weight function, using a Hilbert space basis, is made such that the completeness relation for this basis is satisfied.

In the complex $\mathbb{C}^2$ space the complex variables z and σ are independent and equivalent, then it is logical that the global integration measure can be written as a product of integration measures corresponding to a single complex variable, in a perfect symmetric manner.

$$d\mu(\varepsilon z + \lambda\sigma) = \frac{d\mu(z)}{{}_pF_q(|\varepsilon z|^2)} \times \frac{d\mu(\sigma)}{{}_pF_q(|\lambda\sigma|^2)} \times {}_pF_q\big(|\{\varepsilon z + \lambda\sigma\}|^2\big), \tag{138}$$

where the integration measure for one variable is given by Eq. (3.28), see also [5]:

$$d\mu(z) = d(|z|^2)\, h(|z|^2) \frac{d\phi_z}{2\pi} = \Gamma(\widetilde{a}/\widetilde{b})\, d(|\varepsilon z|^2) \frac{d\phi_z}{2\pi}\, {}_pF_q(|\varepsilon z|^2) \times G_{p,q+1}^{q+1,0}\left(|\varepsilon z|^2 \,\middle|\, \begin{matrix} \{a_i - 1\}_1^p \\ 0,\ \{b_j - 1\}_1^q \end{matrix}\right), \tag{139}$$

and similar for the variable σ.

It can be demonstrated (for details, see Appendix 2) that the coherent states with displaced (shifted) argument satisfy the completeness relation (or decomposition of the identity operator):

$$\int d\mu(\varepsilon z + \lambda\sigma)\, |\varepsilon z + \lambda\sigma\rangle\langle \varepsilon z + \lambda\sigma| = \hat{I}. \tag{140}$$

The proof is presented in the Appendix.

The product $\{Z\}^n\{Z^*\}^m$ appears in many calculations. Let us evaluate it.

$$\{Z\}^n\{Z^*\}^m = \{\varepsilon z + \lambda\sigma\}^n\{\varepsilon z^* + \lambda\sigma^*\}^m = \sum_{s=0}^{n} \binom{\rho(n)}{\rho(s)} (\varepsilon z)^{n-s} (\lambda\sigma)^s \times \sum_{k=0}^{m} \binom{\rho(m)}{\rho(k)} (\varepsilon z^*)^{m-k} (\lambda\sigma^*)^k. \tag{141}$$

The expression of the integration measure can be deduced much more simply if the DOOT rules are used.

After suitable substitutions, the decomposition relation of the unit operator will be (see, Appendix 2)

$$\int \frac{d\mu(Z)}{{}_pF_q(|\{Z\}|^2)} \frac{{}_pF_q(\{Z\}\hat{A}_+)\, {}_pF_q(\{Z^*\}\hat{A}_-)}{{}_pF_q(\hat{A}_+\hat{A}_-)} = \frac{1}{{}_pF_q(\hat{A}_+\hat{A}_-)} \int_0^{\infty} d(|Z|^2)\, h(|Z|^2) \times \int_0^{2\pi} \frac{d\phi_Z}{2\pi}\, {}_pF_q(\{Z\}\hat{A}_+)\, {}_pF_q(\{Z^*\}\hat{A}_-) = 1. \tag{142}$$

The angular integral is

$$\begin{aligned}
&\int_0^{2\pi} \frac{d\phi_Z}{2\pi} {}_pF_q(\{Z\}\hat{A}_+)\, {}_pF_q(\{Z^*\}\hat{A}_-) \\
&= \#\sum_{n=0}^{\infty} \frac{(|\{Z\}|\hat{A}_+)^n}{\rho(n)} \sum_{m=0}^{\infty} \frac{(|\{Z^*\}|\hat{A}_-)^m}{\rho(m)}\# \\
&\quad \times \int_0^{2\pi} \frac{d\phi_Z}{2\pi} \exp\{i(n-m)\phi_Z\} \\
&= \sum_{n=0}^{\infty} \frac{(\hat{A}_+\hat{A}_-)^n}{[\rho(n)]^2} (|\{Z\}|^2)^n\, \delta_{nm}.
\end{aligned} \tag{143}$$

The last angular integral has the value $(|\{Z\}|^2)^l \delta_{ls}$, so that the previous relationship becomes

$$\begin{aligned}
&\frac{1}{{}_pF_q(\hat{A}_+\hat{A}_-)} \sum_{l=0}^{\infty} \frac{(\hat{A}_+\hat{A}_-)^l}{[\rho(l)]^2} \\
&\quad \times \int_0^{\infty} d(|Z|^2)\, h(|Z|)\, (|Z|^2)^l = 1,
\end{aligned} \tag{144}$$

and we obtain the same expression as above and finally, the integration measure.

The decomposition relation of the unit operator will now be written as

$$\begin{aligned}
\int d\mu(Z)\, |Z\rangle\langle Z| &= \frac{1}{{}_pF_q(\hat{A}_+\hat{A}_-)} \\
&\quad \times \int d\mu(Z) \frac{1}{{}_pF_q(|\{Z\}|^2)} \\
&\quad \times {}_pF_q(\{Z\}\hat{A}_+)\, {}_pF_q(\{Z^*\}\hat{A}_-) \\
&= 1.
\end{aligned} \tag{145}$$

In the old variables, this becomes

$$\begin{aligned}
&\int \frac{d^2(\epsilon z + \lambda\sigma)}{\pi} \\
&\quad G^{q+1,0}_{p,q+1}\left(|\{\epsilon z + \lambda\sigma\}|^2 \,\middle|\, \begin{matrix} \{a_i - 1\}_1^p \\ 0,\ \{b_j - 1\}_1^q \end{matrix} \right) \\
&\quad \times {}_pF_q(\{\epsilon z + \lambda\sigma\}\hat{A}_+) \\
&\quad \times {}_pF_q(\{\epsilon z^* + \lambda\sigma^*\}\hat{A}_-) \\
&= {}_pF_q(\hat{A}_+\hat{A}_-).
\end{aligned} \tag{146}$$

In the context of what is indicated above, let us define the generalized analogue of the displacement operator corresponding to the variable Z, as follows:

$$\begin{aligned}
\hat{D}(Z) &= \frac{1}{\sqrt{{}_pF_q(|\{Z\}|^2)}} {}_pF_q(\{Z\}\hat{A}_+), \\
\hat{D}(\varepsilon z + \lambda\sigma) &= \frac{{}_pF_q(\{\varepsilon z + \lambda\sigma\}\hat{A}_+)}{\sqrt{{}_pF_q(|\{\varepsilon z + \lambda\sigma\}|^2)}}.
\end{aligned} \tag{147}$$

so CSs with shifted argument can be defined as follows

$$\begin{aligned}
|Z\rangle &= \hat{D}(Z)|0\rangle \\
&= \frac{1}{\sqrt{{}_pF_q(|\{Z\}|^2)}} {}_pF_q(\{Z\}\hat{A}_+)|0\rangle, \\
|\varepsilon z + \lambda\sigma\rangle &= \hat{D}(\varepsilon z + \lambda\sigma)|0\rangle \\
&= \frac{1}{\sqrt{{}_pF_q(|\{\varepsilon z + \lambda\sigma\}|^2)}} \\
&\quad \times {}_pF_q(\{\varepsilon z + \lambda\sigma\}\hat{A}_+)|0\rangle.
\end{aligned} \tag{148}$$

or, in an equivalent manner

$$|Z\rangle = \frac{1}{\sqrt{{}_pF_q(|\{Z\}|^2)}} \times \sum_{n=0}^{\infty} \frac{\{Z\}^n}{\sqrt{\rho(n)}} |n\rangle, \tag{149}$$

respectively, in the old variables

$$\begin{aligned}
|\varepsilon z + \lambda\sigma\rangle &= \frac{1}{\sqrt{{}_pF_q(|\{\varepsilon z + \lambda\sigma\}|^2)}} \\
&\quad \times \sum_{n=0}^{\infty} \frac{\{\varepsilon z + \lambda\sigma\}^n}{\sqrt{\rho(n)}} |n\rangle.
\end{aligned} \tag{150}$$

In conclusion, bringing together the two ways of defining CSs with displaced argument (i.e. the *Perspective 1* and *2*), we will be able to write

$$|\varepsilon z + \lambda\sigma\rangle = \hat{D}(\varepsilon z + \lambda\sigma)|0\rangle = \hat{D}(\varepsilon z)\hat{D}(\lambda\sigma)|0\rangle. \tag{151}$$

$$\begin{aligned}
|\varepsilon z + \lambda\sigma\rangle &= \frac{{}_pF_q\Big(\{\varepsilon z + \lambda\sigma\}\hat{A}_+\Big)}{\sqrt{{}_pF_q(|\{\varepsilon z + \lambda\sigma\}|^2)}}|0\rangle \\
&= \frac{{}_pF_q(\lambda z\hat{A}_+)}{\sqrt{{}_pF_q(|\varepsilon z|^2)}} \frac{{}_pF_q(\varepsilon\sigma\hat{A}_+)}{\sqrt{{}_pF_q(|\lambda\sigma|^2)}}|0\rangle.
\end{aligned} \tag{152}$$

From here the operational equalities for generalized analogue of the displacement operators follow finally, as a conclusion:

$$\hat{D}(\epsilon z + \lambda\sigma)\,|0\rangle = \hat{D}(\epsilon z)\,\hat{D}(\lambda\sigma)\,|0\rangle. \tag{153}$$

Remark: This relation holds only when the operators act on the vacuum state. As an operator equality, $\hat{D}(\epsilon z + \lambda\sigma) \neq \hat{D}(\epsilon z)\hat{D}(\lambda\sigma)$ in general, because the generalized displacement operators are not unitary and do not form a group representation.

$$\begin{aligned}
&{}_pF_q\Big(\{\varepsilon z + \lambda\sigma\}\hat{A}_+\Big) \\
&= \sqrt{\frac{{}_pF_q(|\{\varepsilon z + \lambda\sigma\}|^2)}{{}_pF_q(|\varepsilon z|^2)\, {}_pF_q(|\lambda\sigma|^2)}} \\
&\quad \times {}_pF_q(\varepsilon z\hat{A}_+)\, {}_pF_q(\lambda\sigma\hat{A}_+).
\end{aligned} \tag{154}$$

In general, a function of two independent variables can be expanded into a power series of the form

$$
\begin{aligned}
&{}_pF_q(\{\epsilon z+\lambda\sigma\}\hat{A}_+)\\
&\quad=\sum_{n=0}^{\infty}\frac{1}{\rho(n)}\left[\sum_{k=0}^{n}\binom{\rho(n)}{\rho(k)}(\epsilon z)^{n-k}(\lambda\sigma)^k\right](\hat{A}_+)^n.
\end{aligned}
\tag{155}
$$

Let us consider an operator-valued function depending on creation and annihilation operators, ordered according to the DOOT technique.

$$
f(\hat{A}_+,\hat{A}_-)=\sum_{k,s=0}^{\infty}C_{ks}\,(\hat{A}_+)^k(\hat{A}_-)^s. \tag{156}
$$

We aim to calculate the expected value of the operator $\hat{D}^\dagger(z)f(\hat{A}_+,\hat{A}_-)\hat{D}(z)$ in the coherent state $|\sigma\rangle$, that is, practically the expected value of the operator $f(\hat{A}_+,\hat{A}_-)$ in the displaced coherent state $|z+\sigma\rangle$. We will have, successively

$$
\begin{aligned}
&\langle\sigma|\hat{D}^\dagger(z)f(\hat{A}_+,\hat{A}_-)\hat{D}(z)|\sigma\rangle\\
&\quad=\langle z+\sigma|f(\hat{A}_+,\hat{A}_-)|z+\sigma\rangle\\
&\quad=\sum_{k,s=0}^{\infty}C_{ks}\langle z+\sigma|(\hat{A}_+)^k(\hat{A}_-)^s|z+\sigma\rangle.
\end{aligned}
\tag{157}
$$

Their expectation value in the coherent states representation is

$$
\begin{aligned}
&\langle\varepsilon z+\lambda\sigma|f(\hat{A}_+,\hat{A}_-)|\varepsilon z+\lambda\sigma\rangle\\
&=\sum_{k,s=0}^{\infty}C_{ks}\langle\varepsilon z+\lambda\sigma|(\hat{A}_+)^k(\hat{A}_-)^s|\varepsilon z+\lambda\sigma\rangle\\
&=\sum_{k,s=0}^{\infty}C_{ks}\{\varepsilon z^*+\lambda\sigma^*\}^k\{\varepsilon z+\lambda\sigma\}^s\\
&=f(\varepsilon z^*+\lambda\sigma^*,\varepsilon z+\lambda\sigma),
\end{aligned}
\tag{158}
$$

because

$$
(\hat{A}_-)^s|\varepsilon z+\lambda\sigma\rangle=\{\varepsilon z+\lambda\sigma\}^s|\varepsilon z+\lambda\sigma\rangle, \tag{159}
$$

$$
\langle\varepsilon z+\lambda\sigma|(\hat{A}_+)^k=\{\varepsilon z^*+\lambda\sigma^*\}^k\langle\varepsilon z+\lambda\sigma|, \tag{160}
$$

and

$$
\begin{aligned}
&\{\varepsilon z^*+\lambda\sigma^*\}^k\{\varepsilon z+\lambda\sigma\}^s\\
&=\sum_{m=0}^{k}\binom{\rho(k)}{\rho(m)}(\varepsilon z^*)^{k-m}(\lambda\sigma^*)^m\\
&\quad\times\sum_{j=0}^{s}\binom{\rho(s)}{\rho(j)}(\varepsilon z)^{s-j}(\lambda\sigma)^j.
\end{aligned}
\tag{161}
$$

Finally, we obtain

$$
\begin{aligned}
&\langle\sigma|\hat{D}^\dagger(z)f(\hat{A}_+,\hat{A}_-)\hat{D}(z)|\sigma\rangle\\
&=\sum_{k,s=0}^{\infty}C_{ks}\sum_{m=0}^{k}\binom{\rho(k)}{\rho(m)}(\varepsilon z^*)^{k-m}(\lambda\sigma^*)^m\\
&\quad\times\sum_{j=0}^{s}\binom{\rho(s)}{\rho(j)}(\varepsilon z)^{s-j}(\lambda\sigma)^j.
\end{aligned}
\tag{162}
$$

or, in a more compact form:

$$
\begin{aligned}
&\langle\sigma|\hat{D}^\dagger(z)f(\hat{A}_+,\hat{A}_-)\hat{D}(z)|\sigma\rangle\\
&\quad=\sum_{k,s=0}^{\infty}C_{ks}\{\varepsilon z^*+\lambda\sigma^*\}^k\{\varepsilon z+\lambda\sigma\}^s\\
&\quad=f(\varepsilon z^*+\lambda\sigma^*,\varepsilon z+\lambda\sigma).
\end{aligned}
\tag{163}
$$

Of particular interest are the following two cases, where the function $f(\hat{A}_+,\hat{A}_-)$ is equal to one of the operators, $\hat{A}_+$ or $\hat{A}_-$ and $\varepsilon=\lambda=1$, $C_{00}=0$.

Case 1 $f(\hat{A}_+)=\hat{A}_+$, $k=1$, $s=0$, $C_{10}=1$

$$
\langle\sigma|\hat{D}^\dagger(z)\hat{A}_+\hat{D}(z)|\sigma\rangle=z^*+\sigma^*. \tag{164}
$$

Case 2 $f(\hat{A}_-)=\hat{A}_-$, $k=0$, $s=1$, $C_{01}=1$

$$
\langle\sigma|\hat{D}^\dagger(z)\hat{A}_-\hat{D}(z)|\sigma\rangle=z+\sigma. \tag{165}
$$

The following formula, satisfied by the canonical displacement operators, is well known from quantum optics textbooks:

$$
\hat{D}^\dagger(z)\hat{a}\hat{D}(z)=\hat{a}+z. \tag{166}
$$

so that we obtain a result similar to the previous one:

$$
\langle\sigma|\hat{D}^\dagger(z)\hat{a}\hat{D}(z)|\sigma\rangle=z+\sigma. \tag{167}
$$

and similar for $\hat{a}^\dagger$.

These results also justify retaining the name the generalized analogue of the displacement operator for $\hat{D}(z)$, despite the fact that this operator is not unitary.

In line with the above, considering an analytical function of operators that takes the normal-ordered product—in the DOOT sense—of generalized creation and annihilation operators as its argument,

$$
f(\hat{A}_+\hat{A}_-)=\sum_{k=0}^{\infty}C_{kk}(\hat{A}_+\hat{A}_-)^k, \tag{168}
$$

we obtain the following relation, the proof being similar:

$$
\begin{aligned}
&\langle\sigma|\hat{D}^\dagger(z)f(\hat{A}_+\hat{A}_-)\hat{D}(z)|\sigma\rangle\\
&\quad=\sum_{k=0}^{\infty}C_{kk}\left(|\{\varepsilon z+\lambda\sigma\}|^2\right)^k\\
&\quad=f\left(|\{\varepsilon z+\lambda\sigma\}|^2\right).
\end{aligned}
\tag{169}
$$

In the particular case, for $\varepsilon = \lambda = 1$, $p = q = 0$, $\widetilde{a} = \widetilde{b}$ and considering σ as a constant, the following equality holds.

$$\begin{aligned}&\int \frac{d^2z}{\pi} G_{0,1}^{1,0}\big(|\{z+\sigma\}|^2|0\big)\\ &\quad\times {}_0F_0\Big(\{z+\sigma\}\hat{A}_+\Big)\,{}_0F_0\Big(\{z^*+\sigma^*\}\hat{A}_-\Big)\\ &= {}_0F_0(\hat{A}_+\hat{A}_-)\\ &= \exp(\hat{A}_+\hat{A}_-).\end{aligned} \tag{170}$$

and, because $\{z+\sigma\} = z+\sigma$, and $\rho(1) = 1$, as well as of the following limits

$$\lim_{\substack{p=q=0\\ a=b}} {}_pF_q(\widetilde{a};\widetilde{b};x) = {}_0F_0(;;x) = e^x. \tag{171}$$

$$\lim_{\substack{p=q=0\\ a=b}} G_{p,q+1}^{q+1,0}\left(x\middle|\begin{matrix}\widetilde{a}-1\\ 0,\widetilde{b}-1\end{matrix}\right) = G_{0,1}^{1,0}(x|0) = e^{-x}. \tag{172}$$

we obtain the same results as in [4].

$$\int \frac{d^2z}{\pi} e^{-|z+\sigma|^2} e^{(z+\sigma)\hat{A}_+} e^{(z^*+\sigma^*)\hat{A}_-} = e^{\hat{A}_+\hat{A}_-}. \tag{173}$$

This integral can be transformed as follows

$$\begin{aligned}&\int \frac{d^2z}{\pi}\,\exp\big\{-|z|^2\\ &\qquad + (\hat{A}_+ - \sigma^*)z + (\hat{A}_- - \sigma)z^*\big\}\\ &= \exp\big\{(\hat{A}_+ - \sigma^*)(\hat{A}_- - \sigma)\big\}.\end{aligned} \tag{174}$$

6 Particular cases

Let us specify the values of the real constants ε and λ.

a) Case $\varepsilon = 1$ ***and*** $\lambda = 0$ ***(generalized CSs with non-displaced argument).*** This is the usual case of obtaining expressions for generalized CSs. Since the variable is no longer a binomial, we can give up the curly brace $\{\cdots\}$, so $\{z\} = z$, and the expansion of the generalized CSs is then

$$\begin{aligned}|z\rangle &= \frac{1}{\sqrt{{}_pF_q(|z|^2)}} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{\rho(n)}}|n\rangle\\ &= \frac{1}{\sqrt{{}_pF_q(|z|^2)}}\,{}_pF_q(z\hat{A}_+)|0\rangle.\end{aligned} \tag{175}$$

with the generalized analogue of the displacement operator

$$\hat{D}(z) = \frac{1}{\sqrt{{}_pF_q(|z|^2)}}\,{}_pF_q(z\hat{A}_+)\,{}_pF_q(z^*\hat{A}_-). \tag{176}$$

The corresponding expression for the integration measure is

$$\begin{aligned}d\mu(z) &= \Gamma(\widetilde{a}/\widetilde{b})\, d(|z|^2)\frac{d\phi_z}{2\pi}\,{}_pF_q(|z|^2)\\ &\quad\times G_{p,q+1}^{q+1,0}\left(|z|^2\,\middle|\,\begin{matrix}\{a_i-1\}_1^p\\ 0,\ \{b_j-1\}_1^q\end{matrix}\right).\end{aligned} \tag{177}$$

guarantees the validity of the decomposition relation of the unit operator, so that the corresponding integral of the Stieltjes moments is

$$\begin{aligned}&\int_0^{\infty} d(|z|^2) G_{p,q+1}^{q+1,0}\left(|z|^2\,\middle|\,\begin{matrix}\{a_i-1\}_1^p\\ 0,\ \{b_j-1\}_1^q\end{matrix}\right)(|z|^2)^l\\ &= \frac{1}{\Gamma(\widetilde{a}/\widetilde{b})}\rho(l) = \frac{1}{\Gamma(\widetilde{a}/\widetilde{b})}\, l!\,\frac{\prod_{j=1}^q \Gamma(b_j+l)}{\prod_{i=1}^p \Gamma(a_i+l)}.\end{aligned} \tag{178}$$

By customizing the indices $p$and q, as well as the sets of coefficients $\widetilde{a}$ and $\widetilde{b}$, numerous examples can be obtained, as well as applications, for the various quantum systems. We continue to reproduce only the case of the pseudoharmonic oscillator (PHO). Let's highlight the following two subcases, which can also be considered as sequels to the previous case.

a1) ***Subcase*** $\varepsilon = 1$, $\lambda = 0$, $p = q$, and $\widetilde{a} = \widetilde{b}$ (*canonical CSs with non-displaced argument*). The canonical CSs are

$$\begin{aligned}|z\rangle &= e^{-\frac{1}{2}|z|^2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}}|n\rangle\\ &= e^{-\frac{1}{2}|z|^2} e^{z\hat{a}^\dagger}|0\rangle.\end{aligned} \tag{179}$$

the integration measure $d\mu(z) = \frac{d^2z}{2\pi} = d(|z|^2)\frac{d\phi_z}{2\pi}$, and the displacement operator is the usual one

$$\begin{aligned}\hat{D}(z) &= \frac{1}{\sqrt{{}_0F_0(|z|^2)}}\,{}_0F_0(z\hat{a}^\dagger)\,{}_0F_0(-z^*\hat{a})\\ &= e^{-\frac{1}{2}|z|^2} e^{z\hat{a}^\dagger - z^*\hat{a}}.\end{aligned} \tag{180}$$

a2) ***Subcase*** $\varepsilon = 1$, $\lambda = 0$, $p = 0$, $q = 1$, $\widetilde{a} = \varnothing$, $\widetilde{b} = \varsigma + 1/2$ (*corresponding to the pseudoharmonic oscillator, PHO*)

The pseudoharmonic oscillator (PHO) is an exactly solvable quantum model, proposed by Goldman and Kryvchenkov [19] and their radial part of potential in radial coordinate r is

$$V(r) = \frac{1}{2}m\omega^2 r^2 + \frac{\hbar^2}{2m}\varsigma(\varsigma-1)\frac{1}{r^2}. \tag{181}$$

where m, ω and ς represent respectively: the mass of particle (or the reduced mass if this potential describe the nuclear motion of diatomic molecule), the frequency, and the strength of the external field.

The corresponding creation / annihilation operators for the PHO are

$$\begin{aligned}\hat{J}_+|n;\varsigma\rangle &= \sqrt{(n+1)\left(n+\varsigma+\frac{1}{2}\right)}\,|n+1;\varsigma\rangle,\\ \hat{J}_-|n;\varsigma\rangle &= \sqrt{n\left(n+\varsigma-\frac{1}{2}\right)}\,|n-1;\varsigma\rangle.\end{aligned} \tag{182}$$

The generalized CSs are defined as

$$\hat{J}_-|z;\varsigma\rangle = z|z;\varsigma\rangle, \tag{183}$$

and are expanded in the Fock-vectors basis as follows:

$$\begin{aligned}|z;\varsigma\rangle &= \frac{1}{\sqrt{{}_0F_1(;\varsigma+\frac{1}{2};|z|^2)}}\\ &\quad\times\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{n!\,(\varsigma+\frac{1}{2})_n}}|n;\varsigma\rangle\\ &= \frac{1}{\sqrt{{}_0F_1(;\varsigma+\frac{1}{2};|z|^2)}}\\ &\quad\times{}_0F_1(;\varsigma+\frac{1}{2};z\hat{J}_+)|0;\varsigma\rangle.\end{aligned} \tag{184}$$

The normalization hypergeometric function is connected with the modified Bessel function of the first kind

$${}_0F_1(;\varsigma+\tfrac{1}{2};|z|^2) = \Gamma(\varsigma+\tfrac{1}{2})\,|z|^{\frac{1}{2}-\varsigma}\,I_{\varsigma-\frac{1}{2}}(2|z|), \tag{185}$$

so that the CSs becomes [16]

$$\begin{aligned}|z;\varsigma\rangle &= \sqrt{\frac{|z|^{\varsigma-\frac{1}{2}}}{I_{\varsigma-\frac{1}{2}}(2|z|)}}\\ &\quad\times\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{n!\,\Gamma(\varsigma+\frac{1}{2}+n)}}|n;\varsigma\rangle.\end{aligned} \tag{186}$$

Then, the corresponding generalized analogue of the displaced operator acting on the vacuum ket state of the PHO, $|0;\varsigma\rangle$ is

$$\hat{D}(z) \equiv \frac{{}_0F_1(;\varsigma+\frac{1}{2};z\hat{J}_+)\,{}_0F_1(;\varsigma+\frac{1}{2};-z^*\hat{J}_-)}{\sqrt{{}_0F_1(;\varsigma+\frac{1}{2};|z|^2)}}. \tag{187}$$

b) Case $\varepsilon=1$ and $\lambda=1$ (*CSs with argument sum of two complex numbers*). The expression for generalized CSs with shifted argument is

$$\begin{aligned}|z+\sigma\rangle &= \frac{1}{\sqrt{{}_pF_q(|\{z+\sigma\}|^2)}}\sum_{n=0}^{\infty}\frac{\{z+\sigma\}^n}{\sqrt{\rho(n)}}|n\rangle\\ &= \frac{1}{\sqrt{{}_pF_q(|\{z+\sigma\}|^2)}}{}_pF_q(\{z+\sigma\}\hat{A}_+)|0\rangle.\end{aligned} \tag{188}$$

and similar their conjugate counterpart.

As we stated previously, the expression in the curly brackets $\{\}$ can be interpreted as a *generalized binomial sum*.

$$\{z+\sigma\}^l \equiv \sum_{n=0}^{l}\binom{\rho(l)}{\rho(n)}z^{l-n}\sigma^n, \tag{189}$$

and similarly their complex conjugate.

It is not difficult to prove that $\langle z+\sigma|z+\sigma\rangle = 1$, and also that the fundamental property of CSs, i.e. the decomposition relation of the unity operator is also satisfied. Since the variables z and σ have "equal rights", the expression must be symmetrical:

$$\iint d\mu(z,\sigma)\,|z+\sigma\rangle\langle z+\sigma| = \hat{I}. \tag{190}$$

Therefore, we will look for the integration measure in the form

$$\begin{aligned}d\mu(z,\sigma) &= \frac{d^2z}{\pi}\,\frac{d^2\sigma}{\pi}\\ &\quad\times h_z(|z|)\,h_\sigma(|\sigma|)\\ &\quad\times{}_pF_q(|\{z+\sigma\}|^2),\end{aligned} \tag{191}$$

and we need to find the weighting functions $h_z(|z|)$ and $h_\sigma(|\sigma|)$. So, we will have

$$\begin{aligned}&\int\frac{d^2z}{\pi}h_z(|z|)\int\frac{d^2\sigma}{\pi}h_\sigma(|\sigma|)\\ &\quad\times\sum_{l=0}^{\infty}\frac{\{z+\sigma\}^l}{\sqrt{\rho(l)}}|l\rangle\sum_{s=0}^{\infty}\frac{\{z^*+\sigma^*\}^s}{\sqrt{\rho(s)}}\langle s| = \hat{I}.\end{aligned} \tag{192}$$

Let's break down the product below

$$\begin{aligned}\{z+\sigma\}^l\,\{z^*+\sigma^*\}^s &= \sum_{n=0}^{l}\binom{\rho(l)}{\rho(n)}z^{l-n}\sigma^n\\ &\quad\times\sum_{m=0}^{s}\binom{\rho(s)}{\rho(m)}(z^*)^{s-m}(\sigma^*)^m.\end{aligned} \tag{193}$$

In order to be able to satisfy the closing relation of the Fock vectors

$$\sum_{l=0}^{\infty}|l\rangle\langle l| = \hat{I}, \tag{194}$$

it is obvious, from now on, that we must have equality between the summation indices $s=l$.

Consequently, we will write

$$\begin{aligned}&\sum_{l=0}^{\infty}\frac{|l\rangle\langle l|}{\rho(l)}\sum_{n=0}^{l}\binom{\rho(l)}{\rho(n)}\int\frac{d^2z}{\pi}h_z(|z|)\,z^{l-n}(z^*)^{l-m}\\ &\quad\times\sum_{m=0}^{l}\binom{\rho(s)}{\rho(m)}\int\frac{d^2\sigma}{\pi}h_\sigma(|\sigma|)\,\sigma^n(\sigma^*)^m = 1.\end{aligned} \tag{195}$$

The angular integrals above will be different from zero only if the powers of the complex variable and the corresponding complex conjugate are equal. For example:

$$\begin{aligned}\int \frac{d^2\sigma}{\pi} h_\sigma(|\sigma|)\, \sigma^n (\sigma^*)^m &= \int_0^\infty d(|\sigma|^2) h_\sigma(|\sigma|) \\ &\times \int_0^{2\pi} \frac{d\phi_\sigma}{2\pi} \sigma^n (\sigma^*)^m \\ &= \int_0^\infty d(|\sigma|) h_\sigma(|\sigma|)\, (|\sigma|^2)^n\, \delta_{nm} \\ &= \Gamma(\widetilde{a}/\widetilde{b})\, \rho(n)\, \delta_{nm}.\end{aligned} \tag{196}$$

Consequently, the relationship becomes

$$\begin{aligned}1 = \sum_{l=0}^{\infty} |l\rangle\langle l|\, [g(l)]! \times \sum_{n=0}^{l} \binom{l}{n} \\ \times \left\{ \frac{1}{(l-n)!\, ([g(l-n)]!)^2}\, \mathrm{Int}_z \right\} \\ \times \left\{ \frac{1}{n!\, ([g(n)]!)^2}\, \mathrm{Int}_\sigma \right\},\end{aligned} \tag{197}$$

where by Int_x we denoted the abbreviated integrals above. For reasons of symmetry, we require that they have the following values:

$$\begin{aligned}\mathrm{Int}_{|z|} &\equiv \frac{1}{\sqrt{2^l}} \int_0^\infty d(|z|^2)\, h_z(|z|)\, (|z|^2)^{l-n} \\ &= \frac{1}{\sqrt{2^l\, [g(l)]!}}\, (l-n)!\, ([g(l-n)]!)^2,\end{aligned} \tag{198}$$

$$\begin{aligned}\mathrm{Int}_{|\sigma|} &\equiv \frac{1}{\sqrt{2^l}} \int_0^\infty d(|\sigma|^2)\, h_\sigma(|\sigma|)\, (|\sigma|^2)^{n} \\ &= \frac{1}{\sqrt{2^l\, [g(l)]!}}\, n!\, ([g(n)]!)^2.\end{aligned} \tag{199}$$

This kind of integral is solved by first changing the exponent of the integration variable, so that it will be $s-1$, with s a complex number:

$$\begin{aligned}\mathrm{Int}_{|z|} &\equiv \int_0^\infty d(|z|^2)\, h_z(|z|)\, (|z|^2)^{s-1}\Big|_{n=l+1-s} \\ &= \frac{1}{\sqrt{2^l\, [g(l)]!}}\, [\Gamma(\widetilde{a}/\widetilde{b})]^2 \\ &\times \frac{\Gamma(s) \prod_{j=1}^q [\Gamma(b_j - 1 + s)]^2}{\prod_{i=1}^p [\Gamma(a_i - 1 + s)]^2},\end{aligned} \tag{200}$$

$$\begin{aligned}\mathrm{Int}_{|z|} &\equiv \int_0^\infty d(|z|^2)\, h_z(|z|)\, (|z|^2)^{s-1}\Big|_{n=l+1-s} \\ &= \frac{1}{\sqrt{2^l\, [g(l)]!}}\, [\Gamma(\widetilde{a}/\widetilde{b})]^2 \\ &\times \frac{\Gamma(s) \prod_{j=1}^q [\Gamma(b_j - 1 + s)]^2}{\prod_{i=1}^p [\Gamma(a_i - 1 + s)]^2},\end{aligned} \tag{201}$$

$$\begin{aligned}\mathrm{Int}_{|\sigma|} &\equiv \int_0^\infty d(|\sigma|^2)\, h_\sigma(|\sigma|)\, (|\sigma|^2)^{s-1}\Big|_{n=s-1} \\ &= \frac{1}{\sqrt{2^l\, [g(l)]!}}\, [\Gamma(\widetilde{a}/\widetilde{b})]^2 \\ &\times \frac{\Gamma(s) \prod_{j=1}^q [\Gamma(b_j - 1 + s)]^2}{\prod_{i=1}^p [\Gamma(a_i - 1 + s)]^2}.\end{aligned} \tag{202}$$

The above indexes were changed in order to be able to use the classical integral of Meijer's G functions [15]:

$$\begin{aligned}\int_0^\infty dx\, x^{s-1}\, G_{p,q}^{m,n}\left(\beta x \,\middle|\, \begin{matrix} \{a_i\}_1^n & \{a_i\}_{n+1}^p \\ \{b_j\}_1^m & \{b_j\}_{m+1}^q \end{matrix} \right) \\ = \frac{1}{\beta^s} \frac{\prod_{j=1}^m \Gamma(b_j + s) \prod_{i=1}^n \Gamma(1 - a_i - s)}{\prod_{j=m+1}^q \Gamma(1 - b_j - s) \prod_{i=n+1}^p \Gamma(a_i + s)}.\end{aligned} \tag{203}$$

As a consequence, the following expressions of the weighting functions are obtained

$$\begin{aligned}h_z(|z|) = \frac{1}{\sqrt{2^l\, [g(l)]!}}\, [\Gamma(\widetilde{a}/\widetilde{b})]^2 \\ \times G_{2p,\,2q+1}^{2q+1,\,0}\left(|z|^2 \,\middle|\, \begin{matrix} \{a_i - 1\}_1^p & \{a_i - 1\}_1^p \\ 0,\ \{b_j - 1\}_1^q & \{b_j - 1\}_1^q \end{matrix} \right),\end{aligned} \tag{204}$$

with the notations

$$\begin{aligned}\{A\}_1^{2p} &\equiv \{a_i - 1\}_1^p,\ \{a_i - 1\}_1^p, \\ \{B\}_1^{2q+1} &\equiv 0,\ \{b_j - 1\}_1^q,\ \{b_j - 1\}_1^q.\end{aligned} \tag{205}$$

and similar for $h_\sigma(|z|)$.

A beautiful symmetry is observed relative to the variables z and σ, which was to be expected.

Substituting these results in the decomposition relation of the unit operator and taking into account the equality

$$\sum_{m=0}^{l} \binom{l}{m} = 2^l, \tag{206}$$

we will find that the closure relation of the Fock vectors is satisfied.

Let us now examine the particular case when $p = q = 0$, $\{a_i\}_1^p = \{b_j\}_1^q$ and, consequently, $\rho(x) = x!$, $[g(0)]! = 1$, which case corresponds to the canonical CSs (characteristics of the HO-1D one-dimensional harmonic oscillator).

These are, keeping in mind that for this case we have ${}_pF_q(\widetilde{a};\widetilde{b};x) = {}_0F_0(;;x) = e^x$, as well as $\{z+\sigma\}^l = (z+\sigma)^l$:

$$
\begin{aligned}
|z+\sigma\rangle &= e^{-\frac{1}{2}|z+\sigma|^2} \sum_{l=0}^{\infty} \frac{(z+\sigma)^l}{\sqrt{l!}} |l\rangle \\
&= e^{-\frac{1}{2}|z+\sigma|^2 + (z+\sigma)\hat{A}_+} |0\rangle.
\end{aligned}
\tag{207}
$$

The integration measure is

$$
d\mu(z,\sigma) = \frac{d^2z}{\pi}\frac{d^2\sigma}{\pi} h_z(|z|)\, h_\sigma(|\sigma|)\, e^{|z+\sigma|^2}, \tag{208}
$$

and the weighting functions are, respectively

$$
\begin{aligned}
h_z(|z|) &= G^{1,0}_{0,1}(|z|^2|0) = e^{-|z|^2}, \\
h_\sigma(|\sigma|) &= G^{1,0}_{0,1}(|\sigma|^2|0) = e^{-|\sigma|^2}.
\end{aligned}
\tag{209}
$$

The decomposition relation of the unity operator is written, therefore

$$
\begin{aligned}
\hat{I} &= \iint d\mu(z,\sigma)\, |z+\sigma\rangle\langle z+\sigma| \\
&= \sum_{l=0}^{\infty} |l\rangle\langle l| \sum_{n=0}^{l} \binom{l}{n} \\
&\quad \times \left\{ \frac{1}{(l-n)!}\, \mathrm{Int}_z \right\} \left\{ \frac{1}{n!}\, \mathrm{Int}_\sigma \right\},
\end{aligned}
\tag{210}
$$

where

$$
\mathrm{Int}_{|z|} = \frac{1}{\sqrt{2^l}}(l-n)!, \qquad \mathrm{Int}_{|\sigma|} = \frac{1}{\sqrt{2^l}} n!, \tag{211}
$$

from which it follows that the equality is satisfied.

7 Concluding remarks

In the paper we obtain the expression of CSs with shifted or displaced argument, whose argument is a linear combination of two different complex variables $Z = \varepsilon z + \lambda\sigma$, where ε and λ are real numbers. In this sense, we used a pair of generalized ladder operators (annihilation and creation) $\hat{A}_- = \hat{a} f(\hat{N})$ and $\hat{A}_+ = f(\hat{N})\hat{a}^\dagger$, where $f(\hat{N})$ is the nonlinearity function. These operators generate different sets of nonlinear CSs which were obtained in two equivalent ways: 1.) the classical method, also used in the case of canonical operators, i.e. by successively applying two displacement operators, each depending on only one complex variable; 2.) the generalized method, by using the generalized analogue of the displacement operators, which were defined by Mojaveri and Dehghani [18], that is, as normalized generalized hypergeometric function, which had $z\hat{A}_+$ as its argument. In both methods, CSs are obtained by applying the normalized displacement operators on the vacuum state ket $|0\rangle$. The general results obtained were verified for several particular cases: the case of generalized CSs with non-displaced argument, with subcases - canonical CSs with non-displaced argument and CSs corresponding to the pseudoharmonic oscillator, PHO, as well as for the case of CSs whose argument was a sum of two complex numbers. This last case is useful for obtaining the expression for the Wigner operator in quantum optics [21]. In all calculations we used the diagonal operator ordering technique (DOOT) which we introduced in [5], and which is a generalization of the integration of an ordered product of operators (IWOP) technique developed by Hong-Yi Fan and applied only to canonical operators [3]. The DOOT technique applies to any pair of non-linear generalized operators $\hat{A}_- = \hat{a} f(\hat{N})$ and $\hat{A}_+ = f(\hat{N})\hat{a}^\dagger$. In essence, the DOOT technique stipulates that the generalized operators $\hat{A}_-$ and $\hat{A}_+$ appearing in expressions within the DOOT brackets $\#\cdots\#$ are commutative and can be treated as simple c-numbers.

Appendix 1

For the sake of brevity, let us denote the completeness relation as follows:

$$
\mathrm{Int}_{(z,\sigma)} \equiv \int d\mu(\varepsilon z + \lambda\sigma)\, |\varepsilon z + \lambda\sigma\rangle\langle \varepsilon z + \lambda\sigma|. \tag{A.1}
$$

After the appropriate substitutions, it becomes

$$
\begin{aligned}
\mathrm{Int}_{(z,\sigma)} &= \int \frac{d\mu(z)}{{}_pF_q(|\epsilon z|^2)} \frac{d\mu(\sigma)}{{}_pF_q(|\lambda\sigma|^2)} \\
&\quad \times {}_pF_q(\{\epsilon z + \lambda\sigma\}\hat{A}_+) \\
&\quad \times {}_pF_q(\{\epsilon z^* + \lambda\sigma^*\}\hat{A}_-) \\
&\quad \times \frac{1}{{}_pF_q(|\{\epsilon z + \lambda\sigma\}|^2)}.
\end{aligned}
\tag{A.2}
$$

$$
\begin{aligned}
\mathrm{Int}_{(z,\sigma)} &= \sum_{n,m=0}^{\infty} \frac{|n\rangle\langle m|}{\sqrt{\rho(n)\rho(m)}} \\
&\quad \times \int \frac{d\mu(z)}{{}_pF_q(|\epsilon z|^2)} \frac{d\mu(\sigma)}{{}_pF_q(|\lambda\sigma|^2)} \\
&\quad \times \{\epsilon z + \lambda\sigma\}^n \{\epsilon z^* + \lambda\sigma^*\}^m.
\end{aligned}
\tag{A.3}
$$

Now,

$$
\begin{aligned}
&\{\varepsilon z + \lambda\sigma\}^n \{\varepsilon z^* + \lambda\sigma^*\}^m \\
&= \sum_{s=0}^{n} \binom{\rho(n)}{\rho(s)} (\varepsilon z)^{n-s} (\lambda\sigma)^s \\
&\quad \times \sum_{k=0}^{m} \binom{\rho(m)}{\rho(k)} (\varepsilon z^*)^{m-k} (\lambda\sigma^*)^k.
\end{aligned}
\tag{A.4}
$$

Substituting (A.4) into (A.3), we get

$$\begin{aligned}\mathrm{Int}_{(z,\sigma)} = &\sum_{n,m=0}^{\infty} \frac{|n\rangle\langle m|}{\sqrt{\rho(n)\rho(m)}} \\ &\times \sum_{s=0}^{n} \binom{\rho(n)}{\rho(s)} \sum_{k=0}^{m} \binom{\rho(m)}{\rho(k)} \\ &\times \mathrm{Int}_z \, \mathrm{Int}_\sigma,\end{aligned} \tag{A.5}$$

where

$$\mathrm{Int}_z \equiv \int \frac{d\mu(z)}{{}_pF_q(|\varepsilon z|^2)} (\varepsilon z)^{n-s} (\varepsilon z^*)^{m-k}, \tag{A.6}$$

and

$$\mathrm{Int}_\sigma \equiv \int \frac{d\mu(\sigma)}{{}_pF_q(|\lambda\sigma|^2)} (\lambda\sigma)^{s} (\lambda\sigma^*)^{k}. \tag{A.7}$$

with the following notations, as well as using the expression for one-variable integration measure.

$$\begin{aligned}d\mu(z) = &\Gamma(\widetilde{a}/\widetilde{b}) d(|z|^2) \frac{d\phi_z}{2\pi} {}_pF_q(|z|^2) \\ &\times G^{q+1,0}_{p,q+1}\left(|z|^2 \;\middle|\; \begin{matrix} \{a_i - 1\}_1^p \\ 0,\ \{b_j - 1\}_1^q \end{matrix} \right).\end{aligned} \tag{A.8}$$

$$\begin{aligned}\mathrm{Int}_{(z)} &\equiv \frac{1}{\sqrt{2^n}} \int \frac{d\mu(z)}{{}_pF_q(|\varepsilon z|^2)} (\varepsilon z)^{n-s} (\varepsilon z^*)^{m-k} \\ &= \frac{1}{\sqrt{2^n}} \Gamma(\widetilde{a}/\widetilde{b}) \\ &\quad \times \int_0^\infty d(|\varepsilon z|^2) G^{q+1,0}_{p,q+1}\left(|\varepsilon z|^2 \;\middle|\; \begin{matrix} \widetilde{a} - 1 \\ 0,\ \widetilde{b} - 1 \end{matrix} \right) \\ &\quad \times \int_0^{2\pi} \frac{d\phi_z}{2\pi} (\varepsilon z)^{n-s} (\varepsilon z^*)^{m-k}.\end{aligned} \tag{A.9}$$

The angular integral is

$$\begin{aligned}\int_0^{2\pi} \frac{d\phi_z}{2\pi} &(\varepsilon z)^{n-s} (\varepsilon z^*)^{m-k} \\ &\text{`} = (|\varepsilon z|^2)^{n-k} \, \delta_{n-s,\, m-k}.\end{aligned} \tag{A.10}$$

and, the radial integral becomes

$$\begin{aligned}&\int_0^\infty d(|\varepsilon z|^2) G^{q+1,0}_{p,q+1}\left(|\varepsilon z|^2 \;\middle|\; \begin{matrix} \widetilde{a} - 1 \\ 0,\ \widetilde{b} - 1 \end{matrix} \right) \\ &\quad \times (|\varepsilon z|^2)^{n-k} \\ &= \Gamma(\widetilde{b}/\widetilde{a}) \, \rho(n-k).\end{aligned} \tag{A.11}$$

Finally, we have

$$\mathrm{Int}_{(z)} = \frac{1}{\sqrt{2^n}} \rho(n-k). \tag{A.12}$$

Analogously, the integral over σ will be

$$\begin{aligned}\mathrm{Int}_{(\sigma)} &\equiv \frac{1}{\sqrt{2^m}} \int \frac{d\mu(\sigma)}{{}_pF_q(|\lambda\sigma|^2)} (\lambda\sigma)^{s} (\lambda\sigma^*)^{k} \\ &= \frac{1}{\sqrt{2^m}} \rho(k) \, \delta_{sk}.\end{aligned} \tag{A.13}$$

Consequently, we have $s = k$, $n - s = m - k$ and finally, $m = n$.

$$\begin{aligned}\mathrm{Int}_{(z,\sigma)} = &\sum_{n,m=0}^{\infty} \frac{|n\rangle\langle m|}{\sqrt{\rho(n)\rho(m)}} \\ &\times \sum_{s=0}^{n} \binom{\rho(n)}{\rho(s)} \sum_{k=0}^{m} \binom{\rho(m)}{\rho(k)} \\ &\times \frac{1}{\sqrt{2^n}} \rho(n-k) \frac{1}{\sqrt{2^m}} \rho(k) \\ = &\sum_{n=0}^{\infty} \frac{|n\rangle\langle n|}{2^n} \sum_{k=0}^{n} \binom{\rho(n)}{\rho(k)}.\end{aligned} \tag{A.14}$$

because it is evident that, in order to accomplish the completeness relation, it must be $m = n$.

Taking into account that

$$(1+1)^n = \sum_{k=0}^{n} \binom{\rho(n)}{\rho(k)} 1^{n-k} 1^k = 2^n, \tag{A.15}$$

finally, we find that the completeness relation is satisfied:

$$\sum_{n=0}^{\infty} |n\rangle\langle n| = \hat{I}. \tag{A.16}$$

and, implicitly, the resolution of the identity operator.

Appendix 2

In many situations, products of two complex-conjugate hypergeometric functions arise—and thus, ultimately, products of two complex-conjugate numbers $\{Z\}$ raised to different powers.

$$\begin{aligned}&{}_pF_q(\{Z\}\hat{A}_+) \, {}_pF_q(\{Z^*\}\hat{A}_-) \\ &= \sum_{n=0}^{\infty} \frac{(\hat{A}_+)^n}{\rho(n)} \sum_{m=0}^{\infty} \frac{(\hat{A}_-)^m}{\rho(m)} \{Z\}^n \{Z^*\}^m.\end{aligned} \tag{A.17}$$

In *Perspective 1* we have:

$$\begin{aligned}\{Z\}^n \{Z^*\}^m &= \{\varepsilon z + \lambda\sigma\}^n \{\varepsilon z^* + \lambda\sigma^*\}^m \\ &= \sum_{s=0}^{n} \binom{\rho(n)}{\rho(s)} \sum_{k=0}^{m} \binom{\rho(m)}{\rho(k)} \\ &\quad \times (\varepsilon z)^{n-s} (\lambda\sigma)^{s} (\varepsilon z^*)^{m-k} (\lambda\sigma^*)^{k}.\end{aligned} \tag{A.18}$$

In an integral over the entire space, of the form

$$\begin{aligned}\mathrm{Int}_{(z,\sigma)}^{(m,n)} &\equiv \int d\mu(z,\sigma) \\ &\quad\times \{\varepsilon z+\lambda\sigma\}^n\{\varepsilon z^*+\lambda\sigma^*\}^m \\ &= \sum_{s=0}^{n}\binom{\rho(n)}{\rho(s)}\sum_{k=0}^{m}\binom{\rho(m)}{\rho(k)} \\ &\quad\times \int d\mu(z,\sigma) \\ &\quad\times (\varepsilon z)^{n-s}(\varepsilon z^*)^{m-k}(\lambda\sigma)^s(\lambda\sigma^*)^k.\end{aligned} \tag{A.19}$$

using the variables $z=|z|e^{i\phi_z}$ and $\sigma=|\sigma|e^{i\phi_\sigma}$, as well as the expression of two complex variable integration measure

$$\begin{aligned}d\mu(Z) &\equiv d\mu(z,\sigma) \\ &= d\mu(\varepsilon z)\,d\mu(\sigma)\,{}_pF_q(|\{\varepsilon z+\lambda\sigma\}|^2) \\ &= d(|\varepsilon z|^2)\,h(|\varepsilon z|^2)\frac{d\phi_z}{2\pi} \\ &\quad\times d(|\lambda\sigma|^2)\,h(|\lambda\sigma|^2)\frac{d\phi_\sigma}{2\pi} \\ &\quad\times {}_pF_q(|\{\varepsilon z+\lambda\sigma\}|^2).\end{aligned} \tag{A.20}$$

we will have

$$\begin{aligned}\left(|Z|^2\right)^l &= \left(|\varepsilon z+\lambda\sigma|^2\right)^l \\ &= (\varepsilon z+\lambda\sigma)^l(\varepsilon z^*+\lambda\sigma^*)^l \\ &= \sum_{n=0}^{l}\binom{l}{n}\sum_{m=0}^{l}\binom{l}{m} \\ &\quad\times (\varepsilon z)^{l-n}(\lambda\sigma)^n(\varepsilon z^*)^{l-m}(\lambda\sigma^*)^m \\ &= \sum_{n=0}^{l}\binom{l}{n}\sum_{m=0}^{l}\binom{l}{m} \\ &\quad\times (\varepsilon|z|)^{2l-n-m}(\lambda|\sigma|)^{n+m} \\ &\quad\times e^{i(m-n)\phi_z}e^{-i(m-n)\phi_\sigma}.\end{aligned} \tag{A.21}$$

Situations involving angular integrations are of particular interest. For example:

From the point of view of *Perspective 1*, we have:

$$\begin{aligned}&\int_0^{2\pi}\frac{d\phi_Z}{2\pi}{}_pF_q([Z]\hat{A}_+)\,{}_pF_q([Z^*]\hat{A}_-) \\ &= \sum_{n,m=0}^{\infty}\frac{(\hat{A}_+)^n}{\rho(n)}\frac{(\hat{A}_-)^m}{\rho(m)}\int_0^{2\pi}\frac{d\phi_Z}{2\pi}([Z])^n([Z^*])^m \\ &= \sum_{n,m=0}^{\infty}\frac{(\hat{A}_+)^n}{\rho(n)}\frac{(\hat{A}_-)^m}{\rho(m)}\#\cdot\mathrm{Int}_{\phi_z}\cdot\mathrm{Int}_{\phi_\sigma}.\end{aligned} \tag{A.22}$$

The last line follows from the fact that the DOOT rules require $m=n$.

$$\begin{aligned}&\mathrm{Int}_{\phi_z} = \sum_{s=0}^{n}\binom{\rho(n)}{\rho(s)}(|\varepsilon z|)^{n-s}(|\varepsilon z|)^{m-k}, \\ &\int_0^{2\pi}\frac{d\phi_z}{2\pi}e^{i(n-s-m+k)\phi_z} = \\ &\qquad\sum_{s=0}^{n}\binom{\rho(n)}{\rho(s)}(|\varepsilon z|^2)^{n-s}\delta_{n-s,\,m-k}.\end{aligned} \tag{A.23}$$

$$\begin{aligned}&\mathrm{Int}_{\phi_\sigma} = \sum_{k=0}^{m}\binom{\rho(m)}{\rho(k)}(|\lambda\sigma|)^s(|\lambda\sigma|)^k, \\ &\int_0^{2\pi}\frac{d\phi_\sigma}{2\pi}e^{i(s-k)\phi_\sigma} = \sum_{k=0}^{m}\binom{\rho(m)}{\rho(k)}(|\lambda\sigma|^2)^k\delta_{sk}.\end{aligned} \tag{A.24}$$

from which it follows that $s=k$ and, consequently, $m=n$. This is in agreement with the DOOT rules, in which, into an ordered product of creation and annihilation operators, inside of sign $\#\cdots\#$, the powers of these operators must be equal. Then, the integral becomes

$$\begin{aligned}&\int_0^{2\pi}\frac{d\phi_Z}{2\pi}{}_pF_q([Z]\hat{A}_+)\,{}_pF_q([Z^*]\hat{A}_-) \\ &= \sum_{n,m=0}^{\infty}\frac{(\hat{A}_+)^n}{\rho(n)}\frac{(\hat{A}_-)^m}{\rho(m)}\# \\ &\quad\times\sum_{k=0}^{n}\binom{\rho(n)}{\rho(k)}(|\varepsilon z|^2)^{n-s}(|\lambda\sigma|^2)^k \\ &\quad\times\sum_{k=0}^{m}\binom{\rho(m)}{\rho(k)}(|\varepsilon z|^2)^{m-k}(|\lambda\sigma|^2)^k \\ &= \sum_{n=0}^{\infty}\frac{(\hat{A}_+\hat{A}_-)^n}{[\rho(n)]^2}\left((|\varepsilon z|^2+|\lambda\sigma|^2)^2\right)^n.\end{aligned} \tag{A.25}$$

The last line results from the following calculation:

$$\begin{aligned}\left[|Z|^2\right]^n &= \big[|\varepsilon z|^2+|\lambda\sigma|^2 \\ &\qquad +2|\varepsilon z||\lambda\sigma|\cos(\phi_z-\phi_\sigma)\big]^n \\ &= \sum_{j=0}^{n}\binom{n}{j}\left(|\varepsilon z|^2+|\lambda\sigma|^2\right)^{n-j} \\ &\quad\times(2|\varepsilon z||\lambda\sigma|)^j\cos^j(\phi_z-\phi_\sigma).\end{aligned} \tag{A.26}$$

$$\begin{aligned}\cos^j(\phi_z-\phi_\sigma) &= \frac{1}{2^j}\left[e^{i(\phi_z-\phi_\sigma)}+e^{-i(\phi_z-\phi_\sigma)}\right]^j \\ &= \frac{1}{2^j}\sum_{k=0}^{j}\binom{j}{k}e^{i(\phi_z-\phi_\sigma)(j-k)}e^{-i(\phi_z-\phi_\sigma)k} \\ &= \frac{1}{2^j}\sum_{k=0}^{j}\binom{j}{k}e^{i\phi_z(j-2k)}e^{-i\phi_\sigma(j-2k)}.\end{aligned} \tag{A.27}$$

$$\begin{aligned}
&\int_0^{2\pi}\frac{d\phi_Z}{2\pi}{}_pF_q([Z]\hat{A}_+)\,{}_pF_q([Z^*]\hat{A}_-)\\
&=\sum_{n=0}^{\infty}\frac{(\hat{A}_+\hat{A}_-)^n}{[\rho(n)]^2}\int_0^{2\pi}\frac{d\phi_Z}{2\pi}\left[|Z|^2\right]^n\\
&=\sum_{n=0}^{\infty}\frac{(\hat{A}_+\hat{A}_-)^n}{[\rho(n)]^2}\,\mathrm{Int}_{|Z|^2},
\end{aligned}\tag{A.28}$$

where

$$\begin{aligned}
\mathrm{Int}_{|Z|^2}&=\int_0^{2\pi}\frac{d\phi_z}{2\pi}\int_0^{2\pi}\frac{d\phi_\sigma}{2\pi}\left[|Z|^2\right]^n\\
&=\sum_{j=0}^{n}\binom{n}{j}\left(|\varepsilon z|^2+|\lambda\sigma|^2\right)^{n-j}(|\varepsilon z||\lambda\sigma|)^j\\
&\quad\times\sum_{k=0}^{j}\binom{j}{k}\int_0^{2\pi}\frac{d\phi_z}{2\pi}e^{i\phi_z(j-2k)}\\
&\quad\times\int_0^{2\pi}\frac{d\phi_\sigma}{2\pi}e^{-i\phi_\sigma(j-2k)}\\
&=\sum_{k=0}^{j}\binom{j}{k}\delta_{j-2k,0}\delta_{j-2k,0}\sum_{k=0}^{2k}\binom{2k}{k}=2^{2k},
\end{aligned}\tag{A.29}$$

because $j=2k$, i.e. j must be an even number.

$$\int_0^{2\pi}\frac{d\phi_z}{2\pi}e^{i\phi_z(j-2k)}=\delta_{j-2k,\,0},\qquad\Rightarrow\qquad j=2k.\tag{A.30}$$

Finally, we obtain

$$\begin{aligned}
&\int_0^{2\pi}\frac{d\phi_Z}{2\pi}{}_pF_q(\{Z\}\hat{A}_+)\,{}_pF_q(\{Z^*\}\hat{A}_-)\\
&=\sum_{n=0}^{\infty}\frac{(\hat{A}_+\hat{A}_-)^n}{[\rho(n)]^2}\sum_{k=0}^{n}\binom{n}{2k}\\
&\quad\times\left(|\varepsilon z|^2+|\lambda\sigma|^2\right)^{n-2k}(2|\varepsilon z||\lambda\sigma|)^{2k}\\
&=\sum_{n=0}^{\infty}\frac{(\hat{A}_+\hat{A}_-)^n}{[\rho(n)]^2}\left(|\varepsilon z|^2+|\lambda\sigma|^2+2|\varepsilon z||\lambda\sigma|\right)^n\\
&=\sum_{n=0}^{\infty}\frac{(\hat{A}_+\hat{A}_-)^n}{[\rho(n)]^2}\left((|\varepsilon z|+|\lambda\sigma|)^2\right)^n\\
&=\sum_{n=0}^{\infty}\frac{(\hat{A}_+\hat{A}_-)^n}{[\rho(n)]^2}\left(|Z|^2\right)^n.
\end{aligned}\tag{A.31}$$

$$\begin{aligned}
&\int_0^{2\pi}\frac{d\phi_Z}{2\pi}{}_pF_q(\{Z\}\hat{A}_+)\,{}_pF_q(\{Z^*\}\hat{A}_-)\\
&=\sum_{n,m=0}^{\infty}\frac{(\hat{A}_+)^n}{\rho(n)}\frac{(\hat{A}_-)^m}{\rho(m)}\#\\
&\quad\times\sum_{k=0}^{n}\binom{\rho(n)}{\rho(k)}(|\varepsilon z|^2)^{n-s}(|\lambda\sigma|^2)^k\\
&\quad\times\sum_{k=0}^{m}\binom{\rho(m)}{\rho(k)}(|\varepsilon z|^2)^{m-k}(|\lambda\sigma|^2)^k\\
&=\sum_{n=0}^{\infty}\frac{(\hat{A}_+\hat{A}_-)^n}{[\rho(n)]^2}\left((|\varepsilon z|+|\lambda\sigma|)^2\right)^n.
\end{aligned}\tag{A.32}$$

In *Perspective 2* we have:

$$\begin{aligned}
\mathrm{Int}_{(Z)}&\equiv\int_0^{2\pi}\frac{d\phi_Z}{2\pi}{}_pF_q(\{Z\}\hat{A}_+)\,{}_pF_q(\{Z^*\}\hat{A}_-)\\
&=\sum_{n,m=0}^{\infty}\frac{(\hat{A}_+)^n}{\rho(n)}\frac{(\hat{A}_-)^m}{\rho(m)}\\
&\quad\times\int_0^{2\pi}\frac{d\phi_Z}{2\pi}([Z])^n([Z^*])^m\\
&=\sum_{n,m=0}^{\infty}\frac{(\hat{A}_+)^n}{\rho(n)}\frac{(\hat{A}_-)^m}{\rho(m)}\\
&\quad\times|Z|^n|Z|^m\int_0^{2\pi}\frac{d\phi_Z}{2\pi}e^{i(n-m)\phi_Z}\\
&=\sum_{n=0}^{\infty}\frac{(\hat{A}_+\hat{A}_-)^n}{[\rho(n)]^2}\left(|Z|^2\right)^n.
\end{aligned}\tag{A.33}$$

In conclusion, the same result is obtained from both *Perspective 1* and *Perspective 2*. However, given that the calculations for *Perspective 2* are simpler and more direct, it is preferable to use that approach.

Acknowledgement

The author is deeply grateful to the (anonymous) reviewer for the pertinent observations regarding the manuscript, which will undoubtedly contribute to improving its quality.

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